RESEARCH

ANALYSIS OF STRATEGIES OF MATHEMATICAL ESTIMATION AMONG SECONDARY STUDENTS CONSIDERED TO BE GOOD ESTIMATORS

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Abstract
A methodology is presented, based on the proposal by Reys (1986), to identify and classify the mental processes and strategies of mathematical estimation used by secondary students considered to be good estimators. Out of a total of 248 students in various schools in Ensenada, Baja California, Mexico, a selection was made of the 5% who obtained the highest scores on an examination given for this specific purpose. The results show that the methodology is adequate for selecting good estimators, identifying the estimative strategies used by students, discovering students' difficulties in attempting to solve this type of problems, and reporting on the way students acquire such skills in daily life. Mention is made of the need for mathematics programs in basic education to emphasize the existing connection between estimation and the development of number sense in students. An important aspect was that students in the United States and in Ensenada who are considered to be good estimators, showed similarities in the use of estimative strategies.
Key words: secondary education, mathematics, mathematical estimation, mental calculation, number sense, Mexico.

Introduction

The development of mathematical notions is a gradual process that children construct, based on their experiences of interacting with the objects in their environments. These experiences allow them to establish diverse relationships and comparisons. By establishing the similarities and differences of attributes, children are able to classify objects and compare order and quantity (Mancera, 1991).

The development of the notion or intuition of quantity is an essential element in order for children to understand and imagine what is happening to numbers when they carry out mathematical operations. This mathematical intuition has been defined as an essential part of *number sense* by the National Council of Teachers of Mathematics (NCTM, 1989).

Diverse authors have defined number sense in different forms. Sowder (1992) refers to number sense as a well organized conceptual network that provides the skill to relate operations and solve numerical problems in flexible, creative ways. Sowder adds that understanding the structure of the numerical system and number sense is associated with skill in mental estimation, understood as the skill to make numerical calculations. Mental estimation obtains an inexact response (within certain established limits) by using only mental procedures; i.e., without using pencil and paper or any calculating or registering device (Hazekamp, 1986; Flores, Reys and Reys,1990; Reys, Reys and Hope, 1993). The procedures of mental calculation are fundamentally different from those carried out when using the algorithms of pencil and paper, and especially for approximate or estimated calculations; exact mental calculations are frequently solved as they would be in written form.

Strengthening computational estimation gives students a broader vision and a better use of the number system; and this condition enables students to develop their own procedures for solving diverse mathematical problems at school and in their lives (Sowder, 1989).

Strategies of Computational Estimation

Diverse studies of computational estimation have addressed the mental processes and strategies that secondary school students use to solve mathematical problems without the help of physical devices; the purpose of these studies has been to discover students' methods and explain them to other students. Reys, *et al.* (1982) found and classified three mental processes of computational estimation: a) reformulation; i.e., changing numerical data and leaving the problem's structure intact; b) translation, which consists of changing these data while modifying the structure of the problem; and c) compensation, which implies making final numerical adjustments to the results of problems in order to approach the exact result.

In terms of the strategies of computational estimation, Reys (1986) and Flores, Reys and Reys (1990) believe that individuals use different strategies of mental calculation to solve arithmetic problems. However, according to these authors, the same strategy is not adequate for all problems, and an important characteristic of a good estimator is knowing how to select and use the adequate strategies in each problem. The strategies that these authors consider most important are: a) digit on the left, b) grouping, c) rounding, d) compatible number, and e) special numbers. Each strategy is described below.

Digit on the Left

This strategy can be adapted to each of the four basic operations, but it is applied mainly in addition. It consists of focusing attention on the digit on the left side of the numbert— he most significant part of the number. For example:

Problem	Step 1	Step 2	Exact Result
260 153 + 99 371 528	The digits on the far left are totaled: 2 + 1 + 3 + 5 = 11 and the corresponding zeroes are added, obtaining 1,100.	The remaining digits are adjusted, making an attempt to combine those that total approximately 100: 71 + 28 are approximately 100 60 + 53 are approximately 100 99 is approximately 100 making a total of 300 + 1100= 1,400	1,411

Grouping

Grouping is used frequently in daily situations when a group of numbers is very close to a common value. Whole numbers, decimals and fractions can be used. For example:

Problem	Step 1	Step 2	Exact Result
92 430 83 658 + 87 199 93 280 94 672 88 390	The value closest to the addends in the problem is estimated. In this case, the estimated value is 90,000.	The estimated value is multiplied by the number of addends. In this case: 6 x 90,000 = 540,000.	539,629

Rounding Numbers

Rounding numbers is a very powerful and efficient strategy for estimating results. The main purpose is to produce numbers that can be handled easily. In the process, rounding is done. Then the rounded number is used. The third step is to adjust the result, depending on if the rounding is above or below the base amount. This strategy is particularly effective with multiplication problems. For example:

Problem	Step 1	Step 2	Exact Result
47 x 66=	In this problem, rounding should be upwards, to 50 x 70 = 3,500.	The adjustment consists of taking into account that the real result is less than the estimated result.	3,102

Compatible Numbers

In this strategy, the student must observe all numbers involved in the problem in an overall manner, and round each number to make them all compatible. Pairs of numbers that give exact results should be considered, since they are very easy to calculate mentally. This strategy is especially effective with division problems. For example:

Problem	Compatible Numbers	Incompatible Numbers
3370 ÷ 7 =	3500 ÷ 7 3200 ÷ 8 4000 ÷ 8	3000 ÷ 7 3300 ÷ 7 3400 ÷ 8

Special Numbers

This strategy combines various points of the previously mentioned strategies. Its primary function is to observe if the numbers in the problem to be solved are similar or very close to numbers that are easier to calculate. If so, they can be replaced. Such special values can be multiples of ten, or common fractions and decimals close to 0.5, 0.24, 0.75 or any whole number. For example:

Problem	Step 1	Step 2	Exact Result
23/45 x 720	23/45 is close to ½	The problem is reduced to calculating half of 720	360

The relation between mental processes and strategies of computational estimation, according to Reys *et al*. (1982) and Reys (1986), can be observed in table 1.

TABLE 1

Mental Processes and Related Strategies

Mental Processes	Strategies of Computational Estimation
Reformulation	* Digit on the left * Rounding * Compatible numbers

Translation	* Grouping * Special numbers
Compensation	* Final adjustment

Stating of Problem

Based on this classification, Cortés (2001) developed a master's thesis to reply to the research of Reys and collaborators, in order to discover the strategies of computational estimation used by secondary school students in Mexico who are considered "good estimators". The results were published in a document (Cortés, Backhoff y Organista, 2002) centered on describing the instruments utilized and the levels of solving problems through computational estimation, as shown by these students.

The purpose of this study is to present the methodology for identifying, characterizing and evaluating the mental processes and strategies of computational estimation used by Mexican students who are considered good estimators.

Method

The research was completed during the 1998-1999 school year, with the participation of students in the second year of secondary school in the city of Ensenada, Baja California, Mexico.

Population Studied

A selection was made of 248 students in the second year of secondary school in the eight urban and rural schools of the municipality of Ensenada, Baja California. These students took a test consisting of 39 problems of computational estimation; the problems were presented one by one on the wall, with a film projector. The twelve students (almost 5%) with the most right answers were selected. The characteristics of this sub-population were: mostly male, an average age of approximately 13, high grades in mathematics, from three types of secondary schools (general, technical and tele-secondary); two-thirds were enrolled in the afternoon shift at school, and the number of their correct answers varied from 20 to 25. Table 2 shows a summary of the characteristics of the selected students and their schools of origin.

TABLE 2

Characteristics of Selected Students and Schools

Students					**School of Origin**		
Identifi-cation	*Right Answers*	*Gender*	*Age*	*Average*	*Identification*	*Type*	*Shift*
A	25	M	13	9.0	1	General	Afternoon
B	25	M	13	10	2	General	Morning
C	24	M	13	9.0	4	Technical	Afternoon
D	23	M	13	10	2	General	Morning
E	23	M	13	9.2	3	Tele-secondary	Morning
F	22	F	14	9.8	3	Tele-secondary	Morning
G	22	F	13	9.2	7	General	Afternoon
H	21	M	13	9.3	4	Technical	Afternoon
I	21	M	13	10	6	General	Afternoon
J	21	M	14	9.2	7	General	Afternoon
K	20	M	14	9.6	4	Technical	Afternoon
L	20	M	12	9.4	7	General	Afternoon

Interviews

To discover how the twelve selected students solved the problems of computational estimation, a second test was used: ten questions, six of which were word problems and four were not (see table 3). Five of the questions were taken from the original examination and the rest were new. Each student had to make the mental calculations and explain aloud the steps he followed to reach the final result.

The interviews were videotaped and transcribed literally according to the procedure developed by Martínez (1998). The interviews were analyzed through these transcriptions, which identified and classified the mental processes and strategies the students employed to solve the exercises. The categorization was based on the models by Reys *et al.* (1982) and Reys (1986).

Table 3
Questions Used on the Test

Item #	Problem	Item #	Problem
1	87,419 92,765 + 90,045 81,974 98,102	6	If 30% of the fans at a Mexican series baseball game buy a soft drink, about how many soft drinks are sold if the attendance is 54,215 people?
2	474, 257 ÷ 8,127 =	7	The concessions of the Mexican baseball series had income in the amount of 21,319,908.00 pesos. If this amount is divided equally among 26 teams, about how much does each team receive?
3	12/13 + 7/8 =	8	This is a grocery bill that has not been added up. About how much is the total? 79+44+130+34 ...
4	486 x 0.24 =	9	Estimate the 15% discount for a jacket that costs 28,000.00 pesos
5	Calculate the approximate area of a rectangle that measures: 28 cm x 47 cm.	10	Which answer is reasonable? $\frac{4}{9}+\frac{5}{10}=1\frac{5}{90}$ $\frac{6}{8}+\frac{4}{7}=\frac{8}{15}$ $\frac{8}{15}+\frac{11}{20}=1\frac{1}{12}$

Procedure

The interviews were managed according to the suggestions of Reys (1982). In an individual manner, the twelve students were asked to solve the ten problems and explain aloud the steps they followed to solve them. The purpose of analyzing the interviews was to study further the responses of students considered good estimators, in order describe in detail their procedures, ideas, strategies and some outside elements of influence on their behavior. A three-member team participated: two people with mathematics degrees who conducted the interviews and took notes, and a high school graduate who made the video recording. Since the study's results were largely dependent on the data compiled in the interviews, each interviewer was required to follow a general guide provided for that purpose.

According to the recommendations of Martínez (1998), the interviews were transcribed integrally on pages divided lengthwise. One section included the transcription, where any indication of a student's strategy was underlined. The other part of the sheet was used to indicate the detected strategies (Reys, 1986) and corresponding thinking processes (Reys *et al.*, 1982), next to the underlined words. The steps to determine the strategies and thinking process were:

1) To reread the interview and underline the most relevant and meaningful words.
2) To transcribe that paragraphs that expressed an idea or central concept and integrate them on separate tables, for each one of the ten questions on the test.
3) To summarize at the end of each table the students' comments on that exercise.
4) If different strategies with varying properties or attributes appeared for the same exercise, to describe them and report the differences.
5) To classify the strategies and thinking processes found.

The procedure for analyzing the results of the interviews, according to Martínez (1998), was:

1) To take into account that data do not have equal importance, since the value and meaning of data depend on the context in which they are generated. Some information will be central for solving the problem presented, while other information will be peripheral and secondary.
2) To integrate the information into a coherent, logical whole.
3) To make a verbal report or synthesis, with some direct texts; in other words, with some textual quotations from the participants.

Results

For each exercise on the questionnaire used in the interviews, a summary of individual responses was prepared; these responses included only the information that describes the strategy of computational estimation used by each student. Subsequently, a chart was prepared to summarize the responses obtained. Lastly, the number of students who answered the exercise correctly was reported, along with the mental processes and strategies used most often. Table 4 illustrates this process with the responses for problem 6. This problem was selected because of the wide variety of responses students presented in their solutions, resulting in three mental processes for the exercise.

The exercise was answered correctly by eleven students. The strategies most used were: special numbers, compatible numbers and final adjustment. Some students centered their attention on the percentage and others on the number, while a third group observed the number and the percentage at the same time.

Table 4
Relevant Aspects of the Answers Given to Question 6

Student	Answer to problem 6. If 30% of the fans at a Mexican series baseball game buy a soft drink, about how many soft drinks are sold if the attendance is 54,215 people?	Strategies Used
A	I see the number and I take half (50%), then half again (25%) and I add a little to make it 30%.	Special numbers, special numbers, final adjustment.
B	I see that 30 is about 1/3 of 100, so I divide the number by three.	Compatible numbers.
C	I think of 54 as 100 (double) and I take 30%, then half of the result.	Digit on the left, compatible numbers.
D	I take 10% and then multiply the result by three.	Special numbers.

E	When they ask for 30%, I take 50% and then half (25%) and then I add on a little.	Special numbers, final adjustment.
F	Did not answer.	
G	I see the number and take 10%, then I multiply by three.	Special numbers.
H	I look at the numbers on the left and I multiply them, then I add the zeroes that are lacking.	Digit on the left, final adjustment.
I	I round the number to a multiple of 1,000 and I multiply it by three.	Rounding, compatible numbers.
J	I divide by two (50%), then by two again (25%) and I add a little to get 30%.	Special numbers, final adjustment.
K	I divide the number by 30, which is less than half, about a fourth less.	Special numbers.
L	I round to a multiple of 10,000 and I multiply it by three.	Rounding, compatible numbers.

Table 5 shows the strategies and mental processes that students use most often in each of the twelve problems, according to the classification of Reys (1986). Also shown is the degree of difficulty of each question; i.e., the proportion of individuals who answered the question correctly, obtained with the formula of Crocker and Algina (1986, quoted in Backhoff, Larrazolo and Rosas, 2000):

$$\boldsymbol{p}_i = \frac{\boldsymbol{A}_i}{\boldsymbol{B}_i}$$

In which:

$\boldsymbol{p}_i$ = index of difficulty of question i

A_i = number of correct answers to question i

B_i = number of correct answers plus the number of errors in question i

TABLE 5

Synthesis of the Students' Responses to the Interview Questions

Pro-blem	**Strategies most used**	**Mental processes most used**	**Difficulty**
1	Digit on the left, rounding and grouping	Reformulation	1.00
2	Digit on the left and rounding	Reformulation	1.00
3	Special numbers	Translation	0.75
4	Special numbers	Translation	0.66
5	Digit on the left and rounding	Reformulation	1.00
6	Rounding, special numbers and compatible numbers	Reformulation, translation and comparison	0.91
7	Digit on the left, rounding and compatible numbers	Reformulation	0.83
8	Digit on the left, rounding and grouping	Reformulation	1.00
9	Rounding, compatible numbers and special numbers	Reformulation and translation	0.91
10	Special numbers	Translation	0.25

Table 6 shows the frequencies of each strategy used, as well as the mental processes involved. It can be observed that students do not always make final adjustments (E6); only student H made adjustments in all the exercises. It is also shown that the strategy used most often was rounding (E2). With respect to mental processes, reformulation was the process handled on most occasions, indicating that most of the participants tried to find data more simple than the original data to solve the exercises.

TABLE 6

Frequency of Mental Processes and Strategies
Used by Interviewed Students

Student	Processes and Strategies								
	Reformulation				*Translation*			*Compensation*	
	E1	E2	E3	Total	E4	E5	Total	E6	Total
A	3	1	3	7	1	4	5	3	3
B	1	3	2	6	2	5	7	3	3
C	2	2	5	9	2	2	4	6	6
D	1	7	4	12	1	5	6	1	1
E	4	4	1	9	1	4	5	2	2
F	1	3	0	4	3	1	4	3	3
G	2	7	3	12	0	5	5	5	5
H	8	4	2	14	1	3	4	10	10
I	3	7	3	13	0	4	4	1	1
J	4	5	1	10	1	4	5	5	5
K	4	7	2	13	0	3	3	2	2
L	4	6	1	11	0	2	2	4	4
Totales	**37**	**56**	**27**	**120**	**12**	**42**	**54**	**45**	**45**

Strategies: E1- Digit on the left; E2- Rounding; E3- Compatible numbers; E4- Grouping
E5- Special numbers; E6- Final adjustment

According to the methodology proposed by Martínez (1998), a verbal report was prepared to analyze student responses in greater detail. Observations in the report describe not only students' ideas, processes and strategies, but also other elements that influenced their responses. The most relevant of these elements are presented below:

- The students selected as good estimators commented that they estimate results outside of school. They affirmed that they do not learn nor are they asked to estimate at school.[1]
- The students interviewed always tried to replace the original data with numbers easier to handle, resulting in faster answers.
- Most of the students interviewed made final adjustments.
- The students selected as good estimators showed feelings of happiness and pride.
- Although the video camera intimidated them initially, they calmed down and answered with confidence.

- The mathematical language they used was imprecise, with expressions such "el de allá" ("the one over there"), "lo de ahí" ("this part here"), "el de acá" ("this one over here") to refer to the positional value of numbers.
- The students interviewed always expressed that computational estimation is important and that they use it constantly in their daily lives.

Conclusions

In this study, we centered on methodology to identify and characterize the processes and strategies of computational estimation used by students in the second year of secondary school to solve problems of mental calculations. This methodology consisted basically of selecting good estimators and asking them to solve a series of arithmetic problems aloud; their answers were recorded for subsequent analysis and classification according to the processes and mental strategies used, based on the systems of Reys *et al.* (1982), Reys (1986) and Martínez (1998). The results indicate that the methodology used was appropriate for:

1) selecting good estimators
2) identifying the processes and strategies of mental calculation that these students use most often
3) identifying some problems faced by students when attempting to do mental estimations
4) obtaining an idea of how and where students acquire strategies for estimating

The twelve students who participated in the study were selected as the best estimators out of a group of 248 who completed an examination of computational estimation. Their high averages in mathematics (between 9/10 and 10/10) ratify an essential assumption of the research in mathematical education; i.e., that computational estimation is fundamental for developing number sense (Hazekamp, 1986; Reys, Reys and Hope, 1993; Sowder, 1989). The reflection of number sense is good performance in mathematics as a subject.

The methodology detected strategies and cognitive processes characteristic of all the good estimators, regardless of their cultural background and teaching system (the students studied by Reys in the United States and the Mexican students of this study). Similarities included the following:

- The strategy most used by the students was rounding.
- The strategy known as the digit on the left, was used in an "intuitive" and correct manner by most of the students.
- The strategy of grouping numbers was used infrequently, since it was easily replaced by the previous two strategies.
- The strategy of compatible numbers required students to observe in an overall manner all the numbers involved in the problem, to change or round each number and make the numbers compatible.
- To utilize the strategy of special numbers, the students had to combine different elements of the strategies mentioned above to determine if the numbers in the problem were similar or close to numbers that were easier to operate, and if so, to replace them.

The methodology was also able to detect some types of recurring estimative errors. The most important include the following:

- Some students, when making a computational estimation, forgot to make the "final adjustment" to improve the response; this lack of fine tuning or closure kept some of the answers out of the pre-established interval of approximation. Only one student used the strategy of final adjustment systematically with all the problems.
- Other students did not identify the whole values close to the fractions in the problems, which would have permitted them to make their mental calculations with greater ease.
- Most of the students did not find it easy to respond out loud to the questions projected on the wall. They showed concern about not being able to check their exercises.
- Many students initially found it difficult to work with fractions and large numbers, although the problem may have been easy to solve.
- Another aspect that pressured the students was the time limit for solving each problem; many students became nervous and were unable to give their answers in the established time.

It is important to state that since one of the objectives of this study was to reply to the research of Reys *et al.* (1982), carried out in the United States, our work represents one link more in the chain of research to validate and generalize the model that explains the strategies, processes and characteristics of good estimators.

Due to the above, the methodology and results of this study can be taken into account when designing strategies and didactic material for teaching and evaluating the computational estimation of students in Mexico's secondary schools. This proposal is reinforced by the fact that not even the best estimators correctly use all estimating strategies, and that students from diverse educational contexts present very similar difficulties when solving this type of problems.

Unfortunately, most secondary school teachers in Mexico center their attention on teaching the calculation of algorithms with the use of paper and pencil. They completely omit computational estimation, although the development of number meaning is included in Mexico's plans and programs of study. If the reasons for this omission are the teacher's lack of teacher knowledge of the importance of computational estimation, and a lack of methodological resources to address it, we hope that this study will contribute to improving the idea and practice of mental calculation in Mexico's classrooms.

Note

[1] Some textual statements were: "I use it when I count money for my mother", "when I have small calculations", "when I go to the store", "when you take a certain amount of money to buy something, and you calculate how much you're spending", "when I get home, my dad always asks me to do some calculations in my head, because he thinks I'm very intelligent", "when I go to the market with my mother, first we choose what is most necessary and I do the calculations in my head to see if we have enough money", "I use it for hard calculations, because it's not necessary for the easy ones", "when I want to do fast calculations at school and at the store".

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