Remarks on the (1+1)-Matrix-Branes, qubit theory and non-compact Hopf maps

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We discuss different aspects of a possible link between the (1+1)-matrix-brane system with qubit theory and non-compact Hopf maps. In these scenarios, the (2+2)-signature plays an important role. We argue that such links may shed some light on the (2+2)-dimensional sector of a (2+10)-dimensional target background.

Keywords: P-branes; qubit theory; non-compact Hopf maps; (2+2)-dimensions.

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Our main goal in this work is to establish links between the (1+1)-matrix-brane system [1] (see also Ref. 2) with qubit theory (see Ref. 3 and references therein) and non-compact Hopf maps [4]. In such connections the (2+2)-signature plays a key role. It turns out that through the years the (2+2)-signature has become a very important notion in different scenarios of mathematics [5-6] (see also Refs. 7 and 8) and physics. In fact, the (2+2)-signature emerges in several physical context, including self-dual gravity *a la* Plebanski (see Ref. 9 and references therein), consistent $N = 2$ superstring theory [10-11], $N = (2,1)$ heterotic string [12-15]. Furthermore, it has been emphasized [16-17] that Majorana-Weyl spinors exist in a spacetime of (2+2)-signature. Using the requirement of the $SL(2,R)$-group and Lorentz symmetries it has been proved [18] that (2+2)-target spacetime of a 0-brane is an exceptional signature. Moreover, the (2+2)-signature arises following an alternative idea to the notion of 'worldsheets for worldsheets' [19] or the 0-branes condensation [20]. Another recent motivation for a physical interest in the (2+2)-signature has emerged via the discovery [21] of hidden symmetries of the Nambu-Goto action. In fact, in this case the Nambu-Goto action in a (2+2)-target spacetime can be written in terms of a hyperdeterminant, revealing apparently new hidden symmetries of such an action.

In this work, we shall explore possible links between the (1+1)-matrix-brane system with qubit theory and non-compact Hopf maps. We start by reviewing the relation between Refs. 1 and 21, concerning the (2+2)-signature. For this purpose we shall consider both the (1+1)-matrix-brane theory and the hyperdeterminant structure approach, focusing on the (2+2)-signature.

First, consider the line element

$$ds^2 = dx^\mu dx^\nu \eta_{\mu\nu}. \tag{1}$$

Here, we shall assume that the indices $\mu, \nu \in \{1,2,3,4\}$ and that the flat metric $\eta_{\mu\nu} = diag(1,1,-1,-1)$ determines the (2+2)-signature. Introducing the matrix

$$x^{ab} = \begin{pmatrix} x^1 + x^3 & x^4 + x^2 \\ x^4 - x^2 & x^1 - x^3 \end{pmatrix}, \tag{2}$$

one finds that (1) can be written as

$$ds^2 = dx^{am} dx^{bn} \varepsilon_{ab} \varepsilon_{mn}, \tag{3}$$

where $a, b, m, n \in \{1,2\}$. Moreover, by defining the alternative matrix

$$\zeta^{pq} = \begin{pmatrix} x^1 & x^3 \\ x^4 & x^2 \end{pmatrix}, \tag{4}$$

it is not difficult to show that (1) can also be written as

$$ds^2 = d\zeta^{am} d\zeta^{bn} \eta_{ab} \eta_{mn}. \tag{5}$$

Here, $\eta_{ab} = \text{diag}(1,-1)$ and $\eta_{mn} = \text{diag}(1,-1)$. This proves that the three line elements (1), (3) and (5) are equivalents. Thus, one can say that these equivalences provide an interesting connection between the signatures (1+1) and (2+2).

It turns out that such equivalences at the level of the line elements (1), (3) and (5) can be transferred to the matrix

$$h_{ab} = \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \eta_{\mu\nu}. \tag{6}$$

In fact, (6) can be written in the following two equivalent forms

$$h_{ab} = \frac{\partial x^{cm}}{\partial \xi^a} \frac{\partial x^{dn}}{\partial \xi^b} \varepsilon_{cd} \varepsilon_{mn} \tag{7}$$

and

$$h_{ab} = \frac{\partial \zeta^{cm}}{\partial \xi^a} \frac{\partial \zeta^{dn}}{\partial \xi^b} \eta_{cd} \eta_{mn}. \tag{8}$$

So, by introducing the quantity

$$v_a{}^{ij} = \frac{\partial x^{ij}}{\partial \xi^a}, \tag{9}$$

it is straightforward to verify that

$$\det(h_{ab}) = \frac{1}{2!} \varepsilon^{ac} \varepsilon^{bd} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \frac{\partial x^\alpha}{\partial \xi^c} \frac{\partial x^\beta}{\partial \xi^d} \eta_{\mu\nu} \eta_{\alpha\beta} \tag{10}$$

can be written as

$$Det(h_{ab}) \equiv \frac{1}{2!}\varepsilon^{ac}\varepsilon^{bd}\varepsilon_{eg}\varepsilon_{fh}\varepsilon_{ik}\varepsilon_{jl}v_a{}^{ef}v_b{}^{gh}v_c{}^{ij}v_d{}^{kl}. \quad (11)$$

Here, the notation $\mathrm{Det}(h_{ab})$ means hyperdeterminant. Thus, one finds that the Nambu-Goto action

$$S = \int d\xi^{(1+1)}\sqrt{-\det(h_{ab})} \quad (12)$$

can also be written as

$$\mathbf{S} = \int d\xi^{(1+1)}\sqrt{-\mathrm{Det}(h_{ab})}. \quad (13)$$

Actually, one has

$$\det(h_{ab}) = Det(h_{ab}). \quad (14)$$

Similarly, introducing the quantity

$$u_a{}^{ij} = \frac{\partial \zeta^{ij}}{\partial \xi^a}, \quad (15)$$

it is also straightforward to see that $\det(h_{ab})$, given in (10), can also be written as

$$\mathcal{D}et(h_{ab}) \equiv \frac{1}{2!}\varepsilon^{ac}\varepsilon^{bd}\eta_{eg}\eta_{fh}\eta_{ik}\eta_{jl}u_a{}^{ef}u_b{}^{gh}u_c{}^{ij}u_d{}^{kl}. \quad (16)$$

This means that the Nambu-Goto action (12) is also equivalent to

$$\mathcal{S} = \int d\xi^{(1+1)}\sqrt{-\mathcal{D}et(h_{ab})}. \quad (17)$$

This shows that in (2+2)-dimensions $S, \mathbf{S}$ and $\mathcal{S}$ are equivalent actions. The interesting thing is that $\mathbf{S}$ reveals new hidden symmetries in the original Nambu-Goto action S [21]. Presumably, the same conclusion can be said in the case of the action $\mathcal{S}$. (Details of the connections between the actions $S, \mathbf{S}$ and $\mathcal{S}$ can be found in Ref. 1.)

In order to related the previous discussion with qubit theory and Hopf maps it is convenient to introduce the mathematical notion of 2 by 2 real matrices $M(2,R)$. It turns out that through the years the importance of $M(2,R)$) has emerged in different scenarios of physics and mathematics, including Clifford algebras [22-23], matroid theory [24-25] (see also Refs. 26 to 32 and references therein), string theory [33], $2d$ gravity [34], $2t$ physics [35], qubit theory (see Refs. 3 and references therein) among others. We argue that these connections may suggest that one may even consider the set $M(2,R)$ as one of the underlaying structures of supersymmetry and M-theory [36]. This last observation is due in part to the fact that $M(2,R)$ is linked to a 2-rank self-dual oriented matroid and to the fact that in both oriented matroid theory and M-theory the duality concept plays a fundamental role. Indeed, it has been proposed [27] that oriented matroid theory may be considered as the underlying mathematical framework for M-theory.

Let us briefly recall some aspects of $M(2,R)$. It is not difficult to see that any matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $M(2,R)$ (with $\mathrm{diag}(a,d)$ and $\mathrm{antidiag}(b,c)$) can be written as

$$M_{ij} = x\delta_{ij} + y\varepsilon_{ij} + r\eta_{ij} + s\lambda_{ij}. \quad (18)$$

Here, $\delta_{ij} = \mathrm{diag}(1,1)$, $\varepsilon_{ij} = \mathrm{antidiag}(1,-1)$, $\eta_{ij} = \mathrm{diag}(1,-1)$ and $\lambda_{ij} = \mathrm{antidiag}(1,1)$ are fundamental 2 by 2 matrices and the quantities x, y, r and s are related to the real quantities a, b, c and d by

$$x = \frac{1}{2}(a+d), \quad y = \frac{1}{2}(b-c),$$
$$r = \frac{1}{2}(a-d), \quad s = \frac{1}{2}(b+c). \quad (19)$$

Our first observation is that a complex number

$$z = x + iy, \quad (20)$$

where x and y are real numbers and $i^2 = -1$, can also be written as [37-38]

$$z_{ij} = x\delta_{ij} + y\varepsilon_{ij}. \quad (21)$$

In this case, the product of two complex numbers corresponds to the usual matrix dot product. According to (18), this is equivalent to set $r = 0$ and $s = 0$. So, from these simply observations one may conclude that the complex structure is contained in $M(2,R)$. If instead, one sets $y = 0$ and $r = 0$ (or $s = 0$) in (18) one arrives to the so called split numbers [39] (or semicomplex numbers (among other alternative names)). These kind of numbers shall play and important role below. But before, we use split algebra let just mention the following. Traditionally, one can not set $x = 0$ and $y = 0$ because the dot product of η_{ik} and λ_{lj} is not closed. In fact, one has

$$\eta_{ik}\delta^{kl}\lambda_{lj} = \varepsilon_{ij}, \quad (22)$$

where the quantity δ^{kl} plays the role of the dot product. However, one may introduce a new kind of product (and therefore new kind of numbers which we shall call "niet" (from dutch word meaning "no")) complex if instead of δ^{kl} one uses η^{kl} in such a way that the product combination of η_{ik} and λ_{lj} is again closed. In fact, in this case one has

$$\eta_{ik}\eta^{kl}\lambda_{lj} = \lambda_{ij}. \quad (23)$$

Denoting the matrices product with η^{kl} as a star $\star$ one sees that (23) becomes

$$\eta \star \lambda = \lambda. \quad (24)$$

Thus, one can show that all possible combinations of δ and ε with the dot product are equivalent to all possible combinations of η and λ with the star product. Therefore, through the prescription

$$\delta \leftrightarrow \eta, \qquad \varepsilon \leftrightarrow \lambda, \qquad \cdot \leftrightarrow \star. \quad (25)$$

one discovers that the niet-complex algebra is isomorphic to the complex structure (see Ref. [40] for more details). On the other hand the split number differ from the complex numbers in a number of facts. First while in the complex numbers $\varepsilon^2 = -1$ in the split numbers $\lambda^2 = 1$. Furthermore, the fact that in the complex numbers one has $zz^* = x^2 + y^2$, in the case of split numbers one has $ww^* = x^2 - y^2$, where $z^* = x - iy$ and $w^* = x - jy$, with $j = \lambda$. Of course, according to the Hurwitz theorem the split number structure does not form a division algebra. One can see this by assuming $y = x$ and noting that in this case $ww^* = 0$. So, split numbers with $y = x$ does not have inverse.

It is worth mentioning the following observations. It is known that the fundamental matrices given in (18) not only form a basis for $M(2, R)$ but also determine a basis for the Clifford algebras $C(2, 0)$ and $C(1, 1)$. In fact one has the isomorphisms $M(2, R) \sim C(2, 0) \sim C(1, 1)$. Moreover, one can show that $C(0, 2)$ can be constructed using the fundamental matrices in (18) and Kronecker products. It turns out that $C(0, 2)$ is isomorphic to the quaternionic algebra H. Thus, it is proved that all the other $C(a, b)$'s can be constructed from the basic building blocks $C(2, 0)$, $C(1, 1)$ and $C(0, 2)$ (see Ref. 41 and references therein).

Let us now briefly describe the connection of the coordinates x^{ij} and ζ^{am} in $M(2, R)$ with qubit theory. Let us first introduce the basis

$$\mid j_1 j_2 ... j_n\rangle = \mid j_1\rangle \otimes \mid j_2\rangle \otimes ... \otimes \mid j_n\rangle. \tag{26}$$

A general qubit can be written as

$$|\Psi\rangle = \sum_{j_1, j_2, ..., j_n = 0}^{1} \psi^{j_1 j_2 ... j_n} \mid j_1 j_2 ... j_n\rangle. \tag{27}$$

For instance, a 3-qubit is expressed by

$$\Psi = \sum_{j_1, j_2, j_3 = 0}^{1} \psi^{j_1 j_2 \cdot j_3} \mid j_1 j_2 j_3\rangle. \tag{28}$$

The central idea is to identify x^{ij} and ζ^{am} with 2-rebits which are the real version of the corresponding 2-qubits $\psi^{j_1 j_2}$ (see Ref. 3 and references therein).

Let us now come back to consider again the three Nambu-Goto type actions S, $\mathbf{S}$ and $\mathcal{S}$, given in (12), (13) and (17), respectively. First of all, the idea is to relate these actions with the Polyakov action

$$S = \frac{1}{2} \int d\xi^{(1+1)} \sqrt{-g} \left[g^{ab} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \eta_{\mu\nu} \right]. \tag{29}$$

It is well known that this action is equivalent to the Nambu-Goto action (12) (and therefore to the other two actions $\mathbf{S}$ and $\mathcal{S}$). Let us recall how this is achieved. Making variations of (29) with respect to g_{ab} one obtains the expression

$$\frac{\partial x^\nu}{\partial \xi^a} \frac{\partial x^\sigma}{\partial \xi^b} \eta_{\nu\sigma} - \frac{1}{2} g_{ab} \left(g^{cd} \frac{\partial x^\mu}{\partial \xi^c} \frac{\partial x^\nu}{\partial \xi^d} \eta_{\mu\nu} \right) = 0, \tag{30}$$

which can be used to substitute g_{ab} in (29) and in that way one obtains the Nambu-Goto action (12). One sees that in order to related (29) with the hyperdeterminant is enough to consider (7). In fact, in this case (29) becomes

$$S = \frac{1}{2} \int d\xi^{(1+1)} \sqrt{-g} \left[g^{ab} \frac{\partial x^{cm}}{\partial \xi^a} \frac{\partial x^{dn}}{\partial \xi^b} \varepsilon_{cd} \varepsilon_{mn} \right], \tag{31}$$

which leads to (13). Of course, this is only true in 2+2-dimensions. Similarly, by writing (29) as

$$S = \frac{1}{2} \int d\xi^{(1+1)} \sqrt{-g} \left[g^{ab} \frac{\partial \zeta^{am}}{\partial \xi^a} \frac{\partial \zeta^{bn}}{\partial \xi^b} \eta_{ab} \eta_{mn} \cdot \right], \tag{32}$$

where (8) was used, one obtains (17), after variations of g_{ab}.

From the above observations one is tempting to raise the question: what could be the role of the matrices $M(2, R)$ in the structure of M-theory? One knows that the duality concept is an essential aspect in M-theory. Similarly, duality is a central notion in oriented matroid theory. This is one of the reasons that oriented matroid theory has been proposed as the underlying mathematical structure of M-theory [27]. In this scenario one observe that $M(2, R)$ describes a self-dual graphic oriented matroid and therefore is in agreement with both M(atroid) theory and M-theory. So, an audacious proposal could be that $M(2, R)$ may be one of the essential building blocks of M-theory. This proposal is reinforce by the fact that $M(2, R)$ is related to qubit theory via (2+2)-dimensions and to supersymmetry via the Clifford algebra.

By further research and in order to related the previous discussion with the qubit theory we shall consider the $2 + 10$-dimensional spacetime. This signature has emerged as one of the most interesting possibilities for the understanding of both supergravity and super Yang-Mills theory in $D = 11$. What it is important for us is that the (2+10)-dimensional theory seems to be the natural background for the (2+2)-brane (see Refs. 1, 10, 11 and references therein). Thus, let us think in the possible transition

$$M^{(2+10)} \rightarrow M^{(2+2)} \times M^{(0+8)}, \tag{33}$$

which, in principle, can be achieved by some kind of symmetry breaking applied to the full metric of the spacetime manifold $M^{(2+10)}$. It has been shown that the symmetry $SL(2, R)$ makes the (2+2)-signature an exceptional one [18]. On the other hand, the (0+8)-signature is Euclidean and in principle can be treated with the traditional methods such as the octonion algebraic approach. In pass, it is interesting to observe that octonion algebra is also exceptional in the sense of the celebrated Hurwitz theorem. Thus, one can say that both (2+2) and (0+8) are exceptional signatures. This means that the transition (33) is physically interesting.

Consider now the action of the (1+1)-matrix-brane in (2+10)-dimensional target spacetime background [1],

$$S = \frac{1}{2} \int d\xi^{(2+2)} \sqrt{-g} \sqrt{-\gamma} \times \left[g^{ab} \gamma^{mn} \frac{\partial x^{\hat{\nu}}}{\partial \xi^{am}} \frac{\partial x^{\hat{\sigma}}}{\partial \xi^{an}} \eta_{\hat{\nu}\hat{\sigma}} - 2 \right], \tag{34}$$

where $\eta_{\hat{\nu}\hat{\sigma}}$ is a flat metric and the indices $\hat{\nu}, \hat{\sigma}$ now run from 1 to 12. Splitting the flat metric $\eta_{\hat{\nu}\hat{\sigma}}$ according to the transition (2+10)→ (2+2)+(0+8) one finds that (34) can be written as

$$S = S_1 + S_2, \tag{35}$$

where

$$S_1 = \frac{1}{2}\int d\xi^{(2+2)}\sqrt{-g}\sqrt{-\gamma} \times \left[g^{ab}\gamma^{mn}\frac{\partial x^A}{\partial\xi^{am}}\frac{\partial x^B}{\partial\xi^{bn}}\eta_{AB}\right] \tag{36}$$

and

$$S_2 = \int d\xi^{(2+2)}\sqrt{-g}\sqrt{-\gamma} \times \left[g^{ab}\gamma^{mn}\frac{\partial x^{\hat{A}}}{\partial\xi^{am}}\frac{\partial x^{\hat{B}}}{\partial\xi^{bn}}\eta_{\hat{A}\hat{B}} - 2\right]. \tag{37}$$

Here, the indices A, B run from 1 to 4 and $\hat{A}, \hat{B}$ run from 5 to 12. Using the change $x^A \to x^{pq}$ one can write S_1 in the form

$$\mathbf{S}_1 = \frac{1}{2}\int d\xi^{(2+2)}\sqrt{-g}\sqrt{-\gamma} \times \left[g^{ab}\gamma^{mn}\frac{\partial x^{cd}}{\partial\xi^{am}}\frac{\partial x^{kl}}{\partial\xi^{bn}}\varepsilon_{ck}\varepsilon_{dl}\right], \tag{38}$$

while if one uses the change $x^A \to \zeta^{pq}$ one has

$$\mathcal{S}_1 = \frac{1}{2}\int d\xi^{(2+2)}\sqrt{-g}\sqrt{-\gamma} \times \left[g^{ab}\gamma^{mn}\frac{\partial\zeta^{ij}}{\partial\xi^{am}}\frac{\partial\zeta^{kl}}{\partial\xi^{bn}}\eta_{ik}\eta_{jl}\right]. \tag{39}$$

Now, both metrics g^{ab} and γ^{mn} 'live' in (1+1)-dimensions. So, according to our previous discussion (38) and (39) can be expressed in terms of the fundamental matrices $\delta_{ij}, \varepsilon_{ij}, \lambda_{ij}$ and η_{ij} which are elements of the basis of $M(2, R)$. This means that we have proved that (2+2)-dimensional sector of $M^{(2+10)}$ can be connected with qubit theory via the elementary basis matrices $\delta_{ij}, \varepsilon_{ij}, \lambda_{ij}$ and η_{ij}.

It is interesting to observe that both actions $\mathbf{S}_1$ and $\mathcal{S}_1$, given in (38) and (39) respectively, are double Weyl invariant in the sense that are invariants with respect to the transformations $g^{ab} \to e^f g^{ab}$ and $\gamma^{mn} \to e^h\gamma^{mn}$, for arbitrary functions f and h. This is quite interesting because as it is known the Weyl invariance of the Polyakov action is linked to the critical dimensions of the target spacetime determined by the metric η_{AB}. If one adds to this observation the fact that the flat target metric η_{AB} in the action (36) is written in terms of either $\varepsilon_{ck}\varepsilon_{dl}$ or $\eta_{ik}\eta_{jl}$ (according to (38) and (39), respectively), which are the qubit inspired metrics, one is tempted to conjecture a link between the critical dimensions and qubit theory in the (1+1)-matrix-brane theory.

Let us now discuss the (1+1)-matrix-brane theory from the perspective of split algebra. First, observe that if one has two split numbers $d\omega^1 = dx^1 + jx^3$ and $d\omega^2 = dx^2 + jdx^4$ (remember; $j = \lambda$ with $j^2 = 1$) then one gets the invariant

$$\begin{aligned} ds^2 &= d\omega^1 d\omega^{1*} + d\omega^2 d\omega^{2*} \\ &= dx^1dx^1 + dx^2dx^2 - dx^3dx^3 - dx^4dx^4, \end{aligned} \tag{40}$$

which determines, in a natural way, a $(2+2)$-signature. This means that the Nambu-Goto action in (2+2)-dimensions can also be written as

$$S = \frac{1}{2}\int d\xi^{(1+1)}\sqrt{-g}\left[g^{ab}\frac{\partial\omega^m}{\partial\xi^a}\frac{\partial\omega^{*n}}{\partial\xi^b}\delta_{mn}\cdot\right], \tag{41}$$

or

$$S{=}\frac{1}{2}\int d\xi^{(1+1)}\sqrt{-g}\left[g^{ab}\frac{\partial\omega^{cmr}}{\partial\xi^a}\frac{\partial\omega^{*dns}}{\partial\xi^b}\delta_{cd}\delta_{mn}\cdot\right], \tag{42}$$

where we wrote ω in terms of δ_{ij} and λ_{ij}. Similarly, in the case of (1+1)-matrix-brane system one must have

$$S_1 = \frac{1}{2}\int d\xi^{(2+2)}\sqrt{-g}\sqrt{-\gamma} \times \left[g^{ab}\gamma^{mn}\frac{\partial\omega^{cir}}{\partial\xi^{am}}\frac{\partial\omega^{*djs}}{\partial\xi^{bn}}\delta_{cd}\delta_{ij}\right]. \tag{43}$$

One of the reason to become interested in the structure (43) is because recently Hasebe [4] has introduced the mathematical concept of non-compact Hopf maps. In fact, in analogy to the Hopf maps (which play a key role in the paralellizabilty of spheres and division algebras [42]) $S^3 \xrightarrow{S^1} S^2$, $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$ and using the split algebra Hasebe introduced the non-compact Hopf maps $H^{2,1} \xrightarrow{H^{1,0}} H^{1,1}$, $H^{4,3} \xrightarrow{H^{2,1}} H^{2,2}$ and $H^{8,7} \xrightarrow{H^{4,3}} H^{4,4}$. Here, $H^{p,q}$ denotes higher dimensional hyperboloids

$$x^A x_A - x^{\hat{A}} x_{\hat{A}} = -1, \tag{44}$$

where $A = 1, ..., p$ and $\hat{A} = 1, ..., q+1$. Indeed, in terms of the signature the non-compact Hopf maps may be also written as $(2+2) \xrightarrow{(1+1)} (1+2)$, $(4+4) \xrightarrow{(2+2)} (2+3)$ and $(8+8) \xrightarrow{(4+4)} (4,5)$. So, the (2+2)-signature and the split algebra play a key role in these developments. Moreover, one may expect this approach to be useful in the context of M-theory since it has been shown [43] that versions of M-theory lead to type IIA string theories in spacetime of signatures (0+10), (1+9), (2+8), (6+4) and (5+5), and to type IIB string theories of signatures (1+9), (3+7) and (5 +5). It turns out that these theories are linked by duality transformations. One notices that the (5+5)-signature is common to both type IIA strings and type IIB strings. So, one wonders whether Hasebe formalism and matrix-brane theory may also be related to the (5+5)-signature.

It is worth remarking that the split quaternions can also be related to the (2+2)-signature in a natural way. In fact, one may reveal split quaternionic structure in the action (43) by writing $dp = d\omega^1 + id\omega^2$ and properly using the algebra between i, j and $k = ij$, with $k^2 = 1$ (see the algebra (4) in Ref. 44). Considering such an algebra it is not difficult to show that dp can also be written as

$$dp = dz^1 + dz^2 j = dx^1 + ix^2 + jdx^3 + kdx^4, \qquad (45)$$

where $dz^1 = dx^1 + ix^2$ and $dz^2 = dx^3 + idx^4$ and therefore one gets $dpdp^* = dz^1 dz^{1*} - dz^2 dz^{2*}$. Thus, one finds that in this case the action (43) becomes

$$S_1 = \frac{1}{2} \int d\xi^{(2+2)} \sqrt{-g}\sqrt{-\gamma} \left[g^{ab}\gamma^{mn} \frac{\partial p}{\partial \xi^{am}} \frac{\partial p^*}{\partial \xi^{bn}} \right]. \quad (46)$$

It turns out that one of the advantage of the formulation (46) is that may shed light on a possible route to supersymmetrize the (1+1)-matrix-brane theory via the proposed 2-spinors over the split quaternions structure [44]. This is particularly interesting because there exist already a formulation of the Dirac equation in terms of the split-quaternions. Moreover, the usual Dirac 4-spinor is replaced by a 2-spinor with split quaternionic components. In this framework, the SO(3, 2;R) symmetry of the Lorentz invariant scalar $\bar{\psi}\psi$ is manifest and therefore there exist a finite unitary representations of the Lorentz group over the split-quaternions (see Ref. [44] for details).

Finally, since part of the motivation of considering non-compact Hopf maps it emerges from the concept of fuzzy spheres [45] it may be interesting for further research to relate the (1+1)-matrix-brane with fuzzy geometry.

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