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On geometro dynamics in atomic stationary states

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In a previous paper (G.Gómez Blanch *et al*, 2018) we defined, in the frame of a geometro-dynamic approach, a metric corresponding to a Lorentzian spacetime where the electron stationary trajectories in a hydrogenoid atom, derived from the de Broglie-Bohm model, are geodesics. In this paper we want to complete this purpose: we will determine the remaining relevant geometrical elements of such an approach, and we will calculate the energetic density component of the energy-momentum tensor. We will discuss the meaning of the obtained results and their relationship with other geometrodynamic approaches. Furthermore, we will derive a more general relationship between the Lorentzian metric tensor and the wave function for general monoelectronic stationary states. In our approach, the electron description by the wave function Ψ in the Euclidean space and time is shown equivalent to the description by a metric tensor in a Lorentzian manifold. The particle acquires a determining role over the wave function, in a similar manner as the wave function determines the movement of the particle. This dialectic approach overcomes the de Broglie-Bohm approach. And furthermore, a non local element (the quantum potential) is introduced in the model, and incorporated in the geometrodynamic description by the metric tensor.

Keywords: de Broglie -Bohm; lorentzial manifold; wave function; metric tensor; scalar curvature; quantum potential; energy moment tensor; numerical methods; geometrodynamics

1. Introduction

One can describe the geometrodynamics with the known affirmation: 'mass-energy 'tells' spacetime how to curve and spacetime 'tells' mass-energy how to move' [1].

In a previous paper (G. Gomez Blanch *et al*, 2018), we started from the de Broglie - Bohm description of an electron trajectory [2] in the hydrogen atom. The trajectories described in the de Broglie-Bohm description belong to an Euclidian space and time. Then, we made the ansatz that this kind of trajectories, corresponding to stationary states, are geodesics of a Lorentzian manifold, so their spacetime is curved at less in the electron entourage. Moreover, an electron in a geodesic does not exert any force and does not lose energy. This fact would explain the stability of the atom, without further quantum considerations. In some way, we also established a relatioship between Quantum Theory and General Relativity.

A Lorentzial manifold has locally the structure of an Euclidian space and time, and therefore we can assimilate the de Broglie-Bohm trajectory equations with the Lorentzian geodesic equations at a differential level. In this way, we can obtain equations that interrelate the metric tensor components.

Then, we searched in the general catalogue of exact solutions of the Einstein field equations, a general metric for dust with cylindrical symmetry [3]. The selected metric model derivates from the van Stockung metric class with some contributions of other authors (King(1974); Winicour (1975); Wishweshwara -Winicour (1977)). We used this mentioned metric, but the results presented some incoherences regarding the de Broglie-Bohm approach. Then we modified this model by transforming a constant parameter into a function of the radius. We came finally over a covariant metric that was consistent with the physics, mainly regarding the velocity and the kinetic moment of the electron.

In the present paper we continue this line of work, by characterizing the relevant elements of the geometrical structure, 'that 'tells' mass-energy the way to go' : contravariant metric, Levi-Civita connectors, Ricci tensor and scalar curvature. We compare this scalar curvature with other geometrodynamical approaches of the literature.

Next we consider how 'the mass-energy tells the spacetime how it wraps'. In order to do this we evaluate an element of the energy-momentum tensor: the one that represents the energy density. From it, we make experimental considerations regarding the affected volume of spacetime.

Finally, we derive a general relationship between the two components of the wave equation in space-time coordinates and the metric tensor for stationary states, and we make an interpretation of the results.

The structure of the paper goes through the phases described above: in Sec. 2 we characterize the geometric elements, discuss about the scalar curvature and compare with another geometro-dynamic interpretation; in Sec. 3 we make the derivation of the energy density component of the energy-momentum tensor and the corresponding considerations; in Sec. 4 we study the relationship between the components of the wave function of the non-relativistic quantum mechanics and the components of the metric tensor. Finally, in Sec. 5 we establish the corresponding conclusions.

2. Geometrical elements

We will describe here the general way of the geometrical calculations. We start from the covariant metrics, previously calculated (G.Gomez Blanch *et al*, 2018), and from there we perform the calculations of the required geometrical objects.

As it is known, the curvature of a Riemann manifold is given by the curvature tensor R_{ijkl}, which appears to us by considering the circulation of a generic vector in a closed contour. The contraction of this tensor leads to the Ricci tensor R_{ij}, that is essential for our calculations. This tensor is given by the following expression, based on the connectors of Levi-Civita (Christoffel symbols of second order) [4]

$$R_{ij} = \partial_h \Gamma^h_{ij} - \partial_j \Gamma^h_{ih} + \Gamma^h_{ij}\Gamma^m_{hm} - \Gamma^m_{ih}\Gamma^h_{jm} \tag{1}$$

where the connectors are given in function of the metric by:

$$\Gamma^j_{ik} = \frac{1}{2} g^{jh}(\partial_k g_{ih} + \partial_i g_{hk} - \partial_h g_{ki}) \tag{2}$$

and ∂_i is partial derivative respect to the i coordinate. The contravariant tensor metric reads:

$$g^{ij} = \frac{\alpha_{ij}}{g} \tag{3}$$

being α_{ij} the adjoint of g_{ij} . The equation that relates the Ricci tensor with the Ricci curvature is simply its contraction:

$$R = g^{ij} R_{ij} \tag{4}$$

2.1. The initial covariant metric

According to our previous paper (G.Gomez Blanch et alii, 2018), we start from the following metric, with x^4 = t:

$$g_{ij} = \begin{pmatrix} e^{-\frac{b^2}{c^4}\left(\frac{lnk\rho}{\rho}\right)^2} & 0 & 0 & 0 \\ 0 & \rho^2 - \frac{b^2}{c^4}(lnk\rho)^2 & 0 & -\frac{b}{c}lnk\rho \\ 0 & 0 & e^{-\frac{b^2}{c^4}\left(\frac{lnk\rho}{\rho}\right)^2} & 0 \\ 0 & -\frac{b}{c}lnk\rho & 0 & -c^2 \end{pmatrix} \tag{5}$$

It must be highlighted that this metric for stationary states is not static, as shows the presence of not nul term g_{24} and the term $g_{24}d\phi dt$ changes of sign with the time sense inversion.

We remember briefly the deduction of this metric. We start with the geodesic equation with the proper time as parameter:

$$\frac{d^2x^j}{dt^2} + \Gamma^j{}_{ik}\frac{dx^i}{dt}\frac{dx^k}{dt} = 0 \tag{6}$$

where we introduce the corresponding velocities of the de Broglie- Bohm approach in polar coordinates: ω angular velocity and c light velocity

$$\omega^2\Gamma^j_{22} - 2\omega c\Gamma^j_{24} + c^2\Gamma^j_{44} = 0 \tag{7}$$

We take into acount the quantisation of the angular momentum as (u azimuthal quantum number):

$$L_z = mv\rho = u\hbar \tag{8}$$

If we introduce the constant b:

$$b = \frac{u\hbar}{m} \tag{9}$$

and take into account (2), the equation in the tensor metric components reads:

$$b^2\partial_1 g_{22} - 2bc\rho^2\partial_1 g_{24} + c^2\rho^4\partial_1 g_{44} = 0 \tag{10}$$

Here we introduce the metric form corresponding to dust particles with cylindrical symmetry that is an exact solution of the Einstein's field equation (10):

$$(ds)^2 = e^{-a^2\rho^2}((d\rho)^2 + (dz)^2) + \rho^2(d\phi)^2 - (cdt + a\rho^2 d\phi)^2 \tag{11}$$

The parameter a is substituted for a function of the radius $y(\rho)$ to fit our problem and avoid physical inconsistencies. Then we get:

$$y' + \frac{2y}{\rho} = -\frac{b}{c^2\rho^3 - b\rho^3 y} \tag{12}$$

that can be simplified, with good approximation to:

$$y' + \frac{2y}{\rho} = -\frac{b}{c^2\rho^3} \tag{13}$$

and that has the solution:

$$y = -\frac{b}{c^2}\rho^{-2}ln(k\rho), \tag{14}$$

where k is a constant that shall be estimated from the consideration of evolution to Minkowskian situation. Replacing that solution in the components of the metric tensor drives us to the proposed metric (5).

2.2. Contravariant metric

We need to start from the connectors calculated according to (2) (See G.Gomez Blanch et alii, 2018). To do that, we need to calculate, in a first step, the determinant g and, from it, the contravariant metric tensor. For a metric tensor like (4), with the form:

$$g_{ij} = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & g_{24} \\ 0 & 0 & g_{33} & 0 \\ 0 & g_{24} & 0 & g_{44} \end{pmatrix} \tag{15}$$

the contravariant metric is obtained from its inverse matrix. The determinant of the covariant matrix is:

$$g = det(g_{ij}) = g_{11}g_{33}(g_{22}g_{44} - g_{24}^2) \tag{16}$$

So, taking into account (4), the matrix g^{ij} as a function of the components of g_{ij} reads:

$$g^{ij} = \begin{pmatrix} \frac{1}{g_{11}} & 0 & 0 & 0 \\ 0 & \frac{g_{44}}{g_{22}g_{44}-g_{24}^2} & 0 & -\frac{g_{24}}{g_{22}g_{44}-g_{24}^2} \\ 0 & 0 & \frac{1}{g_{33}} & 0 \\ 0 & -\frac{g_{24}}{g_{22}g_{44}-g_{24}^2} & 0 & \frac{g_{22}}{g_{22}g_{44}-g_{24}^2} \end{pmatrix} \tag{17}$$

We substitute now the values of (5) in the previous equation, and obtain the contravariant metric tensor:

$$g^{ij} = \begin{pmatrix} e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 & \frac{blnk\rho}{c^3\rho^2} \\ 0 & 0 & e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2} & 0 \\ 0 & \frac{blnk\rho}{c^3\rho^2} & 0 & \frac{b^2(lnk\rho)^2-c^4\rho^2}{c^6\rho^2} \end{pmatrix} \tag{18}$$

2.3. Calculation of Levi-Civita connectors

We will use the connectors of Levi -Civita, with null torsion. The reason for that election is the following. As it is known, there are two kinds of geodesics: an affin geodesic is the curve generated by a vector (*i.e.* the velocity) with parallel transport, and a metric geodesic that connects points by minimising their distance [5]. The Levi-Civita connection unifies the requirements for affin and metric geodesics, and therefore we use it. We need to assure that the velocity vector has parallel transport along a geodesic and therefore we need the affin connection. We also need the trajectory of the particle to be metrically coherent with the variational principle and so we need the geodesic metric.

The relationship of Levi-Civita connector with the metric is given by (2). In the calculation of the mentioned connectors we take into account the symmetry of the metric tensor and the fact that the only variable in the elements of the metric tensor is ρ, that is x_1, in cylindrical coordinates. Therefore:

$$\partial_h g_{ij} = 0, \forall i,j \in (1,4), h \in (2,4) \tag{19}$$

For this calculation we want the partial derivatives of the covariant metric tensor in relation with ρ. We obtain the following values:

$$\partial_1 g_{ij} = \begin{pmatrix} \frac{2b^2lnk\rho(lnk\rho-1)}{c^4\rho^3}e^{-\frac{b^2}{c^4}\left(\frac{lnk\rho}{\rho}\right)^2} & 0 & 0 & 0 \\ 0 & 2\rho - \frac{2b^2ln(k\rho)}{c^4\rho} & 0 & -\frac{b}{c\rho} \\ 0 & 0 & \frac{2b^2lnk\rho(lnk\rho-1)}{c^4\rho^3}e^{-\frac{b^2}{c^4}\left(\frac{lnk\rho}{\rho}\right)^2} & 0 \\ 0 & -\frac{b}{c\rho} & 0 & 0 \end{pmatrix} \tag{20}$$

Next we calculate the connectors from (2). They are symmetric in their subscripts (null torsion). We obtain the following 10 generic expressions of the connectors:

$$\Gamma^j_{11} = \frac{1}{2}g^{jh}(\partial_1 g_{1h} + \partial_1 g_{h1} - \partial_h g_{11}) = \frac{1}{2}g^{j1}\partial_1 g_{11} \tag{21}$$

$$\Gamma^j_{12} = \Gamma^j_{21} = \frac{1}{2}g^{jh}(\partial_2 g_{1h} + \partial_1 g_{h2} - \partial_h g_{21}) = \frac{1}{2}g^{j2}\partial_1 g_{22} + \frac{1}{2}g^{j4}\partial_1 g_{42} \tag{22}$$

$$\Gamma^j_{13} = \Gamma^j_{31} = \frac{1}{2}g^{jh}(\partial_3 g_{1h} + \partial_1 g_{h3} - \partial_h g_{31}) = \frac{1}{2}g^{j3}\partial_1 g_{33} \tag{23}$$

$$\Gamma^j_{14} = \Gamma^j_{41} = \frac{1}{2}g^{jh}(\partial_4 g_{1h} + \partial_1 g_{h4} - \partial_h g_{41}) = \frac{1}{2}g^{j2}\partial_1 g_{24} \tag{24}$$

$$\Gamma^j_{22} = \frac{1}{2}g^{jh}(\partial_2 g_{2h} + \partial_2 g_{h2} - \partial_h g_{22}) = -\frac{1}{2}g^{j1}\partial_1 g_{22} \tag{25}$$

$$\Gamma^j_{23} = \Gamma^j_{32} = \frac{1}{2}g^{jh}(\partial_3 g_{2h} + \partial_2 g_{h3} - \partial_h g_{32}) = 0 \tag{26}$$

$$\Gamma^j_{24} = \Gamma^j_{42} = \frac{1}{2} g^{jh}(\partial_2 g_{4h} + \partial_4 g_{h2} - \partial_h g_{24})$$
$$= -\frac{1}{2} g^{11} \partial_1 g_{24} \quad (27)$$

$$\Gamma^j_{33} = \frac{1}{2} g^{jh}(\partial_3 g_{3h} + \partial_3 g_{h3} - \partial_h g_{33})$$
$$= -\frac{1}{2} g^{j1} \partial_1 g_{33} \quad (28)$$

$$\Gamma^j_{34} = \Gamma^j_{43} = \frac{1}{2} g^{jh}(\partial_3 g_{4h} + \partial_4 g_{h2} - \partial_h g_{34})$$
$$= -\frac{1}{2} g^{11} \partial_1 g_{34} = 0 \quad (29)$$

$$\Gamma^j_{44} = \frac{1}{2} g^{jh}(\partial_4 g_{4h} + \partial_4 g_{h4} - \partial_h g_{44})$$
$$= -\frac{1}{2} g^{11} \partial_1 g_{44} = 0 \quad (30)$$

Now we make the detailed calculation of these connectors, equations from (21) to (30). Those non-null ones, ordered by their upper index are: $\Gamma^1_{11}, \Gamma^1_{22}, \Gamma^1_{24}, \Gamma^1_{33}$ $\Gamma^2_{12}, \Gamma^2_{14}$ Γ^3_{13} $\Gamma^4_{12}, \Gamma^4_{14}$ as well as their symmetrical terms in the lower subscripts. Their values read:

$$\Gamma^1_{11} = \frac{b^2 lnk\rho(lnk\rho - 1)}{c^4 \rho^3} \quad (31)$$

$$\Gamma^1_{22} = \frac{(b^2 lnk\rho - c^4\rho^2) e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2}}{c^4 \rho} \quad (32)$$

$$\Gamma^1_{24} = \Gamma^1_{42} = -\frac{b}{2c\rho} e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2} \quad (33)$$

$$\Gamma^1_{33} = -\frac{b^2 lnk\rho(lnk\rho - 1)}{c^4 \rho^3} \quad (34)$$

$$\Gamma^2_{12} = \Gamma^2_{21} = -\frac{b^2 lnk\rho - 2c^4\rho^2}{2c^4\rho^3} \quad (35)$$

$$\Gamma^2_{14} = \Gamma^2_{41} = \frac{b}{2c\rho^3} \quad (36)$$

$$\Gamma^3_{13} = \Gamma^3_{31} = -\frac{2b^2 lnk\rho(lnk\rho - 1)}{c^4\rho^3} \quad (37)$$

$$\Gamma^4_{12} = \Gamma^4_{21} = -\frac{b^3 ln^2 k\rho - 2bc^4\rho^2 lnk\rho + bc^4\rho^2}{2c^7\rho^3} \quad (38)$$

$$\Gamma^4_{14} = \Gamma^4_{41} = \frac{b^2 lnk\rho}{2c^4\rho^3} \quad (39)$$

Now we can make an additional evaluation of the proposed model. The geodesic equation (6), taking into account our previous results, reads:

$$\frac{d^2x^j}{dt^2} + \omega^2 \Gamma^j_{22} - 2\omega c \Gamma^j_{24} = 0 \quad (40)$$

If we replace the non-null values of Γ^j from our previous calculation, we get the results:

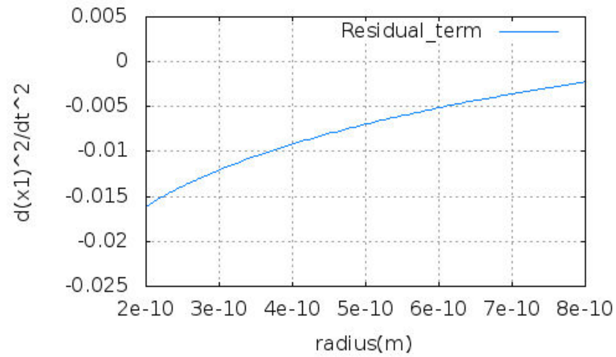

FIGURE 1. Residual term in radial acceleration in terms of the radius.

$$\frac{d^2x^2}{dt^2} = 0 \quad (41)$$

$$\frac{d^2x^3}{dt^2} = 0 \quad (42)$$

And regarding $x^1 = \rho$ we get, taking into account that $b = \omega\rho^2$:

$$\frac{d^2x^1}{dt^2} + \frac{e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2} b^4 ln(k\rho)}{c^4\rho^5}$$
$$+ e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2}(\omega^2\rho - \omega^2\rho) = 0 \quad (43)$$

In the parenthesis of the third term we easily recognize the classical 'centripetal acceleration' and its counterpart, that cancels it. So it stands:

$$\frac{d^2x^1}{dt^2} + e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2} \frac{b^4 ln(k\rho)}{c^4\rho^5} = 0 \quad (44)$$

The second term has a very low value (some 10^{-29} lower than $\omega\rho^2$) in the range of ρ that affects the orbital 2p, and for $k = 10^9$ (as we will establish later on). This term is atributed to the performed approximation in the solution of the differential equation (12) to (13). We can represent it in Fig. 1.

2.4. Ricci tensor

Once the connectors are obtained, we can calculate the Ricci tensor, which allows the determination of the scalar curvature. The equation of definition reads:

$$R_{ij} = \partial_h \Gamma^h_{ij} - \partial_j \Gamma^h_{ih} + \Gamma^h_{ij}\Gamma^m_{hm} - \Gamma^m_{ih}\Gamma^h_{jm} \quad (45)$$

$$R_{11} = -\partial_1(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14}) + \Gamma^2_{12}(\Gamma^1_{11} - \Gamma^2_{12})$$
$$+ \Gamma^4_{14}(\Gamma^1_{11} - \Gamma^2_{14}) - 2\Gamma^4_{12}\Gamma^2_{14} \quad (46)$$

and so, giving only the most significant terms,

$$R_{11} = -\frac{b^2}{c^2}(lnk\rho - 1)\rho^{-4} - \frac{b^4 ln^2 k\rho}{4c^4}\rho^{-6} \quad (47)$$

we follow on with the calculation of the other components of the tensor, and we get:

$$R_{22} = e^{\frac{b^2}{c^4}(\frac{\ln k\rho}{\rho})^2}\left[1 - \frac{2b^2 lnk\rho}{c^4}(lnk\rho - 1)\rho^{-2}\right] \tag{48}$$

$$R_{24} = R_{42} = e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2}\left[\frac{b^4}{c^4}lnk\rho.\rho^{-1} + b\rho^{-2} + \frac{b^4 lnk\rho}{c^4}\rho^{-3} - \frac{b^3 lnk\rho}{c^4}\left(2(lnk\rho - 1) - \frac{b}{2}\right)\rho^{-4}\right] \tag{49}$$

$$R_{33} = \frac{b^2}{c^4}(1 - 4lnk\rho + 2ln^2 k\rho)\rho^{-4} \tag{50}$$

$$R_{44} = e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2}\frac{b^2}{2c^2}\rho^{-4} \tag{51}$$

2.5. Scalar curvature

The scalar curvature is named by the symbol R, as usual. It is calculated by the following equation:

$$R = g^{11}R_{11} + g^{22}R_{22} + 2g^{24}R_{24} + g^{33}R_{33} + g^{44}R_{44} \tag{52}$$

The calculation with all elements of the Ricci tensor without approximations reads:

$$R = \frac{b^2 e^{\frac{b^2}{c^4}(\frac{lnk\rho}{\rho})^2}}{2c^4\rho^4}(12ln^2 k\rho - 20lnk\rho + 5) \tag{53}$$

In our previous paper we made, for heuristic purposes, an estimation of k in the order of 10^6; now, taking into account (53), we can establish $k = 10^9$, that corresponds to a radius limit for the Minkowskian conditions at approximately $\rho = 1,39 \times 10^{-9}$ m.

From this equation of the scalar curvature, we can represent *R* as a function of ρ between 2×10^{-10} and 8×10^{-10} m, corresponding to the interval where the radial probability of the 2p Hydrogen orbital is significant. It is shown in Fig. 2.

There, we can observe that:

- The scalar curvature tends to 0 when $\rho \rightarrow \infty$ (Minkowski).

- The scalar curvature takes infinite value (diverges) when the radius tends to zero. The radius $\rho = 0$, combined with $z = 0$, takes the physical meaning of the mass center of the atomic system; this is, very approximately, the nucleus position. The divergence in the rest of the OZ axis has no physical meaning. Indeed, it suffices to change to spherical coordinates to eliminate this singularity out of the origin. Moreover, we know that, according to the de Broglie-Bohm model, [6], even if the quantum azimuthal number u is 0 and the electron is at rest respect to the nucleus, we can avoid this divergence by excluding the nucleus entourage from our model. Further insights on it will be worked out in future by means of another scalar curvature invariant, as we will explain later on.

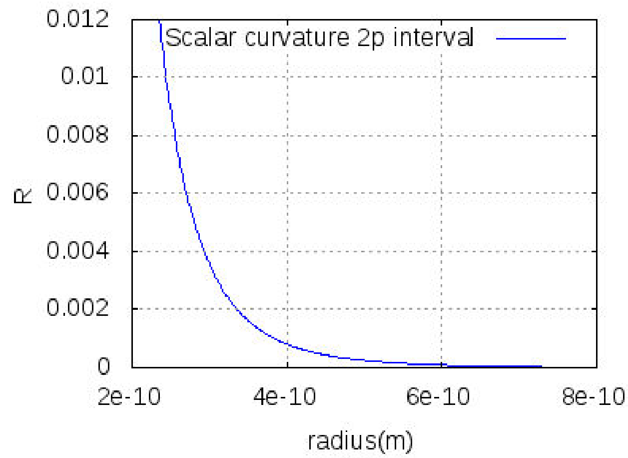

FIGURE 2. Curvature as a function of radius in the maximal probability interval.

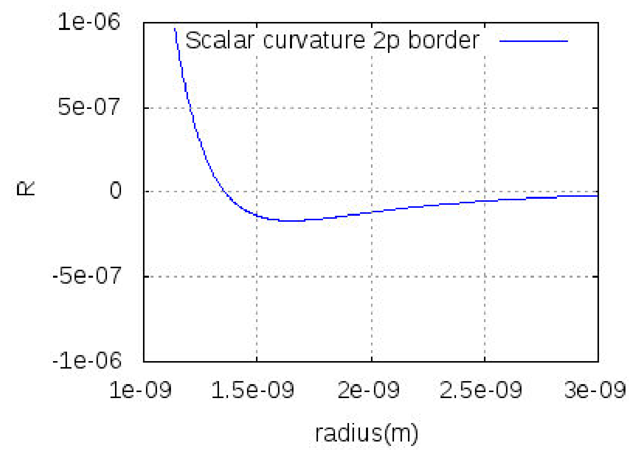

FIGURE 3. Curvature as a function of radius in the upper limit range 4×10^{-9}.

We consider now, in (53), the variation of the scalar curvature with the radial coordinate for a border limit determined by the previously mentioned selection of the constant $k = 10^9$.

We can observe in this border limit, that the curvature decreases when ρ increases until it arrives to 0 (at $1,358 \times 10^{-9}$ m), enters in a zone of negative curvature and returns to the null curvature (at $3,898 \times 10^{-9}$ m). Indeed, the scalar curvature becomes null for the values that are solution of the bracket in Eq (53); which are obtained for the following values of ρ:

$$\rho_{(R=0)} = \frac{e^{\frac{5}{6}\pm\frac{\sqrt{10}}{6}}}{k} \tag{54}$$

Beyond the upper value ($3,898 \times 10^{-9}$ m), the curvature is almost null; it tends to zero asymptotically, as it is shown in Fig. 3.

The physical meaning of that is the following one: an electron that exceeds this critical radius would get in a null curvature region and therefore would escape from the atom. If it gets in the zone where the curvature remains negative, the trajectory would also mean the exit from the atomic system, as it is represented in Fig. 3.

Therefore: if the electron gets out of the zone of positive curvature, a big alteration of its trajectory will happen. This can be the effect of an exterior physical action, that transfers energy to the atomic system. An example is the Frank-Hertz experiment, that is described by D. Bohm in the frame of his interpretation [7].

2.5.1. *Comment on other evaluation of the curvature of space time in microphysical systems*

Now we consider other interesting approach to our subject made by Novello, Salim and Falciano [8].

To see an approximated relationship between the quantum energy and the curvature in our model, [9] we can take the dependence of the curvature, R and the radius according to:

$$R = \frac{\lambda}{\rho^4} \tag{55}$$

where λ is a scalar coefficient. The quantum potential energy, as a difference between the total potential energy of the electron and the kinetic energy reads:

$$E_Q = \frac{me^4}{2\hbar^2 n^2} - \frac{n^2\hbar^2}{2m\rho^2} + \frac{e^2}{4\pi\epsilon_0\sqrt{\rho^2+z^2}} \tag{56}$$

If we replace the previously mentioned approximation of *R*, we get:

$$E_Q = \frac{me^4}{2\hbar^2 n^2} - \frac{n^2\hbar^2}{2m}\lambda^{-\frac{1}{2}}R^{\frac{1}{2}} + \frac{e^2}{4\pi\epsilon_0\sqrt{\lambda^{\frac{1}{2}}R^{-\frac{1}{2}}+z^2}} \tag{57}$$

In this point we remember the assertion of Novello, Salim and Falciano [10] that the quantum potential energy coincides with the curvature of spacetime. We must remark that this affirmation is done within the frame of an approach on Weyl's geometry (3-D Weyl integrable space), very different of the usual Lorentzian, (pseudo-Riemannian) geometry that we use. The geometry that these authors consider has different Γ-affine connections from those of Levi-Civita used by us; therefore, the results obtained by these authors are not comparable to our results, because the scalar curvature is a function of the connections.

Indeed, according to the paper of Novello *et al*. [11], the relationship between the scalar curvature and the quantum potential energy would read:

$$E_Q = -\frac{\hbar}{16m}R \tag{58}$$

very different to Equation (57). It explains that a numerical calculation in our approach yields curvature values different in some orders of magnitude respect to the corresponding to the calculations of Novello *et al*. [12].

Although the expression of the quantum potential energy is particularly simple in the Novello *et al*. approach, our approach allows us to frame our results in standard General Relativity with the use of Levi-Civita connectors, in particular respect to the Einstein field equation and the energy-momentum tensor.

2.5.2. *Other scalar curvature invariants*

The method used until now can be improved by using the Riemann curvature tensor to characterize the curvature in a more detailed way. Indeed, the Ricci tensor can be null and the Riemann tensor, not at all.

In connection with this, one can take into account the Kretschmann curvature, defined as:

$$K = R_{ijkl}R^{ijkl} \tag{59}$$

The use of the Kretschmann scalar curvature can be considered, mainly to detect non physical singularities, as the z=0 axis, out of the nucleus entourage, and tidal effects.

So we made a first approximation to the subject. We made the calculation of the Riemann tensor R^i_{jkl}, that in our case has only 10 non null components. From it, we calculated the completely covariant and the completely contravariant Riemann tensors, R_{ijkl}, R^{ijkl} and from there the Kretschmann scalar curvature. The result had the same divergence features as the R scalar ($\rho- > 0$) by a ρ^{-6} dependence. This fact reinforces our results.

3. Considerations on the energy - momentum tensor. Volume occupied in the spacetime

The deformation of the spacetime previously considered here, derived from the de Broglie-Bohm interpretation, must have a counterpart in the energy momentum tensor. Here we are particularly interested in the correspondence with the energy term of this tensor, which is T^{44}. This term can be explained as an energy density. Let us make the hypothesis that the extension of spacetime affected by the energy momentum tensor is limited to a certain volume; this should be consistent with the deformation stated in the Einstein field equation:

$$R_ij - \frac{1}{2}Rg_{ij} = \frac{8\pi G}{c^4}T_{ij} \tag{60}$$

and from there we obtain [13];

$$T_{ij} = \frac{c^4}{8\pi G}\left(R_{ij} - \frac{1}{2}Rg_{ij}\right). \tag{61}$$

To discuss its physical meaning, we are interested in the contravariant tensor T^{ij}. This can be expressed this way:

$$T^{ij} = g^{ik}g^{jl}T_{kl} \tag{62}$$

Replacing it in the previous equation, we have:

$$T^{ij} = g^{ik}g^{jm}\frac{c^4}{8\pi G}\left(R_{km} - \frac{1}{2}Rg_{km}\right) \tag{63}$$

Or as a function of the covariant Einstein tensor E_{km}:

$$T^{ij} = \frac{c^4}{8\pi G}g^{ik}g^{jm}E_{km} \tag{64}$$

Next we express the component that interest us, comparable to the energy density:

$$T^{44} = \frac{c^4}{8\pi G} g^{4k} g^{4m} E_{km} \tag{65}$$

And developing it we get:

$$T^{44} = \frac{c^4}{8\pi G}\{(g^{24})^2 E_{22} + 2g^{24}g^{44}E_{24} + (g^{44})^2 E_{44}\} \tag{66}$$

The calculation provides the following values for the components of the Einstein tensor (approximating the exponential $e^{-}(b^2/c^4)(ln k\rho/\rho)^2$ to the unit):

$$E_{22} = -\frac{bln(k\rho)(b^3 ln^2(k\rho) - b^3 ln(k\rho) + bc^4\rho^2)}{c^8\rho^4} \tag{67}$$

$$E_{24} = \frac{bln(k\rho)(12b^2 ln^2(k\rho) - 24b^2 ln(k\rho) + 7b^2)}{4c^5\rho^4} - \frac{b(\rho^2 - \frac{b^2}{c^4 ln^2(k\rho)})}{c\rho^4} \tag{68}$$

$$E_{44} = \frac{12b^2 ln^2(k\rho) - 24b^2 ln(k\rho) + 7b^2}{4c^2\rho^4} + \frac{b^2 ln(k\rho)}{c^2\rho^4} \tag{69}$$

Making the calculation of T^{44} we have:

$$T^{44} = \frac{c^4\rho^2 b^2(12ln^2(k\rho) - 28ln(k\rho) + 7)}{32\pi c^6\rho^6 G} + \frac{b^4(24ln^4(k\rho) - 68ln^3(k\rho) + 62ln^2(k\rho) - 14ln(k\rho))}{32\pi c^6\rho^6 G} - \frac{b^6(28ln^5(k\rho) + 56ln^4(k\rho) + 34ln^3(k\rho) - 7ln^2(k\rho))}{32\pi c^{10}\rho^8 G} \tag{70}$$

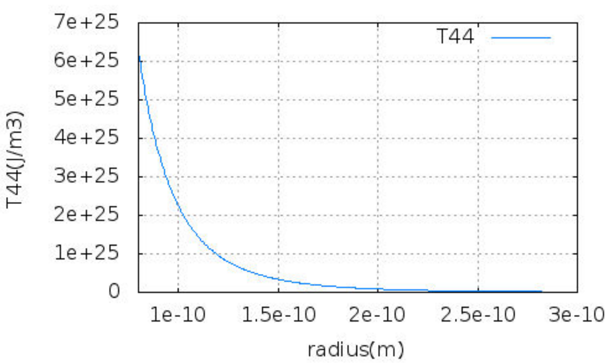

FIGURE 4. T^{44} as a function of radius.

The T^{44} component is interpretable as an energy density. It is plotted in Fig. 4. This component will be related to a certain volume, which will include at least the electron. The energy density, multiplied by this volume –we can suppose the energy density as constant in this volume, given the smallness of this one– will be explainable as the total energy of the nucleus-electron system. This can be made equivalent to the energy derived from the nucleus and electron mass. In this context, the electromagnetic potential energy, the quantum potential energy and the kinetic energy of the electron can be neglected, as a first approximation, and we can write:

$$VT^{44} \simeq (M + m)c^2 \tag{71}$$

The value of the energy of the mass is 1.5×10^{-10} J. The value of T^{44} for $\rho = 10^{-10}$ m is $2,237 \times 10^{25}$ J /m^3. We can, for heuristic purposes, suppose that the volume occupied by the electron and described by T^{44} is shaped like a cylindrical ring around the path, width and height of *h* above it. Its section will be $4h^2$. This volume will be:

$$V = 2\pi\rho 4h^2 = 8\pi\rho h^2 \tag{72}$$

We can match the values and we have:

$$8\pi\rho h^2 T^{44} = (M + m)c^2 \tag{73}$$

and therefore,

$$h = c\sqrt{\frac{M + m}{8\pi\rho T^{44}}} \tag{74}$$

From there we obtain, always for $\rho = 10^{-10}$ m, the width and the height of the ring $2h = 1,0374 \times 10^{-13}$ m. As a reference, let's say that the classic radius of the electron is $2,818 \times 10^{-15}$ m.

This calculation confirms the smallness of the affected width, of the order of 10^{-13} m. Therefore the curvature can be considered constant in the inner of the ring.

A more reasonable approximation by symmetry is that the section of this channel would be circular of radius r (that is to say, that the space considered has a torus form). In this case, a single calculation gives us the value of the diameter of the torus tube: $1,171 \times 10^{-13}$.

In Fig. 5 we represent the radius of the torus based on the radius of the trajectory of the electron.

An alternative view to the kind of spacetime deformation described above could be to consider that it has deformed all the spacetime ring between the radius $\rho_1 = 5 \times 10^{-11}$ and $\rho_2 = 5 \times 10^{-10}$ (interval in which the probability of finding an electron according to quantum mechanics is outstanding) and at a cylinder height of the order of $h_d = 10^{-14}$. If we perform such approach, integrating T^{44} along the radius $\rho_1 = 5 \times 10^{-11}$ and $\rho_2 = 5 \times 10^{-10}$ and at a cylinder height of the order of $h_d = 10^{-14}$, as follows:

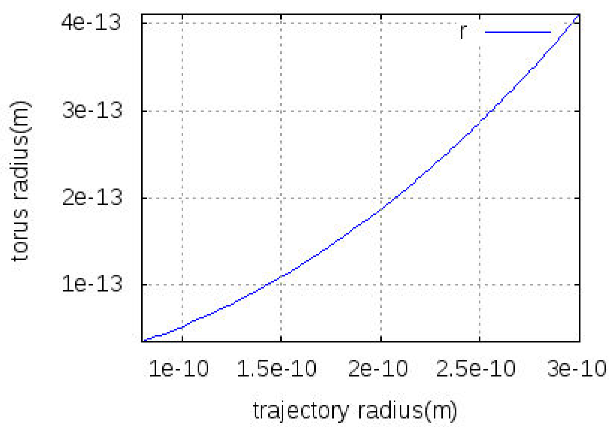

FIGURE 5. Torus radius as a function of the radius of the electron trajectory.

$$\int_{r1}^{r2} 2\pi\rho h_d T^{44}\, d\rho \simeq (M+m)c^2 \tag{75}$$

and we take $k = 10^9$, as we have indicated before; we thus would arrive at a value of 3.2×10^{-8} J, higher than the value of 1.5×10^{-10} J, corresponding to the mass energy of the atomic system. This fact reinforces the assumption of the previous approach, that the shape of the space affected is toroidal around the electron trajectory.

4. Relationship between the wave function and the metric tensor in the hydrogenoid atomic stationary states

We try to go deeper into the relationship between the metric tensor of an atomic hydrogenoid stationary system and the corresponding wave function, expressed with *the same spacial and temporal coordinates*.

We must emphasize again the different mathematical structure regarding the space and time of the de Broglie-Bohm approximation and the General Relativity. While in the de Broglie-Bohm approach we deal with the euclidean E^3 space and time of non local nature, which is evident because of the presence of the quantum potential, in the relativistic approach we deal with a Lorentzial manifold, that is to say, with a curved spacetime of local Minkowskian nature. Sharply said, the wave function “lives” in a pre-relativistic space and time and the metric tensor “lives” in a curved spacetime. Moreover, the wave function concerns the phase space, not the physical spacetime. Although that, as pilot wave, it guides the particle in the physical space and time.

But both models describe the same movement of the particle, at less at differential level, and we can use this to connect both mentioned approaches. So, what we intend to do is to make a coincidence between the tangent space to the manifold in the particle entourage and the space and time of the non relativistic quantum mechanics. That is mathematical coherent due to the very nature of the Lorentzial manifold [14].

To find out this relation, let us start with the de Broglie-Bohm approximation. The pilot wave that governs the movement of the electron, corresponding to the entire system, which we identify as a wave function, is given in polar form like:

$$\Psi(\vec{r}, t) = A e^{\frac{iS}{\hbar}} \tag{76}$$

where A symbolizes the width of the polar form and S is the phase.

In the de Broglie-Bohm approach, the force acting over the electron can be expressed as the gradient of the total potential, that is to say the electrical potential V and the quantum potential [15].

$$m\ddot{\vec{x}} = -\nabla V + \nabla\left(\frac{\hbar^2}{2m}\left(\frac{\nabla^2 A}{A}\right)\right) \tag{77}$$

where the second term of the second member is the quantum potential gradient.

The previous equation is expressed in cartesian coordinates, but must be referred to a general orthogonal reference system in order to express it in components and thus compare it with the geodesic equation.

To express the acceleration in a general orthogonal reference system we will use their corresponding Christoffel symbols $G^{\alpha}_{\beta\gamma}$. The first member of Eq. (77) transforms to:

$$m\left(\frac{d^2x^\alpha}{dt^2} + G^{\alpha}_{\beta\gamma}\partial x^\beta \partial x^\gamma\right) = m\left(\frac{d^2x^\alpha}{dt^2} + w^\alpha\right), \tag{78}$$

where we defined $w^\alpha = G^{\alpha}_{\beta\gamma}\partial x^\beta \partial x^\gamma$ to make the expressions shorter.

Furthermore, concerning the second member: to build the gradient of a function f in general curvilinear, not necessarily orthogonal, coordinates x_α, in a space with metric tensor $g_{\beta\gamma}$ we get the following relationship:

$$(\nabla f)^\alpha = g^{\alpha\beta}\frac{1}{\sqrt{|g^{ii}|}}\partial_\beta f \tag{79}$$

But we will use, as it is usual in such cases, an orthogonal curvilinear reference system, like the cylindrical, the spherical or the cartesian system. Then, Eq. (79) can be simplified by using the so-called scalar functions h_α . These functions can be also derived from the consideration of a scalar vector in E^3 [16];

$$h_\alpha = \partial_\alpha |\vec{r}| \tag{80}$$

And then the gradient of f reads:

$$(\nabla f)_\alpha = h_\alpha^{-1}\partial_\alpha f \tag{81}$$

In cylindrical coordinates we get: h_1=1, h_2=ρ, $h_3 = 1$.

We expand Eq. (77) in terms of the three-dimensional components ($\alpha = 1, 2, 3$), substitute Q by its value and dividing by m; then it reads:

$$\frac{d^2x^\alpha}{dt^2} + w^\alpha + h_\alpha^{-1}\frac{1}{m}\frac{dV}{dx^\alpha} - h_\alpha^{-1}\frac{\hbar^2}{2m^2}\frac{d}{dx^\alpha}\left(\frac{\nabla^2 A}{A}\right) = 0 \tag{82}$$

Now, we will consider the relativistic side. We return to the hypothesis that an electron in an atomic quantum system and in a stationary state, describes a geodesic in the spacetime. The geodesic equation is:

$$\frac{d^2x^j}{ds^2} + \Gamma^j{}_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = 0 \tag{83}$$

If we take the proper time as a parameter, (latin indexes varying between 1 and 4, $x_4 = t$), the previous equation reads:

$$\frac{d^2x^j}{dt^2} + \Gamma^j{}_{ik}\frac{dx^i}{dt}\frac{dx^k}{dt} = 0 \tag{84}$$

From there we separate the equations related to the spatial coordinates ($\alpha, \beta, \gamma = 1, 2, 3$):

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} - 2c\Gamma^\alpha_{\beta 4}\frac{dx^\beta}{dt} + c^2\Gamma^\alpha_{44} = 0 \tag{85}$$

The equation corresponding to $i = 4$ has been excluded from this group, but one must take it into account because it means a relationship that introduces a restriction among the connectors and therefore among the elements of the metric tensor.

As $(d^2x^4/dt^2) = 0$, we get:

$$\Gamma^4_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \tag{86}$$

Here we use the local equivalence of the Euclidean spacetime of the de Broglie-Bohm approach and the spacetime of the Lorentzial manifold, and we make the approximation to identify (d^2x^α/dt^2) in both equations. We proceed to relate Eqs. (82) and (85) and we write:

$$\begin{aligned}\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} - 2c\Gamma^\alpha_{\beta 4}\frac{dx^\beta}{dt} + c^2\Gamma^\alpha_{44} &= h_\alpha^{-1}\frac{1}{m}\frac{dV}{dx^\alpha} \\ &- h_\alpha^{-1}\frac{\hbar^2}{2m^2}\frac{d}{dx^\alpha}\left(\frac{\nabla^2 A}{A}\right) + f^\alpha\end{aligned} \tag{87}$$

Now we must take into account that, in the de Broglie-Bohm approach, the linear momentum of the particle can be expressed as a function of the gradient of the phase S as follows:

$$\vec{p} = \nabla S \tag{88}$$

that in curvilinear orthogonal coordinates can be written, according to Equation (81), as:

$$m\frac{dx^\alpha}{dt} = h_\alpha^{-1}\partial_\alpha S \tag{89}$$

from there we get:

$$\frac{dx^\alpha}{dt} = \frac{1}{mh_\alpha}\partial_\alpha S \tag{90}$$

where we are using the velocity value of the de Broglie-Bohm approach to substitute it in the first member, the 'relativistic side' of the equation. It is a good approximation taking into account the reduced value of the velocity compared with c.

The previous equation allows us to substitute its first term in Eq. (87) and get:

$$\begin{aligned}\frac{1}{m^2h_\beta h_\gamma}\Gamma^\alpha_{\beta\gamma}\partial_\beta S\partial_\gamma S - \frac{2c}{mh_\beta}\Gamma^\alpha_{\beta 4}\partial_\beta S + c^2\Gamma^\alpha_{44} &= \frac{1}{mh_\alpha}\frac{dV}{dx^\alpha} \\ &- \frac{\hbar^2}{2m^2h_\alpha}\frac{d}{dx^\alpha}\left(\frac{\nabla^2 A}{A}\right) + w^\alpha\end{aligned} \tag{91}$$

and in a more convenient and simplified form:

$$\begin{aligned}\frac{1}{mh_\beta h_\gamma}\Gamma^\alpha_{\beta\gamma}\partial_\beta S\partial_\gamma S - \frac{2c}{h_\beta}\Gamma^\alpha_{\beta 4}\partial_\beta S + mc^2\Gamma^\alpha_{44} &= \frac{1}{h_\alpha}\frac{dV}{dx^\alpha} \\ &- \frac{\hbar^2}{2mh_\alpha}\frac{d}{dx^\alpha}\left(\frac{\nabla^2 A}{A}\right) + mw^\alpha.\end{aligned} \tag{92}$$

This expression is relevant to us in order to relate the connectors with the wave function components. But furthermore we directly want to relate the wave function and the metric tensor. For this reason, taking into account Eq. (2), we can write the explicit dependence from the metric tensor by replacing the connectors in Eq. (92)

$$\begin{aligned}&\frac{g^{\alpha h}}{2mh_\beta h_\gamma}(\partial_\gamma g_{\beta h} + \partial_\beta g_{h\gamma} - \partial_h g_{\gamma\beta})\partial_\beta S\partial_\gamma S \\ &\quad - \frac{cg^{\alpha h}}{h_\beta}(\partial_4 g_{\beta h} + \partial_\beta g_{h4} - \partial_h g_{4\beta})\partial_\beta S \\ &\quad + \frac{mc^2}{2}g^{\alpha h}(2\partial_4 g_{4h} - \partial_h g_{44}) \\ &\quad = \frac{1}{h_\alpha}\left(\frac{dV}{dx^\alpha} - \frac{\hbar^2}{2m}\frac{d}{dx^\alpha}\left(\frac{\nabla^2 A}{A}\right)\right) + mw^\alpha\end{aligned} \tag{93}$$

This is the relationship between the components of the metric tensor around a stationary electron integrated into a hydrogenoid system, characterized by a potential V, and whose pilot wave or wave function is given by the components A and S, in polar form, in agreement with the approach of de Broglie-Bohm. This equation represents a kind of bridge between the quantum and the geometro-dynamics descriptions.

A very important feature of that equation is to relate the metric tensor, that is of local character, with a non local entity: the quantum potential, represented by $(\nabla^2 A/A)$. The quantum potential is an element of the Euclidian quantum theory that in our Relativistic approach is incorporated in the geometrodynamics by the metric tensor.

We also note that that equation involves three systems of differential equations (α = 1,2,3) and the condition (86). The wave function has two components and the metric tensor has 10 independent ones. In this way, given a metric tensor in the environment of the electron and with potential V, we could, in principle, reconstruct the wave function that corresponds to it (taking into account integration constants). Nevertheless, from the wave function (and the potential V, although it is already used to define the wave function) there is a lack of definition of the corresponding metric tensor, that must be filled with additional relativistic considerations or hypotheses, consistent with the particular system under study, *i.e.* a metric derived from the Einstein's field equation.

5. Main conclusions

In this article, we have developed the geometrodynamic model applied to a stationary state of the hydrogenoid atomic electron, taking into consideration General Relativity over the particle trajectory defined by the de Broglie-Bohm model, that was initiated in a previous article. We have determined the related Levi-Civita connectors, the contravariant metric tensor, the Ricci tensor and the scalar curvature. We have studied their features and structure and have determined the constant, in such a way that, beyond the experimental value of maximum extension of the corresponding orbital 2p, there will be null curvature (Minkowskian spacetime). A zone beyond the mentioned limit has been evidenced, where the curvature becomes negative and asymptotically null and therefore a zone where the electron is not allowed to be.

We also consider the relationship between the curvature and the quantum potential. Our approach allow us to consider the quantum potential, element of the Euclidean quantum mechanics, incorporated to our Relativistic approach in the geometrodynamics of the system, by the metric tensor.

We have analyzed the relationship of our approach of the quantum potential with another approach of the literature, which has a different geometrical basis (3-D Weyl integrable space).

We have also calculated the energy component of the energy momentum tensor, and we have interpreted its value with regard to the energy content, advancing a hypothesis about its relation with the path of the electron. Given the approximate numerical results, we have advanced the hypothesis that the deformation of spacetime produced by the proton-electron system implies that the volume affected by the energy density consists on a torus that would have the path of the electron in its axis.

It should be noted that the curvature of spacetime is conditioned by the action of electromagnetic interaction between the nucleus and the electron, and by the inertial and kinetic elements of the electron movement. These elements are included in the dynamic equations of the de Broglie-Bohm model that allow the path of the electron to be established (equivalent to the Hamiltonian approach to Standard Quantum Mechanics). The geometrical characterization carried out by us naturally starts from the results of that approach.

We have derived a relationship between the elemental wave function that describes the system in the classical space and time and the metric tensor, that describes it in a Lorentzial manifold. The affirmation that Ψ is a kind of vibration of the empty space takes in our approach a quantitative perspective.

The relationship found is such that, given a metric tensor induced by a trajectory, the wave function that generates it can be calculated. However, the reciprocal relationship is not possible: we can not derive the metric of the spacetime from the wave function if we do not make additional hypotheses. In our case, we have to do hypotheses regarding a cylindrical type metric of dust. In any case, a relation between the guiding character that the wave function has on the particle and the deformation of the spacetime described by the metric is obvious.

There are other approaches that have some common grounds with ours and that are commented in the Appendix.

Our study exceeds the de Broglie-Bohm's theory as our approach establishes a dialectic relationship between particle and wave function, in contrast to the entirely preponderant role that de Broglie-Bohm's theory gives to the wave over the particle, described in the expression 'pilot wave': where the wave determines the movement of the particle, but the particle has no effect on the wave. On the contrary, our approach transfers to our model the interactive nature of Einstein field equations, in which the mass-energy configures the spacetime, as well as spacetime configures the movement of the mass- energy. This interaction should be, in our opinion, the cornerstone of the quantum geometrodynamics.

The global conclusion of it all is that our geometrodinamic approach to the microphysics, coming from the General Relativity and the de Broglie-Bohm interpretation, is physical and mathematicaly coherent and merits further theoretical and even experimental efforts to develop it.

Appendix

A. Some additional considerations

At this point, it is interesting to make some considerations related to other geometrodynamical approaches.

B. Beyond the standard General Relativity

An interesting approach to geometrodinamics in microphysics are based on Weyl geometry. As it is known, the Weyl geometry is a generalization of the Riemann spacetime that focuses in the unification of gravity and electromagnetic field. Two particularly interesting works are the works of A. Shojai and F. Shojai, [15,16] that use the Weyl geometry and Novello, Salim and Falciano, that use the Weyl integrable space. In both cases they go over to conformal transformations in the metrics.

These approaches seem useful mainly for the study of the quantum potential in the frame of geometrodynamics.

However, we use the Levi-Civita connectors that make possible to unify the affin geodesic (parallel transport of the velocity vector) and the metric geodesic (extremal action).

Another possibility is to use approaches like the teleparallelism, in a spacetime without curvature but with torsion. In this approach the gravity is not due to the curvature, but to the torsion of spacetime. This model was used by Einstein at first, in an attempt to unify gravity and electromagnetism and can be useful for this purpose.

C. The geometrothermodynamical approaches

The use of the Riemann manifolds to describe physical systems has been extended to other fields, like thermodynamics. Ruppeiner and Quevedo make interesting approaches to the use of Riemann manifolds for the study of thermodynamics. We would like to comment briefly the adequacy of such approaches.

The difficulties found seemed to be related with the definition of the elements of the Riemann manifold by thermodynamic magnitudes. In particular, it seemed difficult to establish a metric. In General Relativity this function is mainly accomplished by the exact solutions of the Einstein field equation. Also, other issues as the connections should be worked out to build a completely useful structure.

Anyway, if we have a function of some mathematical entities that locally perform in an Euclidean way, we can safely use the Riemannian manifold structure to study such issues.

Needless to say, these approaches seem that would play an important role in the characterization of the thermodynamic systems.

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