On Bell’s theorem*

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Bell’s theorem is a crucial tool to solve the contradiction between local realism conceptions à la Einstein-Podolsky-Rosen and quantum mechanics. In this note we present three versions of the theorem: the original one of Bell of 1964, the Clauser et al stronger version of 1969, and the Wigner’s version of 1970, and show how the simultaneous validity of EPR ideas and quantum theory leads to a contradiction with number theory. The quantum mechanical formula for the correlation between two spin 1/2 particles is derived in an appendix.

Keywords: Bell theorem; quantum mechanics; EPR.

1. Introduction
In 1935, Einstein, Podolsky and Rosen showed that the hypothesis of locality and a properly defined criterion of reality of physical quantities, implied that physical observables with non commuting quantum operators could have exact values simultaneously i.e. simultaneous reality; so, quantum mechanics could not be a complete theory of the physical world, if by a complete theory we understand a theory in which every element of the physical reality has a counterpart in the physical theory.

This discovery led to the proposals of hidden variables theories, local and non local, the best known being the non local theory of Bohm [17,18]. It was not till 1964 when Bell, with his famous theorem, provided a theoretical and experimental tool to decide between local realism, inherent to the EPR approach, and the mathematical theory of quantum mechanics with its usual Copenhagen interpretation. Theoretical because their simultaneous validity leads to a contradiction with number theory (inadmissible!), and experimental because it forced many laboratory tests of the quantum mechanical formulae. Most of these tests were in agreement with the theory, the most celebrated being that of Aspect and collaborators [19, 20]. The conclusion of many authors, shared by the author of this note, is that local realism à la EPR does not hold.

In this note we present three different versions of Bell’s theorem: the 1964 original version, then the 1969 version of Clauser et al, and finally the Wigner’s version of 1970. In an appendix we give a detailed derivation of the formula for the average value of the product of two spin 1/2 particles in arbitrary directions.

*Based mainly in Ref. 1.

2. Introductory comment
According to Bell, von Neumann stated that there can not even exist a non local hidden variables theory (in Ref. 1, Bell occupied himself of local hidden variables theories, proving that they are in contradiction with quantum mechanics). In Ref. 2, Bell showed that von Neumann’s statement was not correct. (In this connection, in Ref. 1, Bell also mentioned the work of J. M. Jauch and C. Piron, Helv. Phys. Acta 36 (1963) 827).

3. Bell’s logic (based on Refs. 3 and 4; see also Ref. 5)

1. One prepares a pair of spin 1/2 particles in the singlet state, and leaves the particles to travel in opposite senses.

2. One measures the spin components \( \vec{\sigma}_1 \cdot \hat{a} \) and \( \vec{\sigma}_2 \cdot \hat{b} \) of the particles 1 and 2 respectively. \( \hat{a} \) and \( \hat{b} \) are unitary vectors in \( \mathbb{R}^3 \) i.e. \( \hat{a}, \hat{b} \in S^2 \). \( \hat{a} \) and \( \hat{b} \) are experimentally controllable parameters. If \( \hat{b} = \hat{a} \) and \( \vec{\sigma}_1 \cdot \hat{a} = +1 \) then \( \vec{\sigma}_2 \cdot \hat{a} = -1 \) and vice versa i.e. \( \vec{\sigma}_1 \cdot \hat{a} = -\vec{\sigma}_2 \cdot \hat{a} \). This is a quantum mechanical result.

3. According to Einstein, the measurement at 1 (2) can not affect the simultaneous measurement at 2 (1): this is the hypothesis of locality. Since, however, according to quantum mechanics, the measurement at 1 (2) predicts the result of the measurement at 2 (1), for not
4. Quantum mechanics (the wave function) however, does not predict the results of individual measurements (e.g. $\sigma^x$) can be $+1$ or $-1$); therefore the predetermination in 3 demands a more complete description of a quantum state.

5. The parameters which effect the more complete description are denoted by $\lambda$, and are called hidden variables. If $\lambda$ specifies the results of the measurements of the spin on 1 and 2, respectively through $A(b, \lambda)$, $\hat{a}$, and $\hat{b}$, with values in $\{+1, -1\}$. If $\rho(\lambda)$ is the probability distribution of $\lambda$

\[
\left( \int_{\Lambda} d\lambda \rho(\lambda) = 1, \rho(\lambda) \geq 0 \right)
\]

then the average of the product of the measurements of $\hat{a} \cdot \hat{b}$ is given by

\[
P(\hat{a}, \hat{b}) = \int_{\Lambda} d\lambda \rho(\lambda) A(\hat{a}, \lambda) B(\hat{b}, \lambda). \quad (1)
\]

In these formulae, $\Lambda$ is the domain of $\lambda$. By their own nature, the $\lambda$ parameters are not experimentally controllable. The functions $P(\hat{a}, \hat{b})$ are called correlation functions between the spins or, simply, correlations. From the mathematical point of view, $A$ and $B$ are functions with domain $S^2 \times \Lambda$ and codomain $R$ or, with more precision, $\{-1, +1\}$ i.e.,

\[
A, B : S^2 \times \Lambda \rightarrow R \text{ or } S^2 \times \Lambda \rightarrow \{-1, +1\}.
\]

For the case discussed in 10,

\[
A, B : S^2 \times \Lambda \rightarrow [-1, +1].
\]

6. In quantum mechanics, the correlation is given by

\[
P(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}. \quad (2)
\]

(See appendix.)

7. **Theorem**: 5. and 6. are contradictory. (See the proof in 9.)

8. **Observation**. The fact that $A$ does not depend on $\hat{b}$ and that $B$ does not depend on $\hat{a}$, is the expression of locality in this context. The variables $\lambda$ are, in principle, global. Also, even if $\lambda$ determines the result of each measurement on 1 and 2, respectively through $A$ and $B$, there is a statistical element present, given by $\rho(\lambda)$. If $\lambda = (\lambda_1, \ldots, \lambda_n)$, it is then possible a "locality" in $\lambda$ with $\lambda_i = \lambda_i^{(A)}$ for

\[
i = 1, 2, \ldots, p, \quad \text{and} \quad \lambda_j = \lambda_j^{(B)}
\]

for $j = p + 1, \ldots, p + q = n$. i.e.

\[
A = A(\hat{a}, \lambda_1^{(A)}, \ldots, \lambda_p^{(A)}) = \pm 1
\]

and $B = B(\hat{b}, \lambda_{p+1}^{(B)}, \ldots, \lambda_{p+q}^{(B)}) = \pm 1$.

9. **Proof of 7**.

If $\hat{b} = \hat{a}$, then $B(\hat{a}, \lambda) = -A(\hat{a}, \lambda)$ for all $\hat{a} \in S^2$ and all $\lambda \in \Lambda$; then $P(\hat{a}, \hat{b}) = -\int_{\Lambda} d\lambda \rho(\lambda) A(\hat{a}, \lambda) A(\hat{b}, \lambda)$. If $\hat{c}$ is another unitary vector, then $P(\hat{a}, \hat{c}) = -\int_{\Lambda} d\lambda \rho(\lambda) A(\hat{a}, \lambda) A(\hat{c}, \lambda)$ and therefore

\[
P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c}) = -\int_{\Lambda} d\lambda \rho(\lambda) (A(\hat{a}, \lambda) A(\hat{b}, \lambda) - A(\hat{a}, \lambda) A(\hat{c}, \lambda));
\]

since $(A(\hat{x}, \lambda))^2 = 1$, one has

\[
P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c}) = -\int_{\Lambda} d\lambda \rho(\lambda) (1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda)) A(\hat{a}, \lambda) A(\hat{b}, \lambda).
\]

Then

\[
|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c})| = \left| -\int_{\Lambda} d\lambda \rho(\lambda) (1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda)) A(\hat{a}, \lambda) A(\hat{b}, \lambda) \right| \leq \int_{\Lambda} \rho(\lambda) |1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda)| A(\hat{a}, \lambda) A(\hat{b}, \lambda) |
\]

\[
= \int_{\Lambda} d\lambda \rho(\lambda) (1 - A(\hat{b}, \lambda) A(\hat{c}, \lambda))
\]

\[
= \int_{\Lambda} d\lambda \rho(\lambda) - \int_{\Lambda} d\lambda \rho(\lambda) A(\hat{b}, \lambda) A(\hat{c}, \lambda) = 1 + P(\hat{b}, \hat{c}),
\]

where we have used that $|A(\hat{a}, \lambda)A(\hat{b}, \lambda)| = 1$ and that $1 - A(\hat{b}, \lambda)A(\hat{c}, \lambda) \geq 0$. Therefore we have obtained

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c})| \leq 1 + P(\hat{b}, \hat{c}), \quad (3)$$

which is the inequality obtained by Bell in Ref. 1. Notice that $\lambda$ does not appear in Eq. (3). This formula appears in Eq. (A.12) in Ref. 7, and in Eq. (8) in Ref. 8.

Let us see how, in some cases, Eqs. (2) and (3) are contradictory: Let $\hat{a}$, $\hat{b}$, and $\hat{c}$ be in the same plane, with $\hat{a} \cdot \hat{b} = 0$ and $\hat{c}$ in the bisection of $\hat{a}$ and $\hat{b}$. Then, according to quantum mechanics,

$$P(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b} = 0, \quad P(\hat{a}, \hat{c}) = -\hat{a} \cdot \hat{c} = -\sqrt{2}/2,$$

and

$$P(\hat{b}, \hat{c}) = -\hat{b} \cdot \hat{c} = -\sqrt{2}/2;$$

if these values are replaced in (3), one obtains

$$|0 + \sqrt{2}/2| \leq 1 + (-\sqrt{2}/2)$$

i.e.

$$\sqrt{2}/2 \leq 1 - \sqrt{2}/2,$$

which amounts to $\sqrt{2} \leq 1$: false! Therefore, Bell’s inequality is violated by quantum mechanics. QED

(Note: In Ref. 11, page 36, Bell argues that the above proof does not restrict to non relativistic quantum mechanics, but only depends on the existence of separated systems "highly correlated" with respect to quantities like the spin.)

10. **Stronger Bell's inequality**

If $A(\hat{a}, \lambda) = \pm 1$ then $|A(\hat{a}, \lambda)| \leq 1$; let us assume this weaker condition. The difference between correlations is given by

$$P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}') = \int \Lambda d\lambda \rho(\lambda)(A(\hat{a}, \lambda)B(\hat{b}, \lambda) - A(\hat{a}, \lambda)B(\hat{b}', \lambda))$$

$$= \int \Lambda d\lambda \rho(\lambda)A(\hat{a}, \lambda)B(\hat{b}, \lambda)(1 \pm A(\hat{a}', \lambda)B(\hat{b}', \lambda)) - \int \Lambda d\lambda \rho(\lambda)A(\hat{a}, \lambda)B(\hat{b}', \lambda)(1 \pm A(\hat{a}', \lambda)B(\hat{b}, \lambda))$$

where we have summed and subtracted the integral

$$\int \Lambda d\lambda \rho(\lambda)A(\hat{a}, \lambda)A(\hat{a}', \lambda)B(\hat{b}, \lambda)B(\hat{b}', \lambda);$$

taking absolute values,

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| \leq \int \Lambda d\lambda \rho(\lambda)|A(\hat{a}, \lambda)||B(\hat{b}, \lambda)||1 \pm A(\hat{a}', \lambda)B(\hat{b}', \lambda)|$$

$$+ \int \Lambda d\lambda \rho(\lambda)|A(\hat{a}, \lambda)||B(\hat{b}', \lambda)||1 \pm A(\hat{a}', \lambda)B(\hat{b}, \lambda)|$$

$$\leq \int \Lambda d\lambda \rho(\lambda)(1 \pm A(\hat{a}', \lambda)B(\hat{b}', \lambda)) + \int \Lambda d\lambda \rho(\lambda)(1 \pm A(\hat{a}', \lambda)B(\hat{b}, \lambda))$$

$$= 2 \pm \left(\int \Lambda d\lambda \rho(\lambda)A(\hat{a}', \lambda)B(\hat{b}', \lambda) + \int \Lambda d\lambda \rho(\lambda)A(\hat{a}', \lambda)B(\hat{b}, \lambda)\right)$$

$$= 2 \pm (P(\hat{a}', \hat{b}') + P(\hat{a}', \hat{b})) \leq 2 \pm |P(\hat{a}', \hat{b}') + P(\hat{a}', \hat{b})|,$$

then

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| \pm |P(\hat{a}', \hat{b}') + P(\hat{a}', \hat{b})| \leq 2$$

and therefore

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| + |P(\hat{a}', \hat{b}') + P(\hat{a}', \hat{b})| \leq 2, \quad (4)$$

which is the desired Bell’s inequality. This formula is eq. (9) in Ref. 9 (Bell, 1971), eq. (11) in Ref. 8 (Jackiw and Shimony, 2001), eq. (1a) in Ref. 10 (Clauser et al, 1969), and eq. (20-6) in Ref. 6 (Ballentine, 1990).

11. **Proposition**: Eq. (4) implies Eq. (3).

Proof. Since Eq. (4) was obtained from a weaker condition, one expects that in a particular case it will re-
duce to Eq. (3). If in Eq. (4) we make \( \hat{a}' = \hat{b} \), then in the second term of the left hand side one has

\[
|P(\hat{b}, \hat{b}') + P(\hat{b}, \hat{b})|,
\]

but

\[
P(\hat{b}, \hat{b}) = \int_{\Lambda} d\lambda \rho(\lambda) A(\hat{b}, \lambda) B(\hat{b}, \lambda)
\]

\[
= - \int_{\Lambda} d\lambda \rho(\lambda) A(\hat{b}, \lambda) A(\hat{b}, \lambda) = -1
\]

if one restricts to \( A(\hat{b}, \lambda) \in \{-1, +1\} \); then

\[
|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| + |1 + P(\hat{b}, \hat{b}')| \leq 2.
\]

On the other hand,

\[
|P(\hat{b}, \hat{b}')| = |\int_{\Lambda} d\lambda \rho(\lambda) A(\hat{b}, \lambda) B(\hat{b}', \lambda)|
\]

\[
\leq \int_{\Lambda} d\lambda \rho(\lambda) |A(\hat{b}, \lambda)||B(\hat{b}', \lambda)|
\]

\[
= \int_{\Lambda} \rho(\lambda) = 1,
\]

and therefore \( -1 + P(\hat{b}, \hat{b}') \leq 0 \), then

\[
| -1 + P(\hat{b}, \hat{b}')| = -(1 - P(\hat{b}, \hat{b}')) = 1 - P(\hat{b}, \hat{b}')
\]

and therefore

\[
|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| + 1 - P(\hat{b}, \hat{b}') \leq 2
\]

implies

\[
|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| \leq 1 + P(\hat{b}, \hat{b}'),
\]

which is eq. (3). QED


Proof. Choose \( \hat{a}, \hat{a}', \hat{b}, \) and \( \hat{b}' \) in the same plane, with \( \hat{b} = \hat{a}' \) and with the angle \( \theta \) between \( \hat{a} \) and \( \hat{a}' \) the same as the angle between \( \hat{a} \) and \( \hat{b} \). Then, using Eq. (2) in Eq. (4), we have:

\[
\frac{1}{\cos^2 \theta} + \frac{1}{\cos^2 \theta} - 1 \leq 2;
\]

\[
| - \cos \theta + \cos 2\theta| + | - \cos \theta - 1| \leq 2;
\]

then

\[
| - \cos \theta + \cos 2\theta| + \cos \theta \leq 1.
\]

If \( \theta \in [0, \pi/2] \), then \( -\cos \theta + \cos 2\theta < 0 \) and therefore

\[
| - \cos \theta + \cos 2\theta| = \cos \theta - \cos 2\theta
\]

which gives \( 2 \cos \theta - \cos 2\theta \leq 1 \). I.e.

\[
2 \cos \theta - \cos^2 \theta + \sin^2 \theta \leq 1,
\]

then

\[
2 \cos \theta - \cos^2 \theta \leq 1 - \sin^2 \theta = \cos^2 \theta.
\]

I.e.

\[
2(\cos \theta - \cos^2 \theta) \leq 0;
\]

so for \( \theta \in (0, \pi/2) \), \( 1 \leq \cos \theta \): false! QED

13. Wigner’s formulation of Bell’s inequalities

Wigner’s formulation is based in probabilities instead of correlations (averages of products of spin projections). By the hypothesis of Einstein’s locality, which together with quantum mechanics, demands the realism hypothesis, each particle of the pair (1,2) has a spin projection along each of the directions \( \hat{a} \equiv a, \hat{b} \equiv b \) and \( c \equiv c \) of \( R^3 \); these projections were determined at the moment in which the particles were together (e.g. before the decay of the original particle or molecule in the singlet state). What in the Bell’s formulation is represented by the probability distribution \( \rho(\lambda) \), in this formulation it is represented by the probabilities

\[
p(a, b, c, a', b', c'), \quad (a' = -a, b' = -b, c' = -c),
\]

where the first three entries refer to particle 1, and the second three entries refer to particle 2:

\[
p(+, +, +, -), p(-, +, +, -),
\]

\[
p(+, -, +, -, ), p(+, +, - , -),
\]

\[
p(- , - , + , - ), p(-, +, - , +),
\]

\[
\]

Since in the singlet state, the second three entries are determined by the first three entries, it is sufficient to denote these probabilities by

\[
p(+, +, +), p(-, +, +), p(+, -, +), p(+, +, -),
\]

\[
p(-, - , +), p(-, -, +), p(-, - , -), p(-, +, -).
\]
Let us consider the quantity \( p_{a,b}(+,-) \): probability that the particle 1 has spin +1 in direction \( a \) and spin -1 in direction \( b \). This quantity (and its similars) can be measured without interference of measurements: one measures on particle 1 the cases with \( \sigma_a = +1 \) and on particle 2 the cases with \( \sigma_b = +1 \) i.e. \( p_{a,b}(+,-) = p(1a+,2b+) \) with an obvious notation. It is clear that
\[
p_{a,b}(+,-) = p(+,-,+) + p(+,-,-),
\]
and from the non negativity of probabilities
\[
p_{a,b}(+,-) \leq p_{a,c}(+,-) + p_{b,c}(+,-), \tag{5}
\]
which is Bell’s inequality in the Wigner’s formulation i.e.
\[
p(1a+,2c+) \leq p(1a+,2b+) + p(1b+,2c+).
\]
Introducing the result of quantum mechanics
\[
p(1a+,2c+) = \frac{1}{2} \sin^2 \left( \frac{\theta_{ac}}{2} \right), \tag{6}
\]
(see appendix) where \( \theta_{ac} \) is the angle between the unitary vectors \( \hat{a} \) and \( \hat{c} \), one obtains
\[
\frac{1}{2} \sin^2 \left( \frac{\theta_{ac}}{2} \right) \leq \frac{1}{2} \sin^2 \left( \frac{\theta_{ab}}{2} \right) + \frac{1}{2} \sin^2 \left( \frac{\theta_{ac}}{2} \right). \tag{7}
\]
For some angles, Eq. (7) is contradictory, what again establishes a contradiction between quantum mechanics and the hypothesis which lead to Eq. (5). This inequality (and its analogous) appears in Eq. (3) in Ref. 12 (Wigner, 1970), eq. (48) in Ref. 13 (Wigner, 1983), eq. in pg. 171 in Ref. 14 (d’Espagnat, 1979), Eq. (3.9.12) in Ref. 15 (Sakurai, 1985), and Eq. (7.7) in Ref. 16 (Teimian, 1999).

Let \( \hat{a}, \hat{b}, \hat{c} \) be in the same plane, with \( \hat{b} \) the bisection of \( \hat{a} \) and \( \hat{c} \), so if \( \theta_{ac} = 2\theta \) then \( \theta_{ab} = \theta_{bc} = \theta \), with \( \theta \in [0, \pi/2] \). Eq. (7) gives
\[
\sin^2 \theta \leq 2 \sin^2 \frac{\theta}{2}.
\]
If \( \theta = 0 \) then \( 0 \leq 0 \); if \( \theta = \pi/2 \) then
\[
1 \leq 2 \times \sin^2 (\pi/4) = 2 \times (1/\sqrt{2})^2 = 1;
\]
let \( \theta \in (0, \pi/2) \),
\[
\sin^2 \theta = 4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}
\]
and therefore \( 2 \cos^2 \frac{\theta}{2} \leq 1 \) which implies \( \cos \frac{\theta}{2} \leq \frac{1}{\sqrt{2}} \) for \( \frac{\theta}{2} \in (0, \pi/4) \): false! Then, with this election of directions, quantum mechanics is contradictory with the Bell’s inequality for all \( \theta \in (0, \pi/2) \).

Appendix

**Proposition:** In quantum mechanics, the probability that two spin 1/2 particles in the singlet state (i.e. with total spin \( S = 0 \)) have their spins in the positive directions of the unit vectors \( \hat{n}_1 \) and \( \hat{n}_2 \), is given by
\[
P_{++} = \frac{1}{2} \sin^2 \left( \frac{\alpha_{12}}{2} \right)
\]
where \( \alpha_{12} \) is the angle between \( \hat{n}_1 \) and \( \hat{n}_2 \).

**Proof.**
\[
P_{++} = | \langle \chi_0, \chi_{n_1}, \chi_{n_2} \rangle |^2
\]
where
\[
\chi_0 = \frac{1}{\sqrt{2}}(\chi_{1+} \chi_{2-} - \chi_{1-} \chi_{2+})
\]
and
\[
\chi_{nk+} = \cos \left( \frac{\theta_k}{2} \right) \chi_{k+} + e^{i\varphi_k} \sin \left( \frac{\theta_k}{2} \right) \chi_{k-}, \quad k = 1, 2.
\]
We shall use the notation:
\[
\sin \theta_1 = s_1, \quad \sin \theta_2 = s_2, \quad \cos \frac{\theta_1}{2} = c_{\frac{\theta_1}{2}},
\]
\[
\sin \frac{\theta_1}{2} = s_{\frac{\theta_1}{2}}, \quad \cos \frac{\theta_2}{2} = c_{\frac{\theta_2}{2}}, \quad \sin \frac{\theta_2}{2} = s_{\frac{\theta_2}{2}},
\]
\[
\cos \varphi_1 = C_1, \quad \sin \varphi_1 = S_1, \quad \cos \varphi_2 = C_2,
\]
\[
\sin \varphi_2 = S_2, \quad \cos (\varphi_1 - \varphi_2) = C_{12}, \quad \cos \varphi_{12} = c_{12},
\]
\[
\sin \left( \frac{\alpha_{12}}{2} \right) = s_{\frac{\alpha_{12}}{2}}, \quad \sin \left( \frac{\theta_1 - \theta_2}{2} \right) = s_{\frac{\theta_1 - \theta_2}{2}},
\]
\[
\cos (\theta_1 - \theta_2) = c_{\frac{\theta_1 - \theta_2}{2}}
\]
and the formulae
\[
\sin \lambda = 2 \sin \frac{\lambda}{2} \cos \frac{\lambda}{2}
\]
and \( \sin^2 \frac{\lambda}{2} = \frac{1}{2} (1 - \cos \lambda) \).
In this notation:

\[ \hat{n}_1 \cdot \hat{n}_2 = c_{12} = (s_1C_1, s_1S_1, c_1)(s_2C_2, s_2S_2, c_2) \]

\[ = s_1s_2(C_1C_2 + S_1S_2) + c_1c_2 \]

\[ = s_1s_2C_{12} + c_1c_2, \]

then

\[ \frac{1}{2} s_1^2 = \frac{1}{4}(1 - c_{12}) = \frac{1}{4}(1 - s_1s_2C_{12} - c_1c_2); \quad (A1) \]

\[ P_{++} = \frac{1}{2} \left( s_1^2 s_2^2 + s_1^2 s_2^2 - e^{i(\varphi_2 - \varphi_1)} s_1^2 s_2^2 - e^{-i(\varphi_2 - \varphi_1)} s_1^2 s_2^2 \right) \]

\[ = \frac{1}{2} \left( s_1^2 s_2^2 + s_1^2 s_2^2 - 2s_1s_2C_{12} \right) \]

\[ = \frac{1}{2} \left( s_1^2 - c_1s_2^2 \right)^2 + (1 - C_{12}) c_1s_2s_1c_2 = \frac{1}{2} \left( s_{12}^2 + 2s_1c_1s_2c_2 (1 - C_{12}) \right) = \frac{1}{2} \left( s_{12}^2 + \frac{1}{2}s_1s_2 - \frac{1}{2}s_1s_2C_{12} \right) \]

\[ = \frac{1}{2} \left( \frac{1}{2} \left( 1 - (c_1s_2 + s_1s_2) \right) + \frac{1}{2}s_1s_2 - \frac{1}{2}s_1s_2C_{12} \right) = \frac{1}{4} (1 - c_1c_2 - C_{12}s_1s_2), \quad (A2) \]

and (A1) = (A2). QED

To simplify, let us write \( \alpha_{12} \equiv \alpha \). From the definition and formula for \( P_{++} \) we obtain the formulae for \( P_{+-}, P_{-+} \) and \( P_{--} \), with obvious interpretation of the notation:

\[ P_{++} = \frac{1}{2} \sin^2 \left( \frac{\pi - \alpha}{2} \right) = \frac{1}{2} \cos^2 \frac{\alpha}{2}, \]

\[ P_{-+} = \frac{1}{2} \sin^2 \left( \frac{\pi - \alpha}{2} \right) = \frac{1}{2} \cos^2 \frac{\alpha}{2}, \quad P_{--} = \frac{1}{2} \sin^2 \frac{\alpha}{2}. \]

Then:

\[ P_{++} + P_{--} = \sin^2 \frac{\alpha}{2} \]

= probability that the product of the spins be equal to +1,

and

\[ P_{+-} + P_{-+} = \cos^2 \frac{\alpha}{2} \]

= probability that the product of the spins be equal to −1.

We have then the

**Corollary:** The average of the product of the spins, \( P(\hat{n}_1, \hat{n}_2) \) is given by

\[ P(\hat{n}_1, \hat{n}_2) = -\hat{n}_1 \cdot \hat{n}_2. \]

**Proof.**

\[ P(\hat{n}_1, \hat{n}_2) = (1) \times \text{prob.}(1) + (-1) \times \text{prob.}(-1) \]

\[ = \sin^2 \frac{\alpha}{2} - \cos^2 \frac{\alpha}{2} = -\cos \alpha = -\hat{n}_1 \cdot \hat{n}_2. \]

QED.

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