Goos-Hänchen effect on a one transverse dimensional Hermite-Gaussian beam

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Received 23 July 2018; accepted 22 November 2018

We study the lateral displacement (Goos-Hänchen effect) of a Hermite-Gaussian beam incident on a dielectric interface of lower index of refraction than the incident media. Unlike previous results on the same subject, the present result can be applied to an infinite family of higher order solutions (or modes) of the Huygens-Fresnel integral. The final theoretical expression is valid for values that are close to the critical angle θ_c. Discussion is made for the behavior of the lateral displacement for different modes of the Hermite-Gaussian beam.

Keywords: Reflection; geometrical optics; wave optics.

PACS: 42.25.Gy; 42.15.-i; 42.25.-p

1. Introduction

The Goos-Hänchen phenomenon refers to a deviation of the classical trajectory of a light beam predicted by the geometrical optics theory, when the light beam is incident upon a dielectric interface at an angle larger than the critical angle (see Fig. 1). The first experimental observation dates from 1947 [1], and since then, many authors have tried to give a theoretical solution to the problem. Lotsch [2] gave an extended discussion on the whole subject, saying that no rigorous solution was known back then, but that an analitical solution definitely exists. Renard [3] assumed that there is an energy flow associated to the evanescent wave that penetrates the dielectric interface, then, using conservation of energy, arrives to an analytical expression that, however, is not valid in a close neighborhood of θ_c. Horowitz and Tamir [4], using direct integration methods, arrived to the first theoretical important result, since they were the first to give an explicit formula valid for angles close to θ_c. Cowan and Aničin [5], using a bounded microwave beam, investigated the lateral displacement, and compared their results with the theoretical results of Horowitz and Tamir. They found that their results disagree with the theoretical curve near the critical angle, they argue that this disagreement may be due to the fact that they were using a short wide beam. Later on, Lai, Cheng and Tang [6] slighty improved the result presented by Horowitz and Tamir, when they used an expansion of the reflectance that retains terms of second order. Their results give a lateral shift that varies continuously and smoothly around the critical angle. Also, Chan and Tamir [7], using the same expansion, investigated the lateral displacement and other effects that were not studied before: focal shift, angular shift and the beam waist modification.

This effect has also been studied from the view point of physical applications. For example, Chiu and J. J. Quinn [8] considered a wave packet and interpret the Goos-Hänchen effect as a time delay scattering process. Kogelnik and Weber [9], investigated the light propagation through a dielectric waveguide. Using ray methods and considering the Goos-Hänchen effect, arrived to a predicted phase and group velocity that agree with the usual energy conservation approach. Since then, various aplications of this physical proccess were given. Some include, aplications in acoustics [10], quantum mechanics [11-13] and nonlinear optics [14]. Using photonic crystals, Soboleva, Moskalenko and Fedyanin [15] enhanced the Goos-Hänchen effect and observed that the displacement is, at least, one order of magnitude larger than in a dielectric surface. Chremmos and Efremidis [16] give the lateral displacement for an Airy beam. Finally, Prajapati and Ranganathan [17], using numerical methods, give the total Goos-Hänchen effect for a three dimensional Hermite-Beam for two orthogonal components.

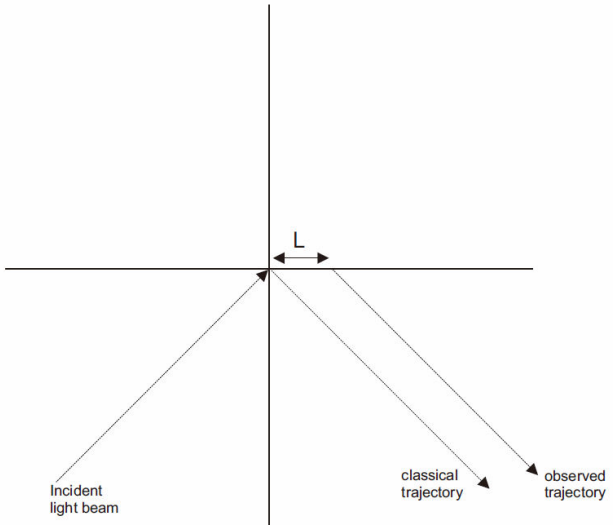

FIGURE 1. Classical trajectory of a light beam and the real trajectory affeceted by the lateral displacement

Despite all the results above, an analytical expression for the lateral shift for a Hermite-Gaussian beam does not exist. Gaussian beam is just one of many solutions (or modes) of the Huygens-Fresnel integral. It is the purpose of the present

paper to show the theoretical behavior of the lateral displacement for a higher mode solution. To this end, we use a higher unidimensional mode solution of the Huygens-Fresnel integral and use the direct integration method used by Horowitz and Tamir to derive the theoretical total displacement of the field incident upon a dielectric interface. The final result is valid for angles that are in a close neighborhood of the critical angle.

In Sec. 2 we give a derivation of the reflected field for a Hermite-Gaussian beam in one transverse dimension. We write the incident field as a normalized one dimensional higher order Hermite-Gaussian field, this field being a solution to the Hygens-Fresnel equation (strictly speaking, it is a eigenfunction of the Hyghens-Frensnel integral equation). Then, we write this field in a simplified form, assuming that the distance of propagation is so small that we can ignore the Guoy phase shift. This approximation allows us to express the field only in terms of the Hermite polynomials and a exponential. Thereafter, we study separately the case for even and odd Hermite polynomials. Then, the field is expressed as a superposition of plane waves using the Fourer Transform. With this expression we write the reflected field in terms of the reflectance and a correction factor. The reason to express the field in such a way is to simplify the integration of the reflected field. In Sec. 3 we determine the general expression for the lateral displacement. This formula shows explicity the dependence with the incidence angle and the mode of the field. In Sec. 4 we present numerical results that show the behavior of the lateral displacement for different modes of the field and for two different beam widths. An interesting result is that, in general, the lateral displacement increases as the mode of the field increases and then decreases again. All important calculations are derived in the Appendix A and B.

2. Reflected Field

We will work under the paraxial approximation. This means that the field is a solution to the Huygens-Fresnel integral. Then, assume a monocromatic one transverse dimensional Hermite-Gaussian beam incident upon a dielectric interface in the plane $y = 0$. In addition, we will assume a linearly polarized field and troughout the discussion a time dependence is implied and supressed. The index of refraction of the dielectric is taken to be $n = k/k_0 > 1$, where k and k_0 are the wavenumbers associated to the incident and dielectric media, respectively. We will make the assumption that the beam is well defined, that is $kw \gg 1$, where w is the beam waist. The geometry of the problem is presented in Fig. 2. We have three set of coordinates: the interface coordinates (x, y), the incident coordinates (x_i, y_i) and the reflected coordinates (x_r, y_r). We are going to work out the problem in the rotated plane that is paralell to the interface plane. By this construction, these set of coordinates are related by

$$\begin{aligned} x_i &= x\cos\theta - y_+\sin\theta, \\ y_i &= x\sin\theta + y_+\cos\theta, \end{aligned} \tag{1}$$

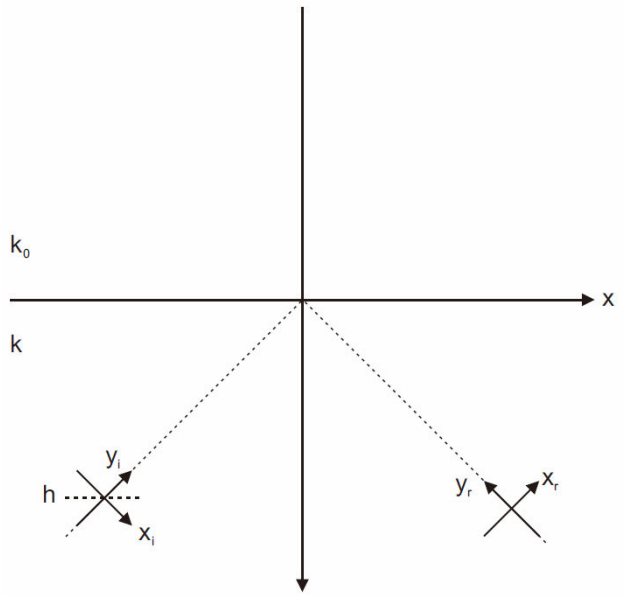

FIGURE 2. The three set of coordinates. The incident axis, the reflected axis and the interface axis. The plane h is the paralell to the interface.

and

$$\begin{aligned} x_r &= x\cos\theta + y_-\sin\theta, \\ y_r &= x\sin\theta - y_-\sin\theta. \end{aligned} \tag{2}$$

where $y_+ = y + h$, $y_- = y - h$ and h is the position of the plane where the beam is and parallel to the dielectric interface.

The incident beam is located at $y_i = 0$. According to Siegman and Sziklas [18] (see also [19] Chapter 16, Eq. 54), a higher-order Hermite Gaussian beam in one transverse dimension, along the $y_i = 0$ axis, can be represented as

$$\psi_{n,inc}(x_i, y) = \left(\frac{2}{\pi}\right)^{1/4}\left(\frac{1}{2^n n! w}\right)^{1/2}\frac{q_0}{q(y)}\left[\frac{q_0}{q_0^*}\frac{q^*(y)}{q(y)}\right]^{n/2} \times H_n\left[\frac{\sqrt{2}x_i}{w(y)}\right]\exp\left[-i\frac{kx_i^2}{2q(y)}\right],$$

where

$$\frac{1}{q(y)} = \frac{1}{R(y)} - i\frac{\lambda}{\pi w^2(y)}, \tag{3}$$

where $R(y)$ is the radius of curvature and $w(y)$ the beam waist as a function of propagation distance. This field is generated by a point source (see [20]), so if we assume that the wave is at long distance from the source we will have $1/R(y) \longrightarrow 0$. We can suppose, also, that the distance of propagation is so small that we can take $q(z) \simeq q_0$ and $w(y) \simeq w$. In the same manner, the coordinate x_i, written in the rotated axis paralell to the dielectric interface, is $x_i = x\cos\theta$. In that way, we can write the equation for a Hermite-Gaussian beam, of order n, as

$$\psi_{n,inc}(x_i, y) = \alpha_n H_n\left[\sqrt{2c^2}x\right]\frac{\exp[-c^2x^2]}{\sqrt{\pi}w}, \tag{4}$$

where

$$\alpha_n = \left(2\pi^3\right)^{1/4}\left(\frac{w}{2^n n!}\right)^{1/2}, \tag{5}$$

$$c = \frac{\cos\theta}{w}. \tag{6}$$

The final field propagating along the y axis will be approximated by adding a plane-wave variation $\exp(iky_i)$ along the same axis. Using Eq. (1), the coordinates along the rotated plane $y = -h$ will give for the field

$$\psi_{n,inc}(x_i, y) = \alpha_n H_n\left[\sqrt{2c^2}x\right]\frac{\exp[-c^2x^2 + i\bar{k}x]}{\sqrt{\pi}w}, \tag{7}$$

where

$$\bar{k} = k\sin\theta. \tag{8}$$

We need now an explicit representation for the Hermite polynomials. Hermite polynomialas can be of even or odd order, for each case we must derive an expression for the lateral displacement. We consider, as a first case, an even order Hermite-Gaussian beam, which can be written as:

$$\psi_{2n,inc}(x_i, y) = \alpha_{2n}\sum_{j=0}^{n} f_j x^{2j}\frac{\exp[-c^2x^2 + i\bar{k}x]}{\sqrt{\pi}w}, \tag{9}$$

where

$$f_j = \frac{(-1)^{n+j}(2n)!}{(2j)!(n-j)!}(2\sqrt{2c^2})^{2j}. \tag{10}$$

The reflected wave can be written as a continuous superposition of plane waves, each of one are affected by the Fresnel reflectance r_{k_x}, *i.e.*,

$$\psi_{2n,r}(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k_x)\widetilde{\psi}_{2n}(k_x) \times \exp[i(k_x x - k_y y_-)]\, dk_x, \tag{11}$$

where

$$r(k_x) = \frac{(k^2 - k_x^2)^{1/2} - m(k_0^2 - k_x^2)^{1/2}}{(k^2 - k_x^2)^{1/2} + m(k_0^2 - k_x^2)^{1/2}}, \tag{12}$$

and k_x, k_y are the components of the wave number k and are related by $k_x^2 + k_y^2 = (2\pi/\lambda)^2$. The term k_0 is $k_0 = k\sin\theta_c$. The constant m is the coefficient that depends on the polarization of the incident wave, *i.e.*,

$$m = 1 \ \text{ or } \ m = n^2, \tag{13}$$

for normal and parallel polarization to the plane of incidence, respectively, and

$$\widetilde{\psi}_{2n}(k_x) = \int_{-\infty}^{\infty}\psi_{2n,inc}(x, y)\exp(-ik_x x)\, dx, \tag{14}$$

is the inverse Fourier Transform. Substituting Eq. (9) into Eq. (14) yields

$$\widetilde{\psi}_{2n}(k_x) = \frac{\alpha_{2n}}{\sqrt{\pi}w}\sum_{j=0}^{n} f_j \int_{-\infty}^{\infty} x^{2j} \times \exp[-c^2x^2 - i(k_x - \bar{k})x]dx, \tag{15}$$

after completig the square and changing variable we will obtain

$$\widetilde{\psi}_{2n}(k_x) = \frac{\alpha_{2n}}{\cos\theta}\sum_{j=0}^{n}\sum_{k=0}^{j} f_j g_k\left(\frac{k_x - k}{c}\right)^{2j-2k} \times \exp\left[-\frac{1}{4}\left(\frac{k_x - k}{c}\right)^2\right], \tag{16}$$

where

$$g_k = \frac{(2j)!}{k!(2j-2k)!}(-i)^{2j-2k}\left(\frac{1}{2c}\right)^{2j}. \tag{17}$$

It is important to note that the lateral displacement will depend upon the polarization of the field through the function $r(k_x)$ (which, in turn, depends on polarization through the constant m). We will make all calculations assuming a field in which the polarization is normal to the plane of incidence. The effect of parallel polarization will only re-scale the graphics and will not affect the shape of the final curves.

Proceeding now like Horowitz and Tamir, we can write the reflectance in such a way that we may extract the part responsible for the geometric-optics result,

$$r(k_x) = r(\bar{k})[1 + r_c(k_x)], \tag{18}$$

where

$$r_c(k_x) = -\left[1 - \frac{r(k_x)}{r(\bar{k})}\right]. \tag{19}$$

The term $r_c(k_x)$ represents a correction term and takes into account the ondulatory behavior of the incident field; that is, the function $r_c(k_x)$ is responsable for the deviation of the Hermite-Gaussian beam from the geometric-optics trajectory. This way, we can write the reflected field like

$$\psi_{2n,r}(x, y) = \psi_{2n,rg}(x, y)[1 + \psi_{2n,ro}(x, y)], \tag{20}$$

where

$$\psi_{2n,rg}(x, y) = \frac{r(\bar{k})}{2\pi\cos\theta}\int_{-\infty}^{\infty}\widetilde{\psi}(k_x) \times \exp[i(k_x x - k_y y_-)]\, dk_x, \tag{21}$$

and

$$\psi_{2n,ro}(x, y) = \frac{r(\bar{k})}{2\pi\psi_{rg}(x, y)}\int_{-\infty}^{\infty} r_c(k_x)\widetilde{\psi}(k_x) \times \exp[i(k_x x - k_y y_-)]\, dk_y. \tag{22}$$

Now, if we substitute Eq. (16) into Eq. (21) we arrive to

$$\psi_{2n,rg}(x,y) = \frac{r(\bar{k})}{2\pi\cos\theta}\sum_{j=0}^{n}\sum_{k=0}^{j} f_j g_k \int_{-\infty}^{\infty}\left(\frac{k_x-k}{c}\right)^{2j-2k} \times \exp\left[-\frac{1}{4}\left(\frac{k_x-k}{c}\right)^2\right] \times \exp[i(k_x x - k_y y_-)]\,dk_x. \quad (23)$$

The reason to express the reflected field $\psi_{2n,r}(x,y)$ in terms of $\psi_{2n,rg}(x,y)$ and $\psi_{2n,ro}(x,y)$ is that in such way we can simplify the integration because it is esier to handle the function $r_c(k_x)$ (see Appendix A.2 where we made the integration using series expansion for $r_c(k_x)$) than the function $r(k_x)$. Of course, we can still make a calculation using $r(k_x)$, but in the present work we only focus in the much simplified version of the problem. In future papers we will handle the problem using $r(k_x)$ an we will compare those results with the ones presented here.

Now, making the integration (see Appendix A.1) we get the following expression for the reflected field (geometric-optical field):

$$\psi_{2n,rg}(x,y) = -\frac{\alpha_{2n}\Pi}{\sqrt{\pi}w_r} \times \sum_{j=0}^{n}\sum_{k=0}^{j} f_j h_k H_{2(j-k)}(x_r/w_r), \quad (24)$$

where $H_{2(j-k)}$ is the Hermite polynomial of order $2(j-k)$ and

$$\Pi = \exp\left[-\left(\frac{x_r}{w_r}\right)^2\right]\exp(iky_r), \quad (25)$$

$$w_r^2 = w^2 - i\frac{2z}{k}\sec\theta, \quad (26)$$

$$h_k = \left(-\frac{2w}{w_r}\right)^{2j-2k} g_k. \quad (27)$$

For the correction field we subsitute Eq. (16) into Eq. (22)

$$\psi_{2n,ro}(x,y) = \frac{r(\bar{k})}{2\pi\psi_{rg}(x,y)} \times \sum_{j=0}^{n}\sum_{k=0}^{j}\int_{-\infty}^{\infty} r_c(k_x)\left(\frac{k_x-k}{c}\right)^{2j-2k} \times \exp\left[-\frac{1}{4}\left(\frac{k_x-k}{c}\right)^2\right] \times \exp[i(k_x x - k_y y_-)]\,dk_x. \quad (28)$$

After performing and substituting Eq. (25) into Eq. (28), (see Appendix B) we will obtain,

$$\psi_{2n,ro}(x,y) = \frac{C_1(\theta)}{P_{2n}(x_r/w_r)}\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{2j-2k} f_j g_k \times \quad (29)$$

$$(kw\delta)^{2j-2k-l}\Big[(-\delta)^{1/2} s_l H_l(\gamma/\sqrt{2}) - \quad (30)$$

$$\beta p_l F_{1/2+l}(\gamma)\Big], \quad (31)$$

where

$$C_1(\theta) = \frac{4m\cos^2\theta_c\sin\theta}{\cos^{1/2}\theta(\sin\theta+\sin\theta_c)^{1/2}} \times \frac{1}{[\cos\theta^2 + m^2(\sin^2\theta - \sin^2\theta_c)]}, \quad (32)$$

$$\delta = (\sin\theta - \sin\theta_c)\sec\theta, \quad (33)$$

$$P_{2n}(x/w) = \sum_{j=0}^{n}\sum_{k=0}^{j} f_j h_k H_{2(j-k)}(x/w), \quad (34)$$

$$s_l = \frac{(2j-2k)!}{l!(2j-2k-l)!}(i)^l, \quad (35)$$

$$p_l = \frac{(2j-2k)!}{l!(2j-2k-l)!}(l)^{l/2}\exp\left[-i\frac{\pi}{2}l\right], \quad (36)$$

$$\beta = \frac{2^{3/4}}{(kw)^{1/2}}\exp(i\pi/4), \quad (37)$$

$$\gamma = \sqrt{2}\left(\frac{ikw_r\delta}{2} - \frac{x_r}{w_r}\right), \quad (38)$$

$$F_{1/2+l}(\gamma) = \exp\left[\gamma^2/4\right] D_{1/2+l}(\gamma), \quad (39)$$

and H_l and $D_{1/2+l}$ are the Hermite polynomial and the Parabolic cylinder function of order l and $1/2+l$, respectevily. The term $P_{2n}(x_r/w_r)$ is just the geometric-optical field without the constant part. Notice that the real part of γ is a measure of the distance between the observation point and the reflected axis, while the imaginary part is a measure of the deviation of the incidence angle from the critical angle.

Now, if we consider the odd Hermite polynomials, the incident field will be

$$\psi_{2n+1,inc}(x,y) = \alpha_{2n+1}\sum_{j=0}^{n}\widehat{f}_j x^{2j+1} \times \frac{\exp[-c^2x^2 + i\bar{k}x]}{\sqrt{\pi}w}, \quad (40)$$

where

$$\widehat{f}_j = \frac{(2n+1)!}{(2j+1)!(n-j)!}(-1)^j(2\sqrt{2c^2})^{2j+1}. \quad (41)$$

Making the same procedure we did before, we arrive to the following expression for the correction term

$$\psi_{2n+1,ro}(x,y) = \frac{A(\theta)}{P_{2n}(x_r/w_r)} \times \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{2j-2k} \widehat{f}_j\widehat{g}_k(kw\delta)^{2j-2k-l} \times \left[(-\delta)^{1/2}s_l H_l(\gamma/\sqrt{2}) - \beta p_l F_{1/2+l}(\gamma)\right], \quad (42)$$

where

$$\widehat{g}_k = \frac{(2j)!}{k!(2j-2k)!}(-1)^k\left(\frac{i}{2c}\right)^{2j}\frac{1}{c}. \quad (43)$$

Equations (29) and (42) are expressions for the correction term; it is not, strictly speaking, the total reflected field. Nevertheless, the reflected field is expressed in terms of the classical reflected field (geometric-optical field) and the correction factor. For our purpouses, the correction term can be used to arrive to an expression for the lateral displacement, as it will be discussed in the following section. Notice that the difference between the correction factor for the even and odd cases are just the summation terms f_j and g_k, the rest of the terms are exactly the same.

3. Lateral Displacement

The objective of the present section is to obtain the lateral shift. To this end we need to rewrite the reflcected field in such way that the equation shows explicitly the trajectory of the beam. In order to obtain this expression (for the even Hermite-Gaussian beam), we first recognize that if the distance of propagation is sufficiently small compared to the beam width, then the beam will remain well collimated, that means $w_r \simeq w$. Then, we rewrite Eq. (20) in the following form

$$\psi_{2n,r}(x,y) = -\frac{\alpha}{\sqrt{\pi}wP_{2n}(x_r/w)}\exp[-(x_r/w)^2 + \ln(1+\psi_{2n,ro})]\exp[iky], \quad (44)$$

if we expand the logarithm into a Taylor series around $(x_r/w) = 0$ ($\gamma = ikw\delta/\sqrt{2}$) we get

$$\ln[1+\psi_{2n,ro}(x,y)] \simeq a_0 + a_1(x_r/w) + a_2(x_r/w) + ..., \quad (45)$$

where

$$a_n = \frac{1}{n!}\frac{d^n}{d(x_r/w)^n}\ln(1+\psi_{2n,ro})\Big|_{(x_r/w)=0}. \quad (46)$$

Making the substitution and after algebraic manipulations, we arrive to the following expression:

$$\exp[-(x_r/w)^2 + \ln[1+\psi_{ro}]] \simeq \exp\left(-\frac{1-a_2}{w^2}\right) \times \exp\left[x_r - L_{2n}\right], \quad (47)$$

where L_{2n} has dimensions of length. Notice now the argument of the exponential, we can interpret L_{2n} as a distance that is displaced with respect to x_r and is, therefore, a quantity that can be interpreted as a lateral displacement for the even Hermite-Gaussian beam. Notice that the important step in obtaining such result is present in Eq. (44), where we rewrite the correction factor as a natural logarithm, this shows the importance of rewriting the reflected field in terms of the correction factor. This allow us to make a series expansion and, finally, re-arrange the terms so that we get the lateral shift L_{2n}. This lateral shift is, then, given by

$$L_{2n} \simeq \frac{w}{2\cos\theta}\left|\mathrm{Re}\left[\frac{a_1}{1-a_2}\right]\right|, \quad (48)$$

where

$$a_1 = \frac{A(\theta)}{P_{2n}(0)}\frac{f_2(\delta)}{f_1(\delta)}, \quad (49)$$

$$a_2 = \frac{A(\theta)}{P_{2n}(0)}\left\{\frac{1}{2}\frac{f_3(\delta)}{f_1(\delta)} - \frac{1}{2}\left[\frac{f_2(\delta)}{f_1(\delta)}\right]^2\right\}, \quad (50)$$

and

$$f_1(\delta) = 1 + \frac{A(\theta)}{P_{2n}(0)} \times \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{2j-2k} f_j g_k (kw\delta)^{2j-2k-l} \times \left[(-\delta)^{1/2}s_l H_l(\gamma_0/\sqrt{2}) - \beta p_l F_{1/2+l}(\gamma_0)\right], \quad (51)$$

$$f_2(\delta) = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{2j-2k} f_j g_k (kw\delta)^{2j-2k-l} \times \left[(-\delta)^{1/2}s_l(2l)H_{l-1}(\gamma_0/\sqrt{2}) - \beta p_l(1/2+l)F_{-1/2+l}(\gamma_0)\right], \quad (52)$$

$$f_3(\delta) = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{2j-2k} f_j g_k (kw\delta)^{2j-2k-l} \times \left[(-\delta)^{1/2}s_l(4l)(l-1)H_{l-2}(\gamma_0/\sqrt{2}) - \beta p_l(-1/4+l^2)F_{-3/2+l}(\gamma_0)\right]. \quad (53)$$

For the lateral displacement for the odd case, we use

$$\ln[1+\psi_{2n+1,ro}(x,y)] \simeq b_0 + b_1(x_r/w) + b_2(x_r/w) + \ldots, \quad (54)$$

where

$$b_n = \frac{1}{n!}\frac{d^n}{d(x_r/w)^n}\ln(1+\psi_{2n+1,ro})\Big|_{(x_r/w)=0}. \quad (55)$$

For this case, L_{2n+1} will be

$$L_{2n+1} \simeq \frac{w}{2\cos\theta}\left|\mathrm{Re}\left[\frac{b_1}{1-b_2}\right]\right|, \tag{56}$$

where

$$b_1 = \frac{A(\theta)}{P_{2n}(0)}\frac{g_2(\delta)}{g_1(\delta)}, \tag{57}$$

$$b_2 = \frac{A(\theta)}{P_{2n}(0)}\left\{\frac{1}{2}\frac{g_3(\delta)}{g_1(\delta)} - \frac{1}{2}\left[\frac{g_2(\delta)}{g_1(\delta)}\right]^2\right\}, \tag{58}$$

and

$$g_1(\delta) = 1 + \frac{A(\theta)}{P_{2n}(0)}\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{2j-2k}\widehat{f}_j\widehat{g}_k(kw\delta)^{2j-2k-l} \times\left[(-\delta)^{1/2}s_l H_l(\gamma_0/\sqrt{2}) - \beta p_l F_{1/2+l}(\gamma_0)\right], \tag{59}$$

$$g_2(\delta) = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{2j-2k}\widehat{f}_j\widehat{g}_k(kw\delta)^{2j-2k-l} \times\left[(-\delta)^{1/2}s_l(2l)H_{l-1}(\gamma_0/\sqrt{2}) - \beta p_l(1/2+l)F_{-1/2+l}(\gamma_0)\right], \tag{60}$$

$$g_3(\delta) = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{2j-2k}\widehat{f}_j\widehat{g}_k(kw\delta)^{2j-2k-l} \times\left[(-\delta)^{1/2}s_l(4l)(l-1)H_{l-2}(\gamma_0/\sqrt{2}) - \beta p_l(-1/4+l^2)F_{-3/2+l}(\gamma_0)\right]. \tag{61}$$

Equations (48) and (56) are our final result. It should be noted that we are taking the positive value, since the displacement is always a positive magnitude. Notice, also, that we take terms of second order in Eqs. (48) and (56) since in our case, the terms a_2 and b_2 cannot be neglected. In the next section, we use Eqs. (48) and (56) with Eqs. (46-50) and (54-58), respectevily, to obtain the lateral displacement for beams of different order.

4. Numerical Results

We did calculations (programming was done using Wolfram Mathematica 7.0) for the lateral displacement for a Hermite-Gaussian beam, of even order and of width $kw = 10000$, reflected from a dielectric interface of index of refraction $n = 1.52$, for the following values: $N = 0$, 2, 4, 6, 8, and 10.

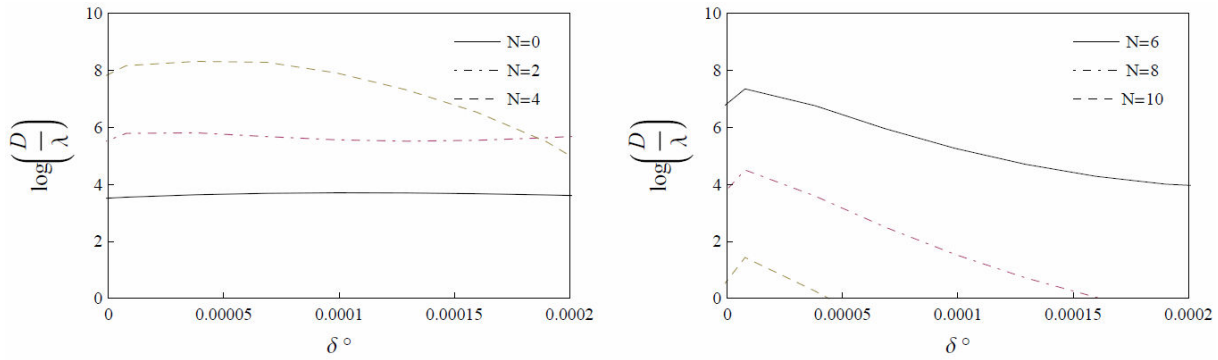

FIGURE 3. Lateral Displacement for an even Hermite-Gaussian Beam, for $n = 1.52$ (glass crown) and a beam width of $kw = 10000$. Only normal polarization is shown. Parallel polarization only has the effect of resclaing the curve. Graphics are shown in semilog scale.

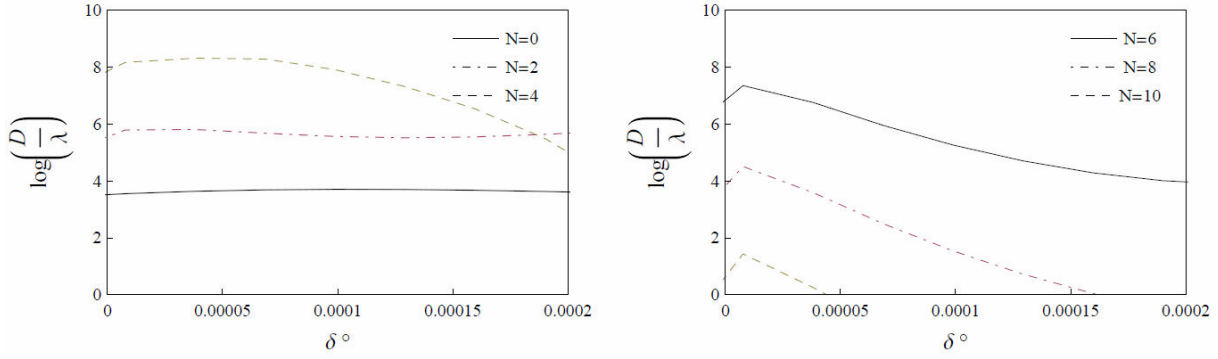

FIGURE 4. Lateral Displacement for an odd Hermite-Gaussian Beam, for n=1.52 (glass crown) and a beam width of $kw = 10000$. Only normal polarization is shown. Parallel polarization only has the effect of resclaing the curve. Graphics are shown in semilog scale.

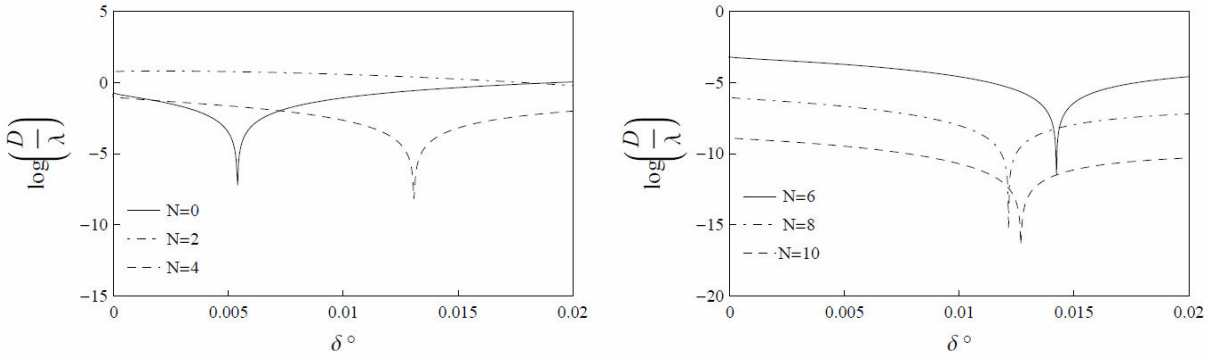

FIGURE 5. Lateral Displacement for an even Hermite-Gaussian Beam, for n=1.52 (glass crown) and a beam width of $kw = 10$. Only normal polarization is shown. Parallel polarization only has the effect of resclaing the curve. Graphics are shown in semilog scale.

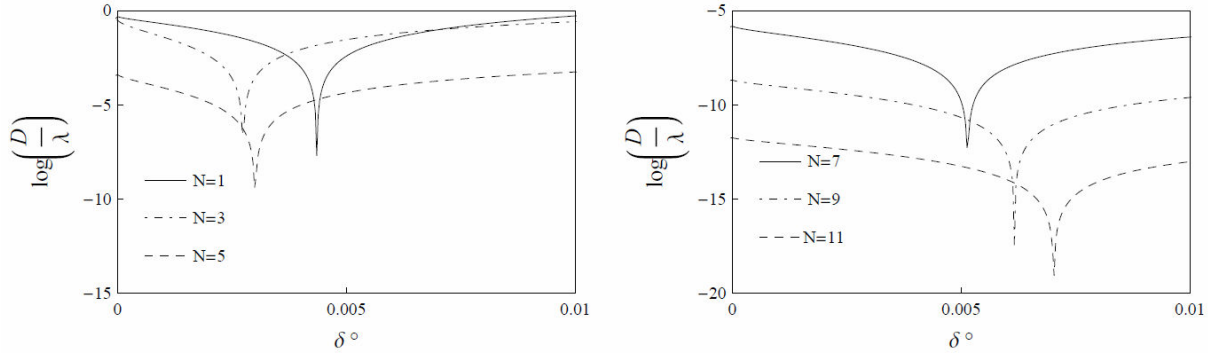

FIGURE 6. Lateral Displacement for an odd Hermite-Gaussian Beam, for n=1.52 (glass crown) and a beam width of $kw = 10$. Only normal polarization is shown. Parallel polarization only has the effect of resclaing the curve. Graphics are shown in semilog scale.

The results are shown in the Fig. 3 in a semilog scale. We see that the lateral displacement increases for $N = 2$ until $N = 4$, and then starts to decrease. This peculiar behavior may be due to the fact that we are cosidering just one dimensional transverse beam and the fact that we are not considering the Guoy phase shift. We also did the calculations for the odd case; that is, for $N = 1$, 3, 5, 7, 9 and 11 (see Fig. 4). We see the same behavior as the even case; that is, an increase in the magnitude of the lateral displacement, and then a decrease in the value of the lateral displacement.

Finally, we did the same calculation, but for a very low beam width $kw = 10$. In Fig. 5 and 6 it is shown the results for $N = 0, 2, 6, 8, 10$ and $N = 1, 3, 5, 7, 9, 11$ in a semilog scale. In this case, we see that the behavior is, except for the case $N = 2$ where the displacement is larger than $N = 0$, a steadily decrease of the lateral displacement as the order of the mode of the beam increases. We see, also, that the maximum displacement occurrs at different angles for the even and odd case. For the even case, the displacement attains its maximum for angles above $\delta = 0.001°$, and for the odd case, for angles below $\delta = 0.001°$.

One point of consideration, as was mentioned before, is that all the present calculations were made assuming a wave which is normal to the plane of incidence, *i.e.*, for the case of normal polarization. This dependence is given by the constant m in the reflectance (see Eq. (13)). For the case of a wave of parallel polarization to the plane of incidence we must include the index of refraction n, this only will re-scale all the curves and wil not affect the shape of these curves.

Take into account that the present simulations are new and, to our knowledge, haven't been made before, so what we do here is only compare the results with the case for a Gaussian beam, result that is known since the results presented by Horowitz and Tamir. In future papers we will make an equivalent procedure to obtain the lateral shift and compare the results with the ones presented here.

5. Conclusions

Previously, Horowitz and Tamir [4] arrived to an analitic equation for the lateral displacement for a Gaussian beam of zero order. Some years after, Lai, Cheng and Chang [6], using an alternative method, arrived to a similar formula. However, until now, no analitic formula has been derived for a more general Hermite-Gaussian beam. By the same methods described above, we could arrive to an expression for the Goos-Hänchen shift for the more general case of a Hermite-Gaussian beam. Using such formula, we gave numerical results of the lateral displacement in fuction of the order of the Hermite-Gaussian beam and the beam width. We found the peculiar behavior that the lateral displacement first increases and then decreases steadily as the order of the beam

increases. Assumptions, such as not taking in account the Gouy phase shift as the beam propagates, has been made. If such phase is taking into account, the mathematical problem becomes, analitically, inmanageable and numerical methods should be used. We will consider, for a future paper, the behavior of the lateral displacement when the beam incides on a finite dielectric interface. Another investigation that will be considered in another paper is the behavior of the lateral displacement when we use the reflectance $r(k_x)$ and not the correction factor $r_c(k_x)$; then, using these results we will make a comparision with the results presented in the present work. Also, we will consider the efect of the Guoy phase shift in the problem; such consideration will requiere the use of numerical methods to solve the integrals.

Appendix

A. Reflected Field - Geometric Optical field

The first integral we need to solve is

$$I_1 = \int_{-\infty}^{\infty} \left(\frac{k_x - k}{c}\right)^{2j-2k} \exp\left[-\frac{1}{4}\left(\frac{k_x - k}{c}\right)^2\right] \times \exp[i(k_x x - k_y y_-)]\, dk_x. \quad \text{(A.1)}$$

To accomplish this, we first need to express k_y as a function of k_x. Since $k_y = (k_0^2 - k_x^2)^{1/2}$ is not managable the way it is, we expand k_y in a Taylor series, to second order, around the critical value $k_0 = k\sin\theta_c$. The result is

$$k_y = k\cos\theta + k\mu\sin\theta - k(\mu^2/2)\sec^2\theta, \quad \text{(A.2)}$$

where

$$\mu = (\sin\theta - k_x/k)/\cos\theta. \quad \text{(A.3)}$$

Substituting Eq. (A.2) into Eq. (A.1), we get

$$I_1 = \Pi(kw)^{2j-2k} \int_{-\infty}^{\infty} \mu^{2j-2k} \times \exp\left[-\left(\frac{kw_r}{2}\right)^2 \left(\mu + i\frac{2x}{w_r^2 k}\right)^2\right] d\mu, \quad \text{(A.4)}$$

where Π and w_r are given by Eqs. 25 and 27. The integral can be solved easily if we make $u = (kw_r/2)[\sigma + i(2x/kw_r^2)]$ and using Newton's binomial,

$$I_1 = \Pi(2/kw_r)(2w/w_r)^{2j-2k} \sum_{l=0}^{2j-2k} \frac{(2j-2k)!}{l!(2j-2k-l)!} \times \left[-i\frac{x_r}{w_r}\right]^{2j-2k-l} \int_{-\infty}^{\infty} u^l \exp(-u^2)\, du. \quad \text{(A.5)}$$

Using now the fact that

$$\int_{-\infty}^{\infty} u^l \exp(-u^2)\, du = \begin{cases} \frac{(2l)!}{2^{2l} l!}\sqrt{\pi} & \text{for } l \text{ even}, \\ 0 & \text{for } l \text{ odd}, \end{cases} \quad \text{(A.6)}$$

we get as a result

$$I_1 = \Pi(2/kw_r)(2w/w_r)^{2j-2k} \times \sum_{l=0}^{j-k} \frac{(2j-2k)!}{l!(2j-2k-2l)!} \left[i\frac{x_r}{w_r}\right]^{2j-2k-2l} \frac{\sqrt{\pi}}{2^{2l}}. \quad \text{(A.7)}$$

We can simplify the previous result by observing that

$$\sum_{l=0}^{j-k} \frac{(2j-2k)!}{l!(2j-2k-2l)!} \left[i\frac{x_r}{w_r}\right]^{2j-2k-2l} \times \frac{\sqrt{\pi}}{2^{2l}} = \sqrt{\pi}\left(-\frac{1}{2}\right)^{2j-2k} \times \sum_{l=0}^{[2(j-k)/2]} \frac{(2j-2k)!}{l!(2j-2k-2l)!}(-1)^l \times \left(2\frac{x_r}{w_r}\right)^{2j-2k-2l}. \quad \text{(A.8)}$$

We recognize that the term in the summatory are the Hermite polynomials of order $2(j-k)$. For this reason, our final result is

$$I_1 = \Pi(2/kw_r)(2w/w_r)^{2j-2k}\sqrt{\pi}\left(-\frac{1}{2}\right)^{2j-2k} \times H_{2(j-k)}(x_r/w_r). \quad \text{(A.9)}$$

B. Reflected Field - Correction Factor

The other integral we need to solve is

$$I_2 = \int_{-\infty}^{\infty} r_c(k_x)\left(\frac{k_x - k}{c}\right)^{2j-2k} \exp\left[-\frac{1}{4}\left(\frac{k_x - k}{c}\right)^2\right] \times \exp[i(k_x x - k_y y_-)]\, dk_x. \quad \text{(B.1)}$$

For this case, it is convenient to use the following variable

$$\nu = \mu - \delta = (\sin_c\theta - k_x/k)/\cos\theta. \quad \text{(B.2)}$$

Since we are interested in a small region of δ, the main contribution to the integral arise for a small values of ν. Also, the principal contribution arise from values of k_y in a neighborhood centered around the value $k_x = k\sin\theta$. In terms of this variable, k_y can be expanded in Taylor series around $\nu = -\delta$ (the axis of the geometric-optics reflected beam) giving

$$k_y = k\cos\theta + k(\nu+\delta)\tan\theta - k[(\nu+\delta)^2/2]\sec\theta^2. \quad \text{(B.3)}$$

We now focus our attention on the functions $r(k_x)$ and $r_c(k_x)$. First, the second term of the numerator and the denomitaror in the reflectance is $(k_0^2 - k_x^2)^{1/2}$. Second, we observe that this function varies rapidly around $k_x \simeq k_0$ due to the singularity at $k_x = k_0$. For this reason it is possible to make the change of variable to $\nu^{1/2}$. We then rewrite the function $r(k_x)$ (see Eq. (20)) as

$$r_c(\nu^{1/2}) = \frac{1}{r(\bar{k})}\frac{p(\nu^{1/2})}{q(\nu^{1/2})} - 1 \tag{B.4}$$

where

$$\begin{aligned} r(\bar{k}) &= r(k\sin\theta) \\ &= \frac{\cos\theta - m(\sin^2\theta_c - \sin^2\theta)^{1/2}}{\cos\theta + m(\sin^2\theta_c - \sin^2\theta)^{1/2}} \end{aligned} \tag{B.5}$$

$$\begin{aligned} p(\nu^{1/2}) = \cos\theta + (\nu+\delta)\sin\theta - \frac{(\nu+\delta)^2}{2}\sec\theta - m\cos\theta \\ \times \left[\xi + 2(\nu+\delta)^2\tan\theta + (\nu+\delta)^2\right]^{1/2}, \end{aligned} \tag{B.6}$$

$$\begin{aligned} q(\nu^{1/2}) = \cos\theta + (\nu+\delta)\sin\theta - \frac{(\nu+\delta)^2}{2}\sec\theta + m\cos\theta \\ \times \left[\xi + 2(\nu+\delta)\tan\theta + (\nu+\delta)^2\right]^{1/2}, \end{aligned} \tag{B.7}$$

and

$$\xi = (\sin\theta_c^2 - \sin\theta^2)\sec\theta^2. \tag{B.8}$$

We now expand $r_c(\nu^{1/2})$ in Tylor series in powers of $\nu^{1/2}$ around $\nu = -\delta$, obtaining (keeping only terms of first order)

$$r_c(\nu) = C_0 - C_1\left[\nu^{1/2} - (-\delta)^{1/2}\right], \tag{B.9}$$

where

$$\begin{aligned} C_0 &= r_c(\nu)\Big|_{nu=-\delta}, \\ C_1 &= -\frac{dr_c(\nu)}{d\nu}\Big|_{\nu=-\delta}. \end{aligned} \tag{B.10}$$

The evaluation of C_0 gives zero. The evaluation of C_1 is easy but cumbersome. First, we obserbe that

$$C_1 = \frac{1}{r(\bar{k})}\frac{p(\nu)q'(\nu) - p'(\nu)q(\nu)}{q(\nu)^2}\Big|_{\nu=-\delta}, \tag{B.11}$$

where

$$p(\nu)\Big|_{\nu=-\delta} = \cos\theta\left[1 - m\xi^{1/2}\right], \tag{B.12}$$

$$q(\nu)\Big|_{\nu=-\delta} = \cos\theta\left[1 + m\xi^{1/2}\right], \tag{B.13}$$

$$p'(\nu)\Big|_{\nu=-\delta} = 2(-\delta)^{1/2}\left[1 - m\xi^{-1/2}\right], \tag{B.14}$$

and

$$q'(\nu)\Big|_{\nu=-\delta} = 2(-\delta)^{1/2}\left[1 + m\xi^{-1/2}\right]. \tag{B.15}$$

Substituting Eqs. (B.12)-(B.15) into Eq. (B.11) gives (after algebraic simplification)

$$\begin{aligned} C_1(\theta) \equiv C_1 = \frac{4m\cos^2\theta_c\sin\theta}{\cos^{1/2}\theta(\sin\theta_c + \sin\theta)^{1/2}} \\ \times \frac{1}{[\cos\theta^2 + m^2(\sin^2\theta - \sin^2\theta_c)]} \end{aligned} \tag{B.16}$$

So, the function $r_c(\nu^{1/2})$ is, finally:

$$r_c(\nu^{1/2}) = -C_1(\theta)[\nu^{1/2} - (-\delta)^{1/2}]. \tag{B.17}$$

The integral I_2 is then

$$\begin{aligned} I_2 = (k\cos\theta)(kw)^{2j-2k}\Pi\exp[\gamma^2/2]C_1(\theta) \\ \times \int_{-\infty}^{\infty}[\nu^{1/2} - (-\delta)^{1/2}](\nu+\delta)^{2j-2k} \\ \times \exp\left[-\left(\frac{kw_r}{2}\right)^2\nu^2 + i\left(\frac{k\gamma w_r}{\sqrt{2}}\right)\nu\right]d\nu. \end{aligned} \tag{B.18}$$

If now we use Newton's binomial we get

$$\begin{aligned} I_2 = (k\cos\theta)(kw)^{2j-2k}\Pi\exp[\gamma^2/2]C_1(\theta) \\ \times \sum_{l=0}^{2j-2k}\frac{(2j-2k)!}{l!(2j-2k-l)!}\delta^{2j-2k-l} \\ \times [J_1 - (-\delta)^{1/2}J_2], \end{aligned} \tag{B.19}$$

where

$$\begin{aligned} J_1 = \int_{-\infty}^{\infty}\nu^{1/2+l} \\ \times \exp\left[-\left(\frac{kw_r}{2}\right)^2\nu^2 + i\left(\frac{k\gamma w_r}{\sqrt{2}}\right)\nu\right]d\nu, \end{aligned} \tag{B.20}$$

and

$$\begin{aligned} J_2 = \int_{-\infty}^{\infty}\nu^{l} \\ \times \exp\left[-\left(\frac{kw_r}{2}\right)^2\nu^2 + i\left(\frac{k\gamma w_r}{\sqrt{2}}\right)\nu\right]d\nu. \end{aligned} \tag{B.21}$$

The integral J_1 can be split into two parts:

$$J_1 = \int_{-\infty}^{0} \nu^{1/2+l} \times \exp\left[-\left(\frac{kw_r}{2}\right)^2 \nu^2 + i\left(\frac{k\gamma w_r}{\sqrt{2}}\right)\nu\right] d\nu + \int_{0}^{\infty} \nu^{1/2+l} \times \exp\left[-\left(\frac{kw_r}{2}\right)^2 \nu^2 + i\left(\frac{k\gamma w_r}{\sqrt{2}}\right)\nu\right] d\nu, \quad \text{(B.22)}$$

after changing variable in the first integral

$$J_1 = (-1)^l i \int_{0}^{\infty} \nu^{1/2+l} \exp\left[-\left(\frac{kw_r}{2}\right)^2 \nu^2 - i\left(\frac{k\gamma w_r}{\sqrt{2}}\right)\nu\right] d\nu + \int_{0}^{\infty} \nu^{1/2+l} \times \exp\left[-\left(\frac{kw_r}{2}\right)^2 \nu^2 + i\left(\frac{k\gamma w_r}{\sqrt{2}}\right)\nu\right] d\nu. \quad \text{(B.23)}$$

Each integral can be found in tables (see 3.462-1 in [21]). We obtain

$$J_1 = \left(\frac{kw_r}{\sqrt{2}}\right)^{-(3/2+l)} \Gamma\left(\frac{3}{2}+l\right) \exp\left(-\frac{\gamma^2}{4}\right) \times \left[(-1)^l i D_{-(3/2+l)}(i\gamma) + D_{-(3/2+l)}(-i\gamma)\right], \quad \text{(B.24)}$$

after using the identity 9.248-1 in [21] we get

$$J_1 = \left(\frac{kw_r}{\sqrt{2}}\right)^{-(3/2+l)} \sqrt{2\pi} \exp\left[-\frac{i\pi}{4}(1-2l)\right] \times \exp\left(-\frac{\gamma^2}{4}\right) D_{1/2+l}(\gamma). \quad \text{(B.25)}$$

The integral J_2 can be solved directly after completing the square an changing the variable of integration. We obtain

$$J_2 = \left(\frac{2}{kw_r}\right)\sqrt{\pi} \exp\left(-\frac{\gamma^2}{2}\right) \times \left(-\frac{i}{kw_r}\right)^l H_l(\gamma/\sqrt{2}). \quad \text{(B.26)}$$

Substitution of Eq. (B.25) and (B.26) into Eq. (B.19) gives the final expression for the integral I_2.

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