The Klein-Gordon equation with the Woods-Saxon potential well

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We solve the Klein-Gordon equation in the presence of a spatially one-dimensional Woods-Saxon potential. The bound state solutions are derived. The pair creation mechanism and the antiparticle bound state are discussed.

Keywords: Klein-Gordon equation; Woods-Saxon potential.

Se resuelve la ecuación de Klein-Gordon para el potencial de Woods-Saxon unidimensional independiente del tiempo. Se derivan las soluciones para los estados ligados. Se discute la creación de pares y los estados ligados de antipartículas.

Descriptores: Ecuación de Klein-Gordon; potencial de Woods-Saxon.

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1. Introduction

The discussion of the overcritical behavior of bosons requires a full understanding of the single particle spectrum. For short range potentials, the solutions of the Klein-Gordon equation can exhibit spontaneous production of antiparticles as the strength of an external potential reaches a certain value V_0 [1]. In 1940, Schiff, Snyder and Weinberg [2] carried out one of the earliest investigations of the solution of the Klein-Gordon equation with a strong external potential. They solved the problem of the square well potential and discovered that there is a critical point V_{cr} where the bound antiparticle mode appears to coalesce with the bound particle. In 1979, Bawin [3] showed that the antiparticle p-wave bound state arises for some conditions on the potential parameters. In the present article, we solve the Klein-Gordon equation for the Woods-Saxon potential well. We show that the antiparticle bound states arise also for the Woods-Saxon potential, which is a smoothed out form of the square well potential. We also show how overcritical effects depend on the shape of the short range potential.

2. The Klein-Gordon equation

The one-dimensional Klein-Gordon equation to solve is, in natural units $\hbar = c = m = 1$ [4],

$$\frac{d^2\phi(x)}{dx^2} + \left[(E - V(x))^2 - 1\right]\phi(x) = 0. \tag{1}$$

3. The Woods-Saxon potential

The Woods-Saxon potential is defined as [5]

$$V(x) = V_0\left[\frac{\Theta(-x)}{1+e^{-a(x+L)}} + \frac{\Theta(x)}{1+e^{a(x-L)}}\right], \tag{2}$$

where V_0 is real and positive for a barrier or negative for a well potential; $a > 0$ and $L > 0$ are real and positive. $\Theta(x)$ is the Heaviside step function. The parameter a defines the shape of the barrier or well. The form of the Woods-Saxon potential is showed in the Fig. 1.

4. Bound states

Consider the bound states solutions for $x < 0$. We solve the diferential equation

$$\frac{d^2\phi_L(x)}{dx^2} + \left[\left(E + \frac{V_0}{1+e^{-a(x+L)}}\right)^2 - 1\right]\phi_L(x) = 0. \tag{3}$$

Upon making the substitution $y^{-1}=1+e^{-a(x+L)}$, Eq. (3) becomes

$$a^2 y(1-y)\frac{d}{dy}\left[y(1-y)\frac{d\phi_L(y)}{dy}\right] + \left[(E+V_0 y)^2 - 1\right]\phi_L(y) = 0. \tag{4}$$

Setting $\phi_L(y) = y^\sigma(1-y)^\gamma h(y)$ and substituting into Eq.(4), we obtain that $h(y)$ satisfies the hypergeometric equation:

$$y(1-y)h'' + [(1+2\sigma) - 2(\sigma+\gamma+1)y]h' - \left(\tfrac{1}{2}+\sigma+\gamma+\lambda\right)\left(\tfrac{1}{2}+\sigma+\gamma-\lambda\right)h = 0, \tag{5}$$

where the primes denote derivates with respect to y and

$$\sigma = \frac{\sqrt{1-E^2}}{a}, \tag{6}$$

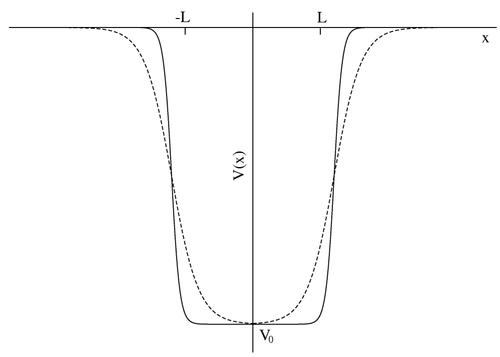

FIGURE 1. The Woods-Saxon potential barrier for $L = 2$ with $a = 10$ (solid line) and $a = 3$ (dotted line).

$$\gamma=\frac{\sqrt{1-(E+V_0)^2}}{a},\lambda=\frac{\sqrt{a^2-4V_0^2}}{2a}. \quad (7)$$

Equation (5) has the general solution [6]

$$h(y)=a_1\,{}_2F_1\left(\frac{1}{2}+\gamma+\sigma-\lambda,\frac{1}{2}+\gamma+\sigma+\lambda,1+2\sigma;y\right)$$
$$+a_2y^{-2\sigma}\,{}_2F_1\left(\frac{1}{2}+\gamma-\sigma-\lambda,\frac{1}{2}+\gamma-\sigma+\lambda,1-2\sigma;y\right), \quad (8)$$

so that

$$\phi_L(y)=a_1y^{\sigma}(1-y)^{\gamma}\,{}_2F_1\times\left(\frac{1}{2}+\gamma+\sigma-\lambda,\frac{1}{2}+\gamma+\sigma+\lambda,1+2\sigma;y\right)$$
$$+a_2y^{-\sigma}(1-y)^{\gamma}\,{}_2F_1\times\left(\frac{1}{2}+\gamma-\sigma-\lambda,\frac{1}{2}+\gamma-\sigma+\lambda,1-2\sigma;y\right). \quad (9)$$

Now we consider the solution for $x>0$. We solve the diferential equation

$$\frac{d^2\phi_R(x)}{dx^2}+\left[\left(E+\frac{V_0}{1+e^{a(x-L)}}\right)^2-1\right]\phi_R(x)=0. \quad (10)$$

After making the substitution $z^{-1}=1+e^{a(x-L)}$, Eq. (10) can be written as:

$$a^2z(1-z)\frac{d}{dz}\left[z(1-z)\frac{d\phi_R(z)}{dz}\right]+\left[(E+V_0z)^2-1\right]\phi_R(z)=0. \quad (11)$$

Introducing $\phi_R(z)=z^{\sigma}(1-z)^{-\gamma}g(z)$ and substituting it into Eq. (11), we obtain that $\phi_R(z)$ satisfies the hypergeometric equation:

$$z(1-z)g''+[(1+2\sigma)-2(\sigma-\gamma+1)z]g'-\left(\tfrac{1}{2}+\sigma-\gamma+\lambda\right)\left(\tfrac{1}{2}+\sigma-\gamma-\lambda\right)g=0, \quad (12)$$

where the prime denotes a derivative with respect to z. The general solution of Eq. (12) is [6]

$$g(z)=b_1\,{}_2F_1\left(\frac{1}{2}-\gamma+\sigma-\lambda,\frac{1}{2}-\gamma+\sigma+\lambda,1+2\sigma;z\right)$$
$$+b_2z^{-2\sigma}\,{}_2F_1\left(\frac{1}{2}-\gamma-\sigma-\lambda,\frac{1}{2}-\gamma-\sigma+\lambda,1-2\sigma;z\right), \quad (13)$$

so that

$$\phi_R(z)=b_1z^{\sigma}(1-z)^{-\gamma}\,{}_2F_1\times\left(\frac{1}{2}-\gamma+\sigma-\lambda,\frac{1}{2}-\gamma+\sigma+\lambda,1+2\sigma;z\right)$$
$$+b_2z^{-\sigma}(1-z)^{-\gamma}\,{}_2F_1\times\left(\frac{1}{2}-\gamma-\sigma-\lambda,\frac{1}{2}-\gamma-\sigma+\lambda,1-2\sigma;z\right). \quad (14)$$

As $x\to-\infty$, $y\to0$ and $x\to\infty$, $z\to0$. We choose the regular wave functions

$$\phi_L(y)=a_1y^{\sigma}(1-y)^{\gamma}\,{}_2F_1\times\left(\frac{1}{2}+\gamma+\sigma-\lambda,\frac{1}{2}+\gamma+\sigma+\lambda,1+2\sigma;y\right)$$
$$\phi_R(z)=b_1z^{\sigma}(1-z)^{-\gamma}\,{}_2F_1\times\left(\frac{1}{2}-\gamma+\sigma-\lambda,\frac{1}{2}-\gamma+\sigma+\lambda,1+2\sigma;z\right) \quad (15)$$

In order to find the energy eigenvalues, we impose the condition that the right and left wave functions and their first derivatives must be matched at $x=0$. This condition leads to

$$\frac{1}{1+2\sigma}\left[\left(\frac{1}{2}+\gamma+\sigma-\lambda\right)\left(\frac{1}{2}+\gamma+\sigma+\lambda\right)\times\frac{F_1\left(\frac{3}{2}+\gamma+\sigma-\lambda,\frac{3}{2}+\gamma+\sigma+\lambda,2+2\sigma,(1+e^{-aL})^{-1}\right)}{F_1\left(\frac{1}{2}+\gamma+\sigma-\lambda,\frac{1}{2}+\gamma+\sigma+\lambda,1+2\sigma,(1+e^{-aL})^{-1}\right)}\right.$$
$$\left.+\left(\frac{1}{2}-\gamma+\sigma-\lambda\right)\left(\frac{1}{2}-\gamma+\sigma+\lambda\right)\times\frac{F_1\left(\frac{3}{2}-\gamma+\sigma-\lambda,\frac{3}{2}-\gamma+\sigma+\lambda,2+2\sigma,(1+e^{-aL})^{-1}\right)}{F_1\left(\frac{1}{2}-\gamma+\sigma-\lambda,\frac{1}{2}-\gamma+\sigma+\lambda,1+2\sigma,(1+e^{-aL})^{-1}\right)}\right]$$
$$+2\sigma\left(1+e^{-aL}\right)=0, \quad (16)$$

which is the eigenvalue condition for energy E. The explicit solutions of Eq. (16), giving E in terms of V_0, can be determined numerically. We consider the range $-1<E<1$ for the values of E. Some aspects of the dependence of the spectrum of bound states on the potential strength V_0 are shown in Figs. 2 and 3. At some value of V_0, a bound antiparticle state appears, it joins with the bound particle state, they form a state with zero norm at $V_0=V_{cr}$, and then both vanish from the spectrum.

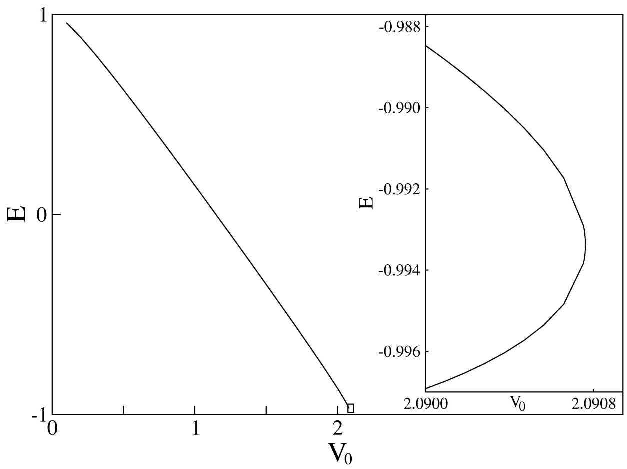

FIGURE 2. Bound-state spectrum for $L=2$, $a=10$.

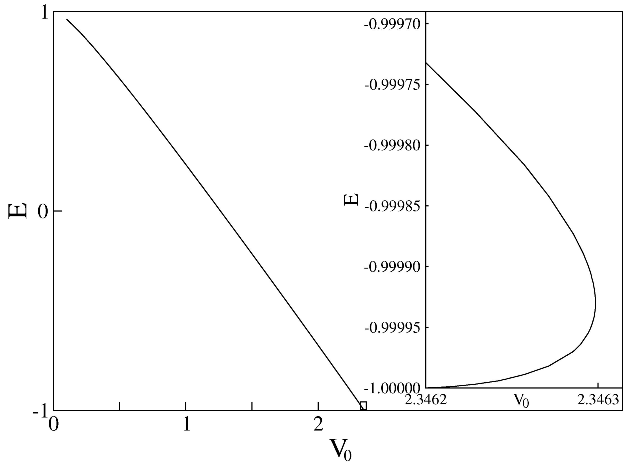

FIGURE 3. Bound-state spectrum for $L = 2$, $a = 2$.

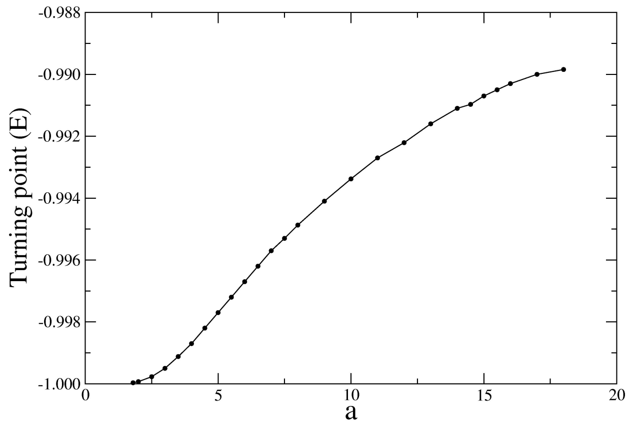

FIGURE 4. Turning point vs a, for $L = 2$.

The normalization of the wave functions (15) is given by

$$N = 2 \int_{-\infty}^{\infty} dx[E - V(x)]\phi(x)^*\phi(x), \qquad (17)$$

The norm of the Klein-Gordon equation vanishes at V_{cr}, where both possible solution $E^{(+)}$ and $E^{(-)}$ meet.

Figure 2 shows that for $2.0900 < V_0 < 2.0908$ two states appear, one with positive energy and another with negative energy. In Fig. 3 the same behavior is observed for $2.3462 < V_0 < 2.3463$. Particle bound states ($E^{(+)}$) and antiparticle bound states ($E^{(-)}$) correspond to $N > 0$ and $N < 0$ respectively. For $N = 0$, both solutions meet and have the same energy. Antiparticle states appear in all the cases considered. For $L = 2$, we moved the shape parameter a from 1 to 18 and, for $L = 1$, we considered $a = 10$. Figures 4 and 5 show the behavior of the turning point (E) versus the potential parameters a and V_0 respectively. Figure 4 shows that, as the value of a increases, the energy value for which antiparticle states appear increases. In Fig. 5 we observe that, as the value of V_0 increases, the energy value for which antiparticle states appear decreases. This behavior indicates that well potentials exhibit antiparticle bound states for values of E larger than for smoothed out potentials.

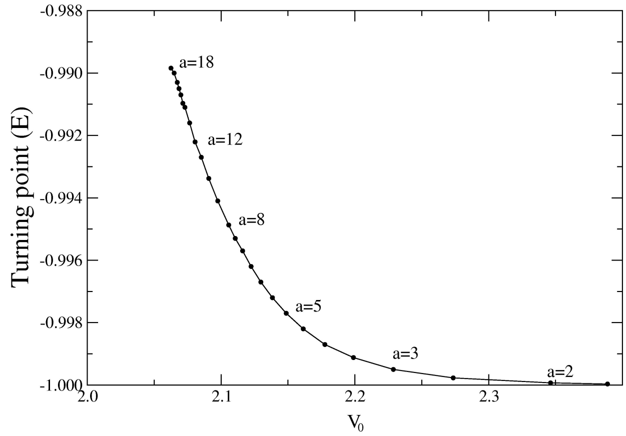

FIGURE 5. Turning point vs V_0, for $L = 2$.

5. Conclusions

The Woods-Saxon potential, analogous to the square well potential, shows antiparticle bound states. The turning point, where the norm of the state is zero, depends on the potential parameters a and V_0. The results reported in this article show that the Woods-Saxon potential exhibits a behavior characteristic of short range potentials [7].

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