On Bell's theorem*

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Bell's theorem is a crucial tool to solve the contradiction between local realism conceptions *à la* Einstein-Podolsky-Rosen and quantum mechanics. In this note we present three versions of the theorem: the original one of Bell of 1964, the Clauser *et al* stronger version of 1969, and the Wigner's version of 1970, and show how the simultaneous validity of EPR ideas and quantum theory leads to a contradiction with number theory. The quantum mechanical formula for the correlation between two spin 1/2 particles is derived in an appendix.

Keywords: Bell theorem; quantum mechanics; EPR.

El teorema de Bell es una herramienta crucial para resolver la contradicción entre concepciones realistas locales *à la* Einstein-Podolsky-Rosen y la mecánica cuántica. En esta nota presentamos tres versiones del teorema: la original de Bell de 1964, la versión más fuerte de Clauser *et al* de 1969 y la versión de Wigner de 1970, y mostramos cómo la validez simultánea de las ideas de EPR y la teoría cuántica conducen a una contradicción con la teoría de números. En un apéndice se deduce la fórmula que da la correlación cuántica entre dos partículas de espín 1/2.

Descriptores: Teorema de Bell; mecánica cuántica; EPR.

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1. Introduction

In 1935, Einstein, Podolsky and Rosen showed that the hypotesis of *locality* and a properly defined criterion of *reality* of physical quantities, implied that physical observables with non commuting quantum operators could have exact values simultaneously *i.e.* simultaneous reality; so, quantum mechanics could not be a *complete* theory of the physical world, if by a complete theory we understand a theory in which every element of the physical reality has a counterpart in the physical theory.

This discovery led to the proposals of *hidden variables theories*, local and non local, the best known being the non local theory of Bohm [17,18]. It was not till 1964 when Bell, with his famous theorem, provided a theoretical and experimental tool to decide between *local realism*, inherent to the EPR approach, and the mathematical theory of *quantum mechanics* with its usual Copenhagen interpretation. Theoretical because their simultaneous validity leads to a contradiction with number theory (inadmissible!), and experimental because it forced many laboratory tests of the quantum mechanical formulae. Most of these tests were in agreement with the theory, the most celebrated being that of Aspect and collaborators [19, 20]. The conclusion of many authors, shared by the author of this note, is that *local realism à la EPR does not hold.*

In this note we present three different versions of Bell's theorem: the 1964 original version, then the 1969 version of Clauser *et al*, and finally the Wigner's version of 1970. In an appendix we give a detailed derivation of the formula for the average value of the product of two spin 1/2 particles in arbitrary directions.

*Based mainly in Ref. 1.

2. Introductory comment

According to Bell, von Newmann stated that there can not even exist a non local hidden variables theory (in Ref. 1, Bell occupied himself of local hidden variables theories, proving that they are in contradiction with quantum mechanics). In Ref. 2, Bell showed that von Neumann's statement was not correct. (In this connection, in Ref. 1, Bell also mentioned the work of J. M. Jauch and C. Piron, *Helv. Phys. Acta* **36** (1963) 827).

3. Bell's logic (based on Refs. 3 and 4; see also Ref. 5)

1. One prepares a pair of spin 1/2 particles in the singlet state, and leaves the particles to travel in opposite senses.

2. One measures the spin components $\vec{\sigma}_1 \cdot \hat{a}$ and $\vec{\sigma}_2 \cdot \hat{b}$ of the particles 1 and 2 respectively. $\hat{a}$ and $\hat{b}$ are unitary vectors in R^3 *i. e.* $\hat{a}, \hat{b} \in S^2$. $\hat{a}$ and $\hat{b}$ are experimentally controllable parameters. If $\hat{b} = \hat{a}$ and $\vec{\sigma}_1 \cdot \hat{a} = +1$ then $\vec{\sigma}_2 \cdot \hat{a} = -1$ and vice versa *i. e.* $\vec{\sigma}_1 \cdot \hat{a} = -\vec{\sigma}_2 \cdot \hat{a}$. This is a quantum mechanical result.

3. According to Einstein, the measurement at 1 (2) can *not* affect the simultaneous measurement at 2 (1): this is the hypothesis of **locality**. Since, however, according to quantum mechanics, the measurement at 1 (2) predicts the result of the measurement at 2 (1), for not

contradicting Einstein's locality one concludes that the measured values must have been **determined** at the moment in which the particles 1 and 2 were in contact: this is the hypothesis of **realism**. In this sense, quantum mechanics and the exigency of locality imply the hypothesis of realism. Quantum mechanics itself, plus locality, predicts or demands realism. Unless locality is wrong, it is quantum mechanics that requires hidden variables. If locality does not hold, then quantum mechanics does not require realism.

4. Quantum mechanics (the wave function) however, does not predict the results of individual measurements (*e.g.* $\sigma_x^{(1)}$ can be +1 or -1); therefore the predetermination in 3 **demands** a more complete description of a quantum state.

5. The parameters which effect the more complete description are denoted by λ, and are called **hidden variables**. If λ specifies the results of the measurements of the spin on 1 and 2, individually these are given, respectively, by functions $A(\hat{a},\lambda)$ and $B(\hat{b},\lambda)$, with values in $\{+1,-1\}$. If $\rho(\lambda)$ is the probability distribution of λ

$$\left(\int_\Lambda d\lambda\rho(\lambda)=1,\rho(\lambda)\geq 0\right)$$

then the **average of the product** of the measurements of $\vec{\sigma}\cdot\hat{a}$ and $\vec{\sigma}\cdot\hat{b}$ is given by

$$P(\hat{a},\hat{b})=\int_\Lambda d\lambda\rho(\lambda)A(\hat{a},\lambda)B(\hat{b},\lambda). \qquad (1)$$

In these formulae, Λ is the domain of λ. By their own nature, the λ parameters are not experimentally controllable. The functions $P(\hat{a},\hat{b})$ are called **correlation functions** between the spins or, simply, **correlations**.

From the mathematical point of view, A and B are functions with domain $S^2\times\Lambda$ and codomain R or, with more precision, $\{-1,+1\}$ *i.e.*,

$$A,B: S^2\times\Lambda\rightarrow R \ \ or \ \ S^2\times\Lambda\rightarrow\{-1,+1\}.$$

For the case discussed in 10,

$$A,B: S^2\times\Lambda\rightarrow[-1,+1].$$

6. In quantum mechanics, the correlation is given by

$$P(\hat{a},\hat{b})=-\hat{a}\cdot\hat{b}. \qquad (2)$$

(See appendix.)

7. **Theorem**: 5. and 6. are contradictory. (See the proof in 9.)

8. *Observation.* The fact that A does not depend on $\hat{b}$ and that B does not depend on $\hat{a}$, is the expression of locality in this context. The variables λ are, in principle, global. Also, even if λ **determines** the result of each measurement on 1 and 2, respectively through A and B, there is a **statistical element** present, given by $\rho(\lambda)$. If $\lambda=(\lambda_1,...,\lambda_n)$, it is then possible a "locality" in λ with $\lambda_i=\lambda_i^{(A)}$ for

$$i=1,2,...,p, \quad \text{and} \quad \lambda_j=\lambda_j^{(B)}$$

for $j=p+1,...,p+q=n$. *I.e.*

$$A=A(\hat{a},\lambda_1^{(A)},...,\lambda_p^{(A)})=\pm 1$$

and $B=B(\hat{b},\lambda_{p+1}^{(B)},...,\lambda_{p+q}^{(B)})=\pm 1.$

9. Proof of 7.

If $\hat{b}=\hat{a}$, then $B(\hat{a},\lambda)=-A(\hat{a},\lambda)$ for all $\hat{a}\in S^2$ and all $\lambda\in\Lambda$; then $P(\hat{a},\hat{b})=-\int_\Lambda d\lambda\rho(\lambda)A(\hat{a},\lambda)A(\hat{b},\lambda)$. If $\hat{c}$ is another unitary vector, then $P(\hat{a},\hat{c})=-\int_\Lambda d\lambda\rho(\lambda)A(\hat{a},\lambda)A(\hat{c},\lambda)$ and therefore

$$P(\hat{a},\hat{b})-P(\hat{a},\hat{c})=-\int_\Lambda d\lambda\rho(\lambda)(A(\hat{a},\lambda)A(\hat{b},\lambda)-A(\hat{a},\lambda)A(\hat{c},\lambda));$$

since $(A(\hat{x},\lambda))^2=1$, one has

$$P(\hat{a},\hat{b})-P(\hat{a},\hat{c})=-\int_\Lambda d\lambda\rho(\lambda)(1-A(\hat{b},\lambda)A(\hat{c},\lambda))A(\hat{a},\lambda)A(\hat{b},\lambda).$$

Then

$$|P(\hat{a},\hat{b})-P(\hat{a},\hat{c})|=|-\int_\Lambda d\lambda\rho(\lambda)(1-A(\hat{b},\lambda)A(\hat{c},\lambda))A(\hat{a},\lambda)A(\hat{b},\lambda)|\leq\int_\Lambda\rho(\lambda)|1-A(\hat{b},\lambda)A(\hat{c},\lambda)||A(\hat{a},\lambda)A(\hat{b},\lambda)|$$

$$=\int_\Lambda d\lambda\rho(\lambda)(1-A(\hat{b},\lambda)A(\hat{c},\lambda))$$

$$=\int_\Lambda d\lambda\rho(\lambda)-\int_\Lambda d\lambda\rho(\lambda)A(\hat{b},\lambda)A(\hat{c},\lambda)=1+P(\hat{b},\hat{c}),$$

where we have used that $|A(\hat{a},\lambda)A(\hat{b},\lambda)| = 1$ and that $1 - A(\hat{b},\lambda)A(\hat{c},\lambda) \geq 0$. Therefore we have obtained

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{c})| \leq 1 + P(\hat{b},\hat{c}), \tag{3}$$

which is the **inequality** obtained by **Bell** in Ref. 1. Notice that λ does not appear in Eq. (3). This formula appears in Eq. (A.12) in Ref. 7, and in Eq. (8) in Ref. 8.

Let us see how, in some cases, Eqs. (2) and (3) are contradictory: Let $\hat{a}$, $\hat{b}$, and $\hat{c}$ be in the same plane, with $\hat{a}\cdot\hat{b} = 0$ and $\hat{c}$ in the bisection of $\hat{a}$ and $\hat{b}$. Then, according to quantum mechanics,

$$P(\hat{a},\hat{b}) = -\hat{a}\cdot\hat{b} = 0, \quad P(\hat{a},\hat{c}) = -\hat{a}\cdot\hat{c} = -\sqrt{2}/2,$$

and

$$P(\hat{b},\hat{c}) = -\hat{b}\cdot\hat{c} = -\sqrt{2}/2;$$

if these values are replaced in (3), one obtains

$$|0 + \sqrt{2}/2| \leq 1 + (-\sqrt{2}/2)$$

i.e.

$$\sqrt{2}/2 \leq 1 - \sqrt{2}/2,$$

which amounts to $\sqrt{2} \leq 1$: false! Therefore, Bell's inequality is violated by quantum mechanics. QED

(Note: In Ref. 11, page 36, Bell argues that the above proof does not restrict to non relativistic quantum mechanics, but only depends on the existence of separated systems "highly correlated" with respect to quantities like the spin.)

10. **Stronger Bell's inequality**

If $A(\hat{a},\lambda) = \pm 1$ then $|A(\hat{a},\lambda)| \leq 1$; let us assume this weaker condition. The difference between correlations is given by

$$\begin{aligned} P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}') &= \int_\Lambda d\lambda\rho(\lambda)(A(\hat{a},\lambda)B(\hat{b},\lambda) - A(\hat{a},\lambda)B(\hat{b}',\lambda)) \\ &= \int_\Lambda d\lambda\rho(\lambda)A(\hat{a},\lambda)B(\hat{b},\lambda)(1 \pm A(\hat{a}',\lambda)B(\hat{b}',\lambda)) - \int_\Lambda d\lambda\rho(\lambda)A(\hat{a},\lambda)B(\hat{b}',\lambda)(1 \pm A(\hat{a}',\lambda)B(\hat{b},\lambda)) \end{aligned}$$

where we have summed and subtracted the integral

$$\int_\Lambda d\lambda\rho(\lambda)A(\hat{a},\lambda)A(\hat{a}',\lambda)B(\hat{b},\lambda)B(\hat{b}',\lambda);$$

taking absolute values,

$$\begin{aligned} |P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')| &\leq \int_\Lambda d\lambda\rho(\lambda)|A(\hat{a},\lambda)||B(\hat{b},\lambda)||1 \pm A(\hat{a}',\lambda)B(\hat{b}',\lambda)| \\ &+ \int_\Lambda d\lambda\rho(\lambda)|A(\hat{a},\lambda)||B(\hat{b}',\lambda)||1 \pm A(\hat{a}',\lambda)B(\hat{b},\lambda)| \\ &\leq \int_\Lambda d\lambda\rho(\lambda)(1 \pm A(\hat{a}',\lambda)B(\hat{b}',\lambda)) + \int_\Lambda d\lambda\rho(\lambda)(1 \pm A(\hat{a}',\lambda)B(\hat{b},\lambda)) \\ &= 2 \pm \left(\int_\Lambda \rho(\lambda)A(\hat{a}',\lambda)B(\hat{b}',\lambda) + \int_\Lambda d\lambda\rho(\lambda)A(\hat{a}',\lambda)B(\hat{b},\lambda)\right) \\ &= 2 \pm (P(\hat{a}',\hat{b}') + P(\hat{a}',\hat{b})) \leq 2 \pm |P(\hat{a}',\hat{b}') + P(\hat{a}',\hat{b})|, \end{aligned}$$

then

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')| \mp |P(\hat{a}',\hat{b}') + P(\hat{a}',\hat{b})| \leq 2$$

and therefore

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')| + |P(\hat{a}',\hat{b}') + P(\hat{a}',\hat{b})| \leq 2, \tag{4}$$

which is the desired Bell's inequality. This formula is eq. (9) in Ref. 9 (Bell, 1971), eq. (11) in Ref. 8 (Jackiw and Shimony, 2001), eq. (1a) in Ref. 10 (Clauser *et al*, 1969), and eq. (20-6) in Ref. 6 (Ballentine, 1990).

11. *Proposition:* Eq. (4) implies Eq. (3).

Proof. Since Eq. (4) was obtained from a weaker condition, one expects that in a particular case it will reduce to Eq. (3). If in Eq. (4) we make $\hat{a}' = \hat{b}$, then in the second term of the left hand side one has

$$|P(\hat{b}, \hat{b}') + P(\hat{b}, \hat{b})|,$$

but

$$\begin{aligned} P(\hat{b}, \hat{b}) &= \int_\Lambda d\lambda \rho(\lambda) A(\hat{b}, \lambda) B(\hat{b}, \lambda) \\ &= -\int_\Lambda d\lambda \rho(\lambda) A(\hat{b}, \lambda) A(\hat{b}, \lambda) = -1 \end{aligned}$$

if one restricts to $A(\hat{b}, \lambda) \in \{-1, +1\}$; then

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| + |-1 + P(\hat{b}, \hat{b}')| \leq 2.$$

On the other hand,

$$\begin{aligned} |P(\hat{b}, \hat{b}')| &= |\int_\Lambda d\lambda \rho(\lambda) A(\hat{b}, \lambda) B(\hat{b}', \lambda)| \\ &\leq \int_\Lambda d\lambda \rho(\lambda) |A(\hat{b}, \lambda)||B(\hat{b}', \lambda)| \\ &= \int_\Lambda \rho(\lambda) = 1, \end{aligned}$$

and therefore $-1 + P(\hat{b}, \hat{b}') \leq 0$, then

$$|-1 + P(\hat{b}, \hat{b}')| = -(-1 + P(\hat{b}, \hat{b}')) = 1 - P(\hat{b}, \hat{b}')$$

and therefore

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| + 1 - P(\hat{b}, \hat{b}') \leq 2$$

implies

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| \leq 1 + P(\hat{b}, \hat{b}'),$$

which is eq. (3). QED

12. *Proposition:* (4) contradicts quantum mechanics.

Proof. Choose $\hat{a}, \hat{a}', \hat{b}$, and $\hat{b}'$ in the same plane, with $\hat{b} = \hat{a}'$ and with the angle θ between $\hat{a}$ and $\hat{a}'$ the same as the angle between $\hat{a}'$ and $\hat{b}'$. Then, using Eq. (2) in Eq. (4), we have:

$$|-cos\theta + cos2\theta| + |-cos\theta - 1| \leq 2;$$

$$|-cos\theta - 1| = |(-1)(1 + cos\theta)| = |1 + cos\theta| = 1 + cos\theta,$$

then

$$|-cos\theta + cos2\theta| + cos\theta \leq 1.$$

If $\theta \in [0, \pi/2]$, then $-cos\theta + cos2\theta < 0$ and therefore

$$|-cos\theta + cos2\theta| = cos\theta - cos2\theta$$

which gives $2cos\theta - cos2\theta \leq 1$. *I.e.*

$$2cos\theta - cos^2\theta + sen^2\theta \leq 1,$$

then

$$2cos\theta - cos^2\theta \leq 1 - sen^2\theta = cos^2\theta.$$

I.e.

$$2(cos\theta - cos^2\theta) \leq 0;$$

so for $\theta \in (0, \pi/2)$, $1 \leq cos\theta$: false! QED

13. **Wigner's formulation of Bell's inequalities**

Wigner's formulation is based in probabilities instead of correlations (averages of products of spin projections). By the hypothesis of Einstein's locality, which together with quantum mechanics, demands the realism hypothesis, each particle of the pair (1,2) has a spin projection along each of the directions $\hat{a} \equiv a$, $\hat{b} \equiv b$ and $\hat{c} \equiv c$ of R^3; these projections were determined at the moment in which the particles were together (e.g. before the decay of the original particle or molecule in the singlet state). What in the Bell's formulation is represented by the probability distribution $\rho(\lambda)$, in this formulation it is represented by the probabilities

$$p(a, b, c, a', b', c'), \ \ (a' = -a, \ b' = -b, \ c' = -c),$$

where the first three entries refer to particle 1, and the second three entries refer to particle 2:

$$\begin{aligned} &p(+,+,+,-,-,-), \ p(-,+,+,+,-,-), \\ &p(+,-,+,-,+,-), \ p(+,+,-,-,-,+), \\ &p(-,-,+,+,+,-), \ p(-,+,-,+,-,+), \\ &p(+,-,-,-,+,+), \ p(-,-,-,+,+,+). \end{aligned}$$

Since in the singlet state, the second three entries are determined by the first three entries, it is sufficient to denote these probabilities by

$$\begin{aligned} &p(+,+,+), \ p(-,+,+), \ p(+,-,+), \ p(+,+,-), \\ &p(-,-,+), \ p(-,+,-), \ p(+,-,-), \ p(-,-,-). \end{aligned}$$

Let us consider the quantity $p_{a,b}(+,-)$: probability that the particle 1 has spin +1 in direction a and spin -1 in direction b. This quantity (and its similars) can be measured without interference of measurements: one measures on particle 1 the cases with $\sigma_a = +1$ and on particle 2 the cases with $\sigma_b = +1$ *i.e.* $p_{a,b}(+,-) = p(1a+, 2b+)$ with an obvious notation. It is clear that

$$p_{a,b}(+,-) = p(+,-,+) + p(+,-,-),$$

analogously

$$p_{b,c}(+,-) = p(+,+,-) + p(-,+,-),$$

and

$$p_{a,c}(+,-) = p(+,+,-) + p(+,-,-).$$

We see that

$$p_{a,b}(+,-) + p_{b,c}(+,-)$$
$$= p_{a,c}(+,-) + p(+,-,+) + p(-,+,-)$$

and from the non negativity of probabilities

$$p_{a,c}(+,-) \leq p_{a,b}(+,-) + p_{b,c}(+,-), \qquad (5)$$

which is **Bell's inequality** in the Wigner's formulation *i.e.*

$$p(1a+, 2c+) \leq p(1a+, 2b+) + p(1b+, 2c+).$$

Introducing the result of quantum mechanics

$$p(1a+, 2c+) = \frac{1}{2} sin^2 \left(\frac{\theta_{ac}}{2}\right), \qquad (6)$$

(see appendix) where θ_{ac} is the angle between the unitary vectors $\hat{a}$ and $\hat{c}$, one obtains

$$\frac{1}{2} sin^2 \left(\frac{\theta_{ac}}{2}\right) \leq \frac{1}{2} sin^2 \left(\frac{\theta_{ab}}{2}\right) + \frac{1}{2} sin^2 \left(\frac{\theta_{bc}}{2}\right). \qquad (7)$$

For some angles, Eq. (7) is contradictory, what again establishes a contradiction between quantum mechanics and the hypothesis which lead to Eq. (5). This inequality (and its analogous) appears in Eq. (3) in Ref. 12 (Wigner, 1970), eq. (48) in Ref. 13 (Wigner, 1983), eq. in pg. 171 in Ref. 14 (d'Espagnat, 1979), Eq. (3.9.12) in Ref. 15 (Sakurai, 1985), and Eq. (7.7) in Ref. 16 (Treiman, 1999).

Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be in the same plane, with $\hat{b}$ the bisection of $\hat{a}$ and $\hat{c}$, so if $\theta_{ac} = 2\theta$ then $\theta_{ab} = \theta_{bc} = \theta$, with $\theta \in [0, \pi/2]$. Eq. (7) gives

$$sin^2\theta \leq 2 sin^2 \frac{\theta}{2}.$$

If $\theta = 0$ then $0 \leq 0$; if $\theta = \pi/2$ then

$$1 \leq 2 \times sin^2(\pi/4) = 2 \times (1/\sqrt{2})^2 = 1;$$

let $\theta \in (0, \pi/2)$,

$$sin^2\theta = 4 sin^2 \frac{\theta}{2} cos^2 \frac{\theta}{2}$$

and therefore $2cos^2\frac{\theta}{2} \leq 1$ which implies $cos\frac{\theta}{2} \leq \frac{1}{\sqrt{2}}$ for $\frac{\theta}{2} \in (0, \pi/4)$: false! Then, with this election of directions, quantum mechanics is contradictory with the Bell's inequality for all $\theta \in (0, \pi/2)$.

Appendix

Proposition: In quantum mechanics, the probability that two spin 1/2 particles in the singlet state (*i.e.* with total spin $S = 0$) have their spins in the positive directions of the unit vectors $\hat{n}_1$ and $\hat{n}_2$, is given by

$$P_{++} = \frac{1}{2} sin^2 \left(\frac{\alpha_{12}}{2}\right)$$

where α_{12} is the angle between $\hat{n}_1$ and $\hat{n}_2$.

$$(\hat{n}_k = (sin\theta_k cos\varphi_k, sin\theta_k sin\varphi_k, cos\theta_k)).$$

Proof.

$$P_{++} = | < \chi_0, \chi_{n_{1+}} \chi_{n_{2+}} > |^2$$

where

$$\chi_0 = \frac{1}{\sqrt{2}}(\chi_{1+}\chi_{2-} - \chi_{1-}\chi_{2+})$$

and

$$\chi_{n_{k+}} = cos\left(\frac{\theta_k}{2}\right)\chi_{k+} + e^{i\varphi_k} sin\left(\frac{\theta_k}{2}\right)\chi_{k-}, \quad k = 1, 2.$$

We shall use the notation:

$$sin\theta_1 = s_1, \quad sin\theta_2 = s_2, \quad cos\frac{\theta_1}{2} = c_{\frac{1}{2}},$$
$$sin\frac{\theta_1}{2} = s_{\frac{1}{2}}, \quad cos\frac{\theta_2}{2} = c_{\frac{2}{2}}, \quad sin\frac{\theta_2}{2} = s_{\frac{2}{2}},$$
$$cos\varphi_1 = C_1, \quad sin\varphi_1 = S_1, \quad cos\varphi_2 = C_2,$$
$$sin\varphi_2 = S_2, \quad cos(\varphi_1 - \varphi_2) = C_{12}, cos\alpha_{12} = c_{12},$$
$$sin(\frac{\alpha_{12}}{2}) = s_{\frac{12}{2}}, \quad sin(\frac{\theta_1 - \theta_2}{2}) = s_{\frac{1-2}{2}},$$
$$cos(\theta_1 - \theta_2) = c_{1-2};$$

and the formulae

$$sin\lambda = 2 sin\frac{\lambda}{2} cos\frac{\lambda}{2}$$

and $sin^2\frac{\lambda}{2} = \frac{1}{2}(1 - cos\lambda)$.

In this notation:

$$\hat{n}_1 \cdot \hat{n}_2 = c_{12} = (s_1 C_1, s_1 S_1, c_1)(s_2 C_2, s_2 S_2, c_2) = s_1 s_2 (C_1 C_2 + S_1 S_2) + c_1 c_2 = s_1 s_2 C_{12} + c_1 c_2,$$

then

$$\frac{1}{2} s^2_{\frac{12}{2}} = \frac{1}{4}(1 - c_{12}) = \frac{1}{4}(1 - s_1 s_2 C_{12} - c_1 c_2); \qquad \text{(A1)}$$

from

$$\chi_{n_1+} = c_{\frac{1}{2}} \chi_{1+} + e^{i\varphi_1} s_{\frac{1}{2}} \chi_{1-}$$

and

$$\chi_{n_2+} = c_{\frac{2}{2}} \chi_{2+} + e^{i\varphi_2} s_{\frac{2}{2}} \chi_{2-}$$

we have

$$< \chi_0, \chi_{n_1+} \chi_{n_2+} >= \frac{1}{\sqrt{2}} (e^{i\varphi_2} c_{\frac{1}{2}} s_{\frac{2}{2}} - e^{i\varphi_1} s_{\frac{1}{2}} c_{\frac{2}{2}}),$$

and therefore

$$\begin{aligned} P_{++} &= \frac{1}{2} \left(c^2_{\frac{1}{2}} s^2_{\frac{2}{2}} + s^2_{\frac{1}{2}} c^2_{\frac{2}{2}} - e^{i(\varphi_2 - \varphi_1)} c_{\frac{1}{2}} s_{\frac{1}{2}} s_{\frac{2}{2}} c_{\frac{2}{2}} - e^{-i(\varphi_2 - \varphi_1)} s_{\frac{1}{2}} c_{\frac{1}{2}} c_{\frac{2}{2}} s_{\frac{2}{2}} \right) \\ &= \frac{1}{2} \left(c^2_{\frac{1}{2}} s^2_{\frac{2}{2}} + s^2_{\frac{1}{2}} c^2_{\frac{2}{2}} - 2C_{12} c_{\frac{1}{2}} s_{\frac{1}{2}} s_{\frac{2}{2}} c_{\frac{2}{2}} \right) \\ &= \frac{1}{2} \left(s_{\frac{1}{2}} c_{\frac{2}{2}} - c_{\frac{1}{2}} s_{\frac{2}{2}} \right)^2 + (1 - C_{12})\, c_{\frac{1}{2}} s_{\frac{2}{2}} s_{\frac{1}{2}} c_{\frac{2}{2}} = \frac{1}{2} \left(s_{\frac{1-2}{2}} + 2 s_{\frac{1}{2}} c_{\frac{1}{2}} s_{\frac{2}{2}} c_{\frac{2}{2}} \,(1 - C_{12}) \right) = \frac{1}{2} \left(s^2_{\frac{1-2}{2}} + \frac{1}{2} s_1 s_2 - \frac{1}{2} s_1 s_2 C_{12} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} (1 - c_{1-2}) + \frac{1}{2} s_1 s_2 - \frac{1}{2} s_1 s_2 C_{12} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} (1 - (c_1 c_2 + s_1 s_2)) + \frac{1}{2} s_1 s_2 - \frac{1}{2} s_1 s_2 C_{12} \right) = \frac{1}{4} (1 - c_1 c_2 - C_{12} s_1 s_2)\ , \end{aligned} \qquad \text{(A2)}$$

and $(A1) = (A2)$. QED

To simplify, let us write $\alpha_{12} \equiv \alpha$. From the definition and formula for P_{++} we obtain the formulae for P_{+-}, P_{-+} and P_{--}, with obvious interpretation of the notation:

$$P_{+-} = \frac{1}{2} sin^2 \left(\frac{\pi - \alpha}{2} \right) = \frac{1}{2} cos^2 \frac{\alpha}{2},$$

$$P_{-+} = \frac{1}{2} sin^2 \left(\frac{\pi - \alpha}{2} \right) = \frac{1}{2} cos^2 \frac{\alpha}{2},\ P_{--} = \frac{1}{2} sin^2 \frac{\alpha}{2}.$$

Then:

$$\begin{aligned} P_{++} + P_{--} &= sin^2 \frac{\alpha}{2} \\ &= probability\ that\ the\ product\ of\ the \\ &\quad spins\ be\ equal\ to\ +1, \end{aligned}$$

and

$$\begin{aligned} P_{+-} + P_{-+} &= cos^2 \frac{\alpha}{2} \\ &= probability\ that\ the\ product\ of\ the \\ &\quad spins\ be\ equal\ to\ -1. \end{aligned}$$

We have then the

Corollary: The average of the product of the spins, $P(\hat{n}_1, \hat{n}_2)$ is given by

$$P(\hat{n}_1, \hat{n}_2) = -\hat{n}_1 \cdot \hat{n}_2.$$

Proof.

$$\begin{aligned} P(\hat{n}_1, \hat{n}_2) &= (+1) \times prob.(+1) + (-1) \times prob.(-1) \\ &= sin^2 \frac{\alpha}{2} - cos^2 \frac{\alpha}{2} = -cos\alpha = -\hat{n}_1 \cdot \hat{n}_2. \end{aligned}$$

QED.

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