The Klein-Gordon equation with the Woods-Saxon potential well

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We solve the Klein-Gordon equation in the presence of a spatially one-dimensional Woods-Saxon potential. The bound state solutions are derived. The pair creation mechanism and the antiparticle bound state are discussed.

Keywords: Klein-Gordon equation; Woods-Saxon potential.

Se resuelve la ecuación de Klein-Gordon para el potencial de Woods-Saxon unidimensional independiente del tiempo. Se derivan las soluciones para los estados ligados. Se discute la creación de pares y los estados ligados de antipartículas.

Descripciones: Ecuación de Klein-Gordon; potencial de Woods-Saxon.

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1. Introduction

The discussion of the overcritical behavior of bosons requires a full understanding of the single particle spectrum. For short range potentials, the solutions of the Klein-Gordon equation can exhibit spontaneous production of antiparticles as the strength of an external potential reaches a certain value \( V_0 \) [1]. In 1940, Schiff, Snyder and Weinberg [2] carried out one of the earliest investigations of the solution of the Klein-Gordon equation with a strong external potential. They solved the problem of the square well potential and discovered that there is a critical point where the bound antiparticle mode appears to coalesce with the bound particle. In 1979, Bawin [3] showed that the antiparticle p-wave bound state arises for some conditions on the potential parameters. In the present article, we solve the Klein-Gordon equation for the Woods-Saxon potential well. We show that the antiparticle bound states arise also for the Woods-Saxon potential, which is a smoothed out form of the square well potential. We also show how overcritical effects depend on the shape of the short range potential.

2. The Klein-Gordon equation

The one-dimensional Klein-Gordon equation to solve is, in natural units \( \hbar = c = m = 1 \) [4],

\[
\frac{d^2 \phi(x)}{dx^2} + \left[ (E - V(x))^2 - 1 \right] \phi(x) = 0. \quad (1)
\]

3. The Woods-Saxon potential

The Woods-Saxon potential is defined as [5]

\[
V(x) = V_0 \left[ \frac{\Theta(-x)}{1 + e^{-a(x+L)}} + \frac{\Theta(x)}{1 + e^{a(x-L)}} \right], \quad (2)
\]

where \( V_0 \) is real and positive for a barrier or negative for a well potential; \( a > 0 \) and \( L > 0 \) are real and positive. \( \Theta(x) \) is the Heaviside step function. The parameter \( a \) defines the shape of the barrier or well. The form of the Woods-Saxon potential is showed in the Fig. 1.

4. Bound states

Consider the bound states solutions for \( x < 0 \). We solve the differential equation

\[
\frac{d^2 \phi_L(x)}{dx^2} + \left[ (E + \frac{V_0}{1 + e^{-a(x+L)}})^2 - 1 \right] \phi_L(x) = 0. \quad (3)
\]

Upon making the substitution \( y^{-1} = 1 + e^{-a(x+L)} \), Eq. (3) becomes

\[
a^2 y(1-y) \frac{d}{dy} \left[ y(1-y) \frac{d \phi_L(y)}{dy} \right] + \left[ (E + V_0y)^2 - 1 \right] \phi_L(y) = 0. \quad (4)
\]

Setting \( \phi_L(y) = y^{\sigma} (1-y)^{\gamma} h(y) \) and substituting into Eq. (4), we obtain that \( h(y) \) satisfies the hypergeometric equation:

\[
y(1-y)h'' + [(1 + 2\sigma) - 2(\sigma + \gamma + 1)y]h' - (\sigma + \gamma + \lambda) (\sigma + \gamma + \lambda - \lambda) h = 0, \quad (5)
\]

where the primes denote derivates with respect to \( y \) and

\[
\sigma = \sqrt{1 - E^2/a^2}, \quad (6)
\]

![Figura 1. El potencial de Woods-Saxon para \( L = 2 \) con \( a = 10 \) (línea sólida) y \( a = 3 \) (línea punteada).](image)
\[ \gamma = \sqrt{1 - (E + V_0)^2}, \quad \lambda = \sqrt{a^2 - 4V_0^2}. \] (7)

Equation (5) has the general solution [6]

\[ h(y) = a_1 y F_1 \left( \frac{1}{2} + \gamma + \sigma - \lambda, \frac{1}{2} + \gamma + \sigma + \lambda, 1 + 2\sigma; y \right) \]

\[ + a_2 y^{-2\sigma} F_1 \left( \frac{1}{2} + \gamma - \sigma - \lambda, \frac{1}{2} + \gamma - \sigma + \lambda, 1 - 2\sigma; y \right), \] (8)

so that

\[ \phi_L(y) = a_1 y^{\gamma} (1-y)^{1+\lambda} F_1 \]

\[ \times \left( \frac{1}{2} + \gamma + \sigma - \lambda, \frac{1}{2} + \gamma + \sigma + \lambda, 1 + 2\sigma; y \right) \]

\[ + a_2 y^{-\sigma} (1-y)^{1+\lambda} F_1 \]

\[ \times \left( \frac{1}{2} + \gamma - \sigma - \lambda, \frac{1}{2} + \gamma - \sigma + \lambda, 1 - 2\sigma; y \right). \] (9)

Now we consider the solution for \( x > 0 \). We solve the differential equation

\[ \frac{d^2 \phi_R(x)}{dx^2} + \left[ \left( E + \frac{V_0}{1 + e^{a(x-L)}} \right)^2 - 1 \right] \phi_R(x) = 0. \] (10)

After making the substitution \( z^{-1} = 1 + e^{a(x-L)} \), Eq. (10) can be written as:

\[ a^2 z(1-z) \frac{d^2 \phi_R(z)}{dz^2} \]

\[ + \left[ (E + V_0 z)^2 - 1 \right] \phi_R(z) = 0. \] (11)

Introducing \( \phi_R(z) = z^\sigma (1-z)^{-\gamma} g(z) \) and substituting it into Eq. (11), we obtain that \( \phi_R(z) \) satisfies the hypergeometric equation:

\[ z(1-z) g'' + \left[ (1 + 2\sigma) - 2(\sigma - \gamma + 1) \right] g' \]

\[ - (\frac{1}{2} + \sigma - \gamma + \lambda)(\frac{1}{2} + \sigma - \gamma - \lambda) g = 0, \] (12)

where the prime denotes a derivative with respect to \( z \). The general solution of Eq. (12) is [6]

\[ g(z) = b_1 F_1 \left( \frac{1}{2} - \gamma + \sigma - \lambda, \frac{1}{2} - \gamma + \sigma + \lambda, 1 + 2\sigma; z \right) \]

\[ + b_2 z^{-2\sigma} F_1 \left( \frac{1}{2} - \gamma - \sigma - \lambda, \frac{1}{2} - \gamma - \sigma + \lambda, 1 - 2\sigma; z \right), \] (13)

so that

\[ \phi_R(z) = b_1 z^{\sigma} (1-z)^{-\gamma} F_1 \]

\[ \times \left( \frac{1}{2} - \gamma + \sigma - \lambda, \frac{1}{2} - \gamma + \sigma + \lambda, 1 + 2\sigma; z \right) \]

\[ + b_2 z^{-\sigma} (1-z)^{-\gamma} F_1 \]

\[ \times \left( \frac{1}{2} - \gamma - \sigma - \lambda, \frac{1}{2} - \gamma - \sigma + \lambda, 1 - 2\sigma; z \right). \] (14)

As \( x \to -\infty, y \to 0 \) and \( x \to \infty, z \to 0 \). We choose the regular wave functions

\[ \phi_L(y) = a_1 y^{\gamma} (1-y)^{1+\lambda} F_1 \]

\[ \times \left( \frac{1}{2} + \gamma + \sigma - \lambda, \frac{1}{2} + \gamma + \sigma + \lambda, 1 + 2\sigma; y \right) \]

\[ \phi_R(z) = b_1 z^{\sigma} (1-z)^{-\gamma} F_1 \]

\[ \times \left( \frac{1}{2} - \gamma + \sigma - \lambda, \frac{1}{2} - \gamma + \sigma + \lambda, 1 + 2\sigma; z \right) \] (15)

In order to find the energy eigenvalues, we impose the condition that the right and left wave functions and their first derivatives must be matched at \( x = 0 \). This condition leads to

\[ \frac{1}{1 + 2\sigma} \left[ \left( \frac{1}{2} + \gamma + \sigma - \lambda \right) \left( \frac{1}{2} + \gamma + \sigma + \lambda \right) \right. \]

\[ \times F_1 \left( \frac{3}{2} + \gamma + \sigma - \lambda, \frac{3}{2} + \gamma + \sigma + \lambda, 2 + 2\sigma, (1 + e^{-aL})^{-1} \right) \]

\[ \left. + \left( \frac{1}{2} - \gamma + \sigma - \lambda \right) \left( \frac{1}{2} - \gamma + \sigma + \lambda \right) \right] \]

\[ \times F_1 \left( \frac{3}{2} - \gamma + \sigma - \lambda, \frac{3}{2} - \gamma + \sigma + \lambda, 2 + 2\sigma, (1 + e^{-aL})^{-1} \right) \]

\[ + 2\sigma (1 + e^{-aL}) = 0, \] (16)

which is the eigenvalue condition for energy \( E \). The explicit solutions of Eq. (16), giving \( E \) in terms of \( V_0 \), can be determined numerically. We consider the range \(-1 < E < 1\) for the values of \( E \). Some aspects of the dependence of the spectrum of bound states on the potential strength \( V_0 \) are shown in Figs. 2 and 3. At some value of \( V_0 \), a bound antiparticle state appears, it joins with the bound particle state, they form a state with zero norm at \( V_0 = V_{cr} \), and then both vanish from the spectrum.
The normalization of the wave functions (15) is given by

\[ N = 2 \int_{-\infty}^{\infty} dx [E - V(x)] \phi(x)^* \phi(x), \quad (17) \]

The norm of the Klein-Gordon equation vanishes at \( V_{cr} \), where both possible solution \( E^{(+)} \) and \( E^{(-)} \) meet.

Figure 2 shows that for \( 2.0900 < V_0 < 2.0908 \) two states appear, one with positive energy and another with negative energy. In Fig. 3 the same behavior is observed for \( 2.3462 < V_0 < 2.3463 \). Particle bound states \( (E^{(+)}) \) and antiparticle bound states \( (E^{(-)}) \) correspond to \( N > 0 \) and \( N < 0 \) respectively. For \( N = 0 \), both solutions meet and have the same energy. Antiparticle states appear in all the cases considered. For \( L = 2 \), we moved the shape parameter \( a \) from 1 to 18 and, for \( L = 1 \), we considered \( a = 10 \). Figures 4 and 5 show the behavior of the turning point \( (E) \) versus the potential parameters \( a \) and \( V_0 \) respectively. Figure 4 shows that, as the value of \( a \) increases, the energy value for which antiparticle states appear increases. In Fig. 5 we observe that, as the value of \( V_0 \) increases, the energy value for which antiparticle states appear decreases. This behavior indicates that well potentials exhibit antiparticle bound states for values of \( E \) larger than for smoothed out potentials.

5. Conclusions

The Woods-Saxon potential, analogous to the square well potential, shows antiparticle bound states. The turning point, where the norm of the state is zero, depends on the potential parameters \( a \) and \( V_0 \). The results reported in this article show that the Woods-Saxon potential exhibits a behavior characteristic of short range potentials [7].

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2. L.I. Schiff, H. Snyder, and J. Weinberg, Phys. Rev. 57 (1940) 315.