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A conjecture for the algorithmic decomposition of paths over an $SU(3)$ ADE graph

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Through a geometric understanding of the creation, cap, annihilation and cup operators for ADE graphs in $SU(3)$ we propose the first steps towards an algorithm that would allow one to write an arbitrary elementary path as an ordered combination of creation and cap operators acting upon an essential path. We propose a sketch of a proof and use our proposal for some examples for the A_2 and E_5 graphs of the $SU(3)$ family. Attaining this decomposition is an important step in obtaining the path formulation of the quantum Algebra of a modular invariant RCFT.

Keywords: Rational conformal field theory; ADE classification; essential paths; $SU(3)$ Temperley-Lieb algebra; Ocenanu cells; quantum groups; graph theory; integrable systems.

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1. Introduction

Over the years Mathematical Physics has proven to be the driving force of important and exciting new ideas in fundamental science, both in physics and mathematics. Lie groups and Lie algebras in particular appear as some of the most successful abstract subjects that while originally studied due to the intrinsic interest of its structure, have become a fundamental piece of modern physics. Here we address a problem in a topic which generalizes the theory of semi-simple Lie algebras in a way that is still to be fully understood, but that, in contrast to semi-simple Lie algebras, originally arises from a purely physical problem: the algebraic foundation of the ADE classification of WZW Conformal Field Theories (CFT) modular invariant partition functions [1].

Today this subject has developed into a topic of independent study in physics and mathematics: Fusion and Module Categories. The efforts of both communities have established connections between a huge list of topics which in principle might seem disconnected: statistical mechanics, string theory, quantum gravity, conformal field theory, re-normalization in quantum field theory, theory of bimodules, Von Neumann algebras, sector theory, (weak) Hopf algebras, modular categories (see [2-4] and references therein.) These manifold connections suggest that the study of Quantum Groups and their associated Hopf algebras is at least part of a new, general and fundamental theory underlying all these fields of mathematics and physics. Additionally this algebraic classification turns out to be intimately connected with graphical methods such as braids or spider webs, so relevant today for their power to represent highly complex models.

Here, combining ideas in [4,5] we address some important problems in this topic in a way as geometric and as intuitive as possible. Essential paths over a generalized $SU(N)$ ADE graph G are used as the fundamental algebraic-geometric objects connecting the $SU(N)$ ADE families of graphs to the $\hat{su}(N)$ WZW Conformal Field Theories (CFT) modular invariants [6,8]. Defining an elementary path on a graph as a sequence of connected vertices it is possible to transform this purely geometric object into a algebraic one through the definition of the vector space of paths over the graph $\mathcal{P}$. This is a vector space over the complex numbers that is spanned by the elementary paths. A subspace of interest is the one constituted by the so called *essential paths* $\mathcal{E}$ [6] which can be roughly defined as a paths that lack backtracking segments [6,9,5].

The connection with WZW theories is obtained by constructing the bialgebra of endomorphisms of essential paths $\mathcal{B}$ and its dual $\hat{\mathcal{B}}$. These pair of bialgebras are both semisimple and cosemisimple and they have two separate sets of characters associated to the corresponding simple blocks. The character algebras, giving the generalized decomposition of tensor product representations as direct sum of irreps, are called the Fusion Algebra $A(G)$ and the Ocneanu Algebra of G $Oc(G)$. In this way characters of the simple blocks m of $\mathcal{B}$ are elements of $\hat{\mathcal{B}}$ and characters of the simple blocks x of $\hat{\mathcal{B}}$ are elements of $\mathcal{B}$.

Given this structure, the simple blocks of $A(G)$, m, n have a double action that is compatible over the simple blocks of $Oc(G)$, x and y, ($m\,x\,n = \sum_y (W_{xy})_{m\,n} y$). The structure constants of this bimodule define the set of Toric Matrices W_{xy}. The matrix W_{00} is modular invariant and gives the partition function of the corresponding RCFT with chiral algebra $\hat{sl}(N)$ in terms of the Virasoro characters, and the remaining matrices $W_{x,y}$ define twisted partition functions with one $(x,0)$, $(0,y)$ or two (x,y) topological defect lines, labeled by the indices x, y [10].

In practice, the products on $\mathcal{B}$ and its dual are naturally defined through a pair of basis which are not dual to each other. We call these basis the "vertical" and "horizontal" basis [11,12]. As a consequence of this, the coproduct in $\mathcal{B}$ is naturally defined in a basis that is not the same as the one that gives the straightforward definition of the product on $\mathcal{B}$. This in turn represents a computational complication that should not be underestimated, even more so as this problem is repeated for the coproduct in the dual space. The change of basis between the vertical and horizontal basis of the bialgebra is given by the set of Ocneanu cells (a kind of generalized 6j-symbols) [11]. The explicit calculation of these cells can be, computationally speaking, extremely demanding, primarily because dimensionality increases rapidly once one gets past the first simplest cases. Only for a few examples [13,14] the complete Hopf algebra has been computed.

In Ref. 9 it was shown that for any $SU(2)$ ADE graph, the quantum grupoid can be obtained directly from the properties of the essential paths subspace without having to calculate Ocneanu cells. The key ingredient is the decomposition of the space of paths as a direct sum of sub-spaces which are: either the subspace of essential paths of length n, or orthogonal subspaces constructed by recurrent applications of the corresponding creation operator $C^\dagger$ on subspaces of essential paths of shorter length. This decomposition and the corresponding orthogonal projectors, are sufficient to compute the quantum grupoid.

In this work we propose an algorithm that, using the definition of essential paths for $SU(3)$ graphs proposed in Refs. 5 and 15, allows one to obtain a similar decomposition for a given path over an (oriented) ADE $SU(3)$ graph. This is an important step in order to achieve the full construction of the bialgebra without the need to calculate Ocneanu cells in $SU(3)$, which is a much more challenging task than in the case of $SU(2)$.

2. The space of paths over an $SU(3)$ graph

We first require the definitions of elementary and essential paths:

Definition 1 *An elementary path is a sequence of vertices $\eta = v_0v_1\ldots v_{i-1}v_iv_{i+1}\ldots v_n$ connected by arrows which may belong to either one graph or its conjugate. It is also possible to define an elementary path as a series of consecutive edges that can go with or against the orientation of the edges of the graph.*

Paths form an inner product vector space $\mathcal{P}$ of which the elementary paths are an orthonormal basis. This space is naturally graded by the length of a path.

For $SU(3)$ ADE graphs, vertices are connected by oriented edges. This implies that, for each ADE $SU(3)$ graph, one can find a conjugate graph by changing the orientation of all edges (cf. *e.g.* Fig. 1). This is due to the existence of the conjugate fundamental representations 1 and $\overline{1}$ which generate oriented edges through the generators σ and $\overline{\sigma}$. This introduces some ambiguity when measuring the length of a path, as it is necessary to explicitly convey the number of edges in a path that go with or against the direction of the arrows.

Definition 2 *The length of a path is $n = (\alpha + \beta)$, where (α, β) are integers that give the number of edges generated by σ and $\overline{\sigma}$ respectively, this is, the number of edges in the path going with or against the orientation of the edges of the graph. As usual, this is equivalent to the total number of vertices minus 1. This means that paths of the same length can have differing values for α and β and thus be obtained by different combinations of the generators.*

2.1. Path creation and annihilation operators

We now introduce the creation, annihilation, cup and cap operators:

Definition 3 *Given a path $\eta = v_0v_1\ldots v_{i-1}v_iv_{i+1}\ldots v_n$*

$$C_i(\eta) = \frac{T_{i-1,i,i+1}}{\sqrt{[i-1][i+1]}}v_0v_1\ldots v_{i-1}v_{i+1}\ldots v_n, \tag{1}$$

$$C_i^\dagger(\eta) = \sum_{b\, n.n.\, v_i} \frac{\overline{T_{i-1,b,i}}}{\sqrt{[i-1][i]}}v_0v_1\ldots v_{i-1}v_bv_i\ldots v_n. \tag{2}$$

$$\cup_i(\eta) = \frac{T_{i-1,i,i+1}}{\sqrt{[i-1][i+1]}} \times \delta_{i-1,i+1}\; v_0v_1\ldots v_{i-1}v_{i+2}\ldots v_n, \tag{3}$$

$$\cap_i(\eta) = \sum_{v_b\, n.n.\, v_i} \frac{\overline{T_{i-1,b,i-1}}}{\sqrt{[i-1][i-1]}} \times v_0v_1\ldots v_{i-1}v_bv_{i-1}v_i\ldots v_n. \tag{4}$$

Here the prefactors $T_{i-1,i,i+1}/\sqrt{[i-1][i+1]}$ ensure that these operators satisfy the Temperley-Lieb algebra. The coefficients $T_{i-1,i,i+1}$ are elementary triangular cells and have been calculated in Refs. 2 and 4.

A special subspace of $\mathcal{P}$ is that spanned by the essential paths. Here one such space of essential paths of length n connecting vertices a and b will be noted by $\mathcal{E}_{ab}^{(n)}$.

Definition 4 *A path η is essential if:*

$$C_i(\eta) = 0, \quad \cup_i(\eta) = 0, \qquad \textit{both for all } i < n. \tag{5}$$

and its length is given by a vertex of the A-type graph that shares the same generalized Coxeter number of the original graph G.

3. Decomposition of elementary paths: a conjecture

In a previous work [5], it was shown that the space of paths of length n, connecting vertices a and b of a simply connected $SU(3)$ ADE graph can be decomposed thusly:

$$\mathcal{P}_{ab}^{(n)} = \mathcal{E}_{ab}^{(n)} \bigoplus_{i\leq n-2} (C^\dagger + \cap)_i \left(\mathcal{E}_{ab}^{(n-1)}\right) \bigoplus_{i_1<i_2\leq n-2} (C^\dagger + \cap)_{i_2}(C^\dagger + \cap)_{i_1} \left(\mathcal{E}_{ab}^{(n-2)}\right) \bigoplus \cdots$$
$$\cdots \bigoplus_{i_1<i_2\cdots<i_{[n/2]}\leq n-2} (C^\dagger + \cap)_{i_{[n/2]}}(C^\dagger + \cap)_{i_{[n/2]-1}} \ldots (C^\dagger + \cap)_{i_1} \left(\mathcal{E}_{ab}^{(1|0)}\right). \tag{6}$$

What the above decomposition does not provide, is an explicit way of writing a given path in terms of a linear combination of elementary paths of equal length or of shorter length on which one has acted with a specific ordering of creation or cap operators. It is with this in mind that we propose an algorithm that takes a path and explicitly deconstructs it in elements of the subspaces described above. A similar algorithm has already been fully explored for $SU(2)$ in Ref. 9 Eq. 5.7. A geometric description of Trinchero's algorithm may be enlightening: the nonessential path η must be such that beyond the i-th vertex it behaves like an essential path up until the last two vertices (*i.e.* there is no backtracking). The algorithm then takes the path and eliminates the backtracking "kink" in the i-th vertex and creates kinks further down the way of the path multiplied by some coefficients, thus "running" the kinks throughout the remaining length of the path. The task then of those coefficients is to ensure the cancellation of the spurious paths created by the algorithm in such a way that only two paths remain: the original path η and a path with a backtracking "kink" in the $i-1$-th vertex that is to be eliminated with an extra path $\xi^{(i)}$ (which is precisely a path with a "kink" in the $i-1$-th vertex and placed "by hand" to ensure the cancellation.) Our proposal is specifically for $SU(3)$, the difference hinges upon the fact that graphs in $SU(3)$ are oriented and belong in the $SU(3)$ weight lattice.

Conjecture: *Let η be a path of length n, not necessarily elementary, such that $C_i(\eta) \neq 0$ and $\cup_i(\eta) \neq 0$ for some i and $C_j(\eta) = 0$ and $\cup_j(\eta) = 0$ $\forall j$ such that $i < j < n-2$ and $\xi^{(i)}$ a path satisfying $(C|\cup)_j(\xi^{(i)}) = 0$ $\forall j$ such that $i \leq j < n-2$. One can decompose these paths using:*

$$\eta = \sum_{k=i}^{n-2} \alpha_k ((C^\dagger|\cap)_k (C|\cup)_k (C^\dagger|\cap)_{k-1} (C|\cup)_{k-1} \ldots$$
$$(C^\dagger|\cap)_{i+1}(C|\cup)_{i+1}(C^\dagger|\cap)_i(C|\cup)_i)(\eta) + \xi^{(i)}, \tag{7}$$
$$\eta = \sum_{k=i}^{n-2} \alpha_k \prod_{m=i}^{k} ((C^\dagger|\cap)_m (C|\cup)_m(\eta)) + \xi^{(i)}. \tag{8}$$

Where the α_k are constants given by the linear combination of elementary paths that define the path η. There are some differences with the algorithm for $SU(2)$, however we can see that this procedure actually performs an analogous task as it "runs" pairs of edges that form the sides of a triangle in $SU(3)$ (*i.e.* edges that survive the application of C_i) which serve as the backtracking kinks in a $SU(3)$ path.

We should explain the action of our proposed algorithm on a path of length n. If the path has no triangular or back-and-forth sequences then it either is an essential path or is constructed through a concatenation of paths in such a way that it is longer than the maximum length of an essential path. This then means that the path belongs to the first subspace of the decomposition (6). For a nonessential path we can ensure that at least one sequence of vertices $v_{i-1}v_iv_{i+1}$ that is triangular or back-and-forth and thus when acted upon by the annihilation or cup operator first and the creation or cap operator second, both times on the i-th vertex, we get not just the original path (up to a constant) but we also get another path due to the action of the creation or cap operator (cf. Eq. 2). The second iteration of the algorithm now requires we act upon the $i+1$-th vertex of these resulting paths. If the sequence of vertices that result from the first step is not triangular or back-and-forth then the algorithm stops and the path is decomposed in such a way that the ξ path, if necessary, cancels out the additional path created by the previous step. This means that the path belongs at least to the second subspace of $\mathcal{P}_n$, that is it acts like an essential path of length $n-1$ on which one has acted with one creation or cap operator in the i-th position. If the sequence is triangular or back-and-forth however, this results in the nonessential sequence moving one step down the length of the path since by hypothesis the path's last triangular sequence did not involve the $i+2$-th vertex. We can see then that this procedure can be repeated and its result is to run the nonessential "kink" from the i-th position towards the end of the sequence of vertices, stopping (at most) before reaching the $n-2$-th vertex of the path.

Once this is completed we could go back in the path to the next nonessential sequence, now in a position $j < i$ and act on this new sequence of nonessential vertices. That is to say we can have the algorithm run the last nonessential sequence of a path to its last possible position, then go back further in the path to the next nonessential sequence and repeating the procedure, eventually ending with nonessential sequences separated by at most 2 vertices. This in turn means that the resulting path would belong to a subspace of $\mathcal{P}_n$ in which one has acted over essential paths of length $n-2$ with two creation or cap operators in two positions separated by 2 vertices. From such an algorithm then the desired decomposition of the space of paths of a given size thus follows.

In order to prove the above one needs to show that there always exists a unique set of α_k coefficients that allows one to write a path η following the above prescription, meaning

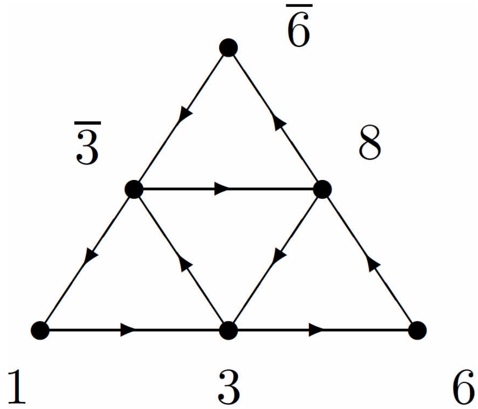

FIGURE 1. The $SU(3)$ $\mathcal{A}$-type graph of level 2. One can find the conjugate graph by reversing the orientations of the arrows.

that our proposed decomposition provides a finite algorithm for the decomposition of paths into essential paths acted upon by an unique sequence of creation operators. Although a formal proof is left for a future work, the arguments above and the examples below make us confident that our algorithm is a step in the right direction.

We will now present a list of examples for the decomposition of elementary, nonessential paths in the A_2 graph of $SU(3)$ starting from length zero up to length four. The list is fully developed for lengths zero through three and we provide an example for a path of length four.

For paths of length $n = 0 + 0$, $0 + 1$ and $1 + 0$ all paths are essential and the decomposition is trivial.

For paths of length $n = 2 + 0$ we find the following nonessential, elementary paths:
$(13\overline{3})$, $(3\overline{3}1)$, $(3\overline{3}8)$, (368), $(\overline{3}13)$, $(\overline{3}8\overline{6})$, $(\overline{3}83)$, (683), $(8\overline{6}\overline{3})$, (836), $(83\overline{3})$, $(\overline{6}\overline{3}8)$.

One can see that some of these paths appear in the list of essential paths in linear combinations but they are not, by themselves, essential. The decomposition of some of these paths as a direct sum is:

$$(13\overline{3}) = \frac{\sqrt{[1][\overline{3}]}}{\overline{T_{13\overline{3}}}}C_1^{\dagger}(1\overline{3}), \quad (3\overline{3}1) = \frac{\sqrt{[3][1]}}{\overline{T_{1\overline{3}3}}}C_1^{\dagger}(31),$$
$$\begin{aligned}(\overline{3}13) &= \alpha C_1^{\dagger}(\overline{3}3) + \xi = \alpha C_1^{\dagger}(\overline{3}3) + A((\overline{3}83) - \sqrt{\beta}(\overline{3}13)) \\ &= \alpha\left(\frac{T_{\overline{3}13}}{\sqrt{[3][3]}}(\overline{3}13) + \frac{T_{\overline{3}83}}{\sqrt{[3][3]}}(\overline{3}83)\right) \\ &\quad + A((\overline{3}83) - \sqrt{\beta}(\overline{3}13)).\end{aligned} \tag{9}$$

Here α is the constant in Eq. 7, A is a normalization factor that ensures that the ξ path is capable of performing the appropriate cancellations and β is the Perron-Frobenius eigenvalue calculated from the graph's adjacency matrix. The previous calculation then implies:

$$\left(A + \alpha\frac{\overline{T_{\overline{3}83}}}{\sqrt{[3][3]}}\right) = 0,$$
$$\left(\alpha\frac{\overline{T_{\overline{3}13}}}{\sqrt{[3][3]}} + \sqrt{\beta}\frac{\overline{T_{\overline{3}83}}}{\sqrt{[3][3]}}\right) = 1.$$

It is important to note that the path ξ that was added above is the path that has the triangular "kinks" in the correct location and also is the essential path connecting both vertices 3 and $\overline{3}$. As we can see the path $(\overline{3}13)$ has been decomposed in an essential path of the length of the original path and a path of shorter length on which one has acted upon with the creation operator. Please note that for paths obtained using conjugation and the Z_3 symmetry of the graph from this example one will find similar results with an analogous procedure.

For length $n = 3 + 0$ the possible cases for nonessential paths can be obtained by rotations of the following set of paths:
(1368), $(13\overline{3}8)$, $(13\overline{3}1)$, (3683), $(368\overline{6})$, $(3\overline{3}13)$, $(3\overline{3}83)$, $(3\overline{3}8\overline{6})$

We can decompose some of these paths with our proposal in the following way:

$$(1368) = \frac{1}{\beta}C_1^{\dagger}(1\overline{3}8) + \sqrt{\beta}C_2^{\dagger}(138),$$
$$(13\overline{3}8) = \frac{1}{\sqrt{\beta}}C_1^{\dagger}(1\overline{3}8),$$
$$(368\overline{6}) = \frac{1}{\sqrt{\beta}}\left(C_1^{\dagger}(38\overline{6}) - \frac{1}{\beta}C_2^{\dagger}(3\overline{3}\overline{6})\right),$$
$$(13\overline{3}1) = \frac{1}{\sqrt{\beta}T_{131}}C_2^{\dagger}(C_0^{\dagger}(1)).$$

For the elementary and nonessential path $\eta = (1368\overline{6})$ we find (this path is clearly nonessential since the sequence (368) is a triangular sequence in the order $v_1 = 3$, $v_2 = 6$, $v_3 = 8$ and therefore gives a nonzero result when acted upon with the annihilation operator):

$$C_2(\eta) = \frac{T_{368}}{\sqrt{[3][8]}}\,138\overline{6},$$
$$C_2^{\dagger}C_2(\eta) = \frac{T_{368}}{\sqrt{[3][8]}}\left(\frac{\overline{T_{3\overline{3}8}}}{\sqrt{[3][8]}}\,13\overline{3}8\overline{6} + \frac{\overline{T_{368}}}{\sqrt{[3][8]}}\,1368\overline{6}\right).$$

Here our algorithm removes the triangular sequence (368) and then adds a vertex that produces a linear combination of two elementary paths: one is the original path $(1368\overline{6})$ and the other, the path $(13\overline{3}8\overline{6})$. This last path has now a triangular sequence in the $v_2 = \overline{3}$, $v_3 = 8$, $v_4 = \overline{6}$, which has been displaced by one position in the order of the sequence of vertices of the elementary path.

In order to continue the algorithm we must now apply $C_3^{\dagger}C_3$ to our previous result, and we get:

$$C_3(\eta') = \frac{T_{368}\overline{T_{3\bar{3}8}}T_{\bar{3}8\bar{6}}}{\sqrt{[3][8][3][8][\bar{3}][\bar{6}]}}\, 13\bar{3}\bar{6},$$

$$C_3^{\dagger}C_3(\eta') = \frac{T_{368}\overline{T_{3\bar{3}8}}T_{\bar{3}8\bar{6}}}{\sqrt{[3][8][3][8][\bar{3}][\bar{6}]}} \left(\frac{\overline{T_{\bar{3}8\bar{6}}}}{\sqrt{[\bar{3}][\bar{6}]}}\, 13\bar{3}8\bar{6}\right).$$

We can get the values for the α_k constants by cancelling the appropriate terms, getting then

$$\alpha_1 = \frac{[3][8]}{|T_{368}|^2}, \quad \alpha_2 = -\frac{[3][\bar{3}][8][\bar{6}]}{|T_{368}|^2|T_{\bar{3}8\bar{6}}|^2}.$$

We can see that the iterated use of creation and annihilation operators allows us to obtain an arbitrary path in this example.

E_5 **graph:** We will now present a list of examples for the decomposition of elementary, nonessential paths in the E_5 graph of $SU(3)$ starting from length zero up to length three. The list is exhaustively calculated for lengths zero through two and we provide two examples of paths of length three.

Once again, for paths of length $n = (0,0)$, $(0,1)$ and $(1,0)$ all paths are essential and the decomposition is trivial.

For paths of length $n = (2,0)$ we find the following nonessential, elementary paths:

$$\begin{matrix} (1_i 2_{i+1} 2_{i+2}) & (2_i 1_{i+4} 2_{i+5}) & (2_i 2_{i+4} 2_{i+5}) \\ (2_i 2_{i+4} 2_{i+2}) & (2_i 2_{i+1} 1_{i+5}) & (2_i 2_{i+1} 2_{i+2}) \end{matrix}$$

For paths of length $n = (1,1)$ we find either essential paths or elementary back and forth paths that require the use of the deformed triangular cells discussed above. Since at this point, their values have yet to be calculated, we will not deal with them here.

For paths of length $n = (0,3)$ the list of nonessential, elementary paths follows:

$$\begin{matrix} (1_i 2_{i+2} 1_{i+1} 2_{i+3}) & (1_i 2_{i+2} 2_{i+1} 2_i) & (1_i 2_{i+2} 2_{i+1} 2_{i+3}) & (1_i 2_{i+2} 2_{i+4} 2_i) \\ (1_i 2_{i+2} 2_{i+4} 2_{i+3}) & (2_i 1_{i+5} 2_{i+1} 1_i) & (2_i 2_{i+5} 2_{i+4} 1_{i+3}) & (2_i 2_{i+5} 2_{i+1} 2_{i+3}) \\ (2_i 1_{i+5} 2_{i+1} 2_{i+3}) & (2_{i+5} 2_{i+5} 2_{i+4} 2_{i+3}) & (2_i 2_{i+2} 2_{i+4} 1_{i+3}) & (2_i 2_{i+2} 2_{i+4} 2_{i+3}) \\ (2_i 2_{i+5} 2_{i+1} 1_i) & (2_i 2_{i+2} 1_{i+1} 2_{i+3}) & (2_i 2_{i+2} 2_{i+1} 2_{i+3}) & (2_i 2_{i+2} 2_{i+1} 1_i) \end{matrix}$$

Let us show a few selected examples of the workings of our proposal for a selection of families of paths. First for a family of paths of length $n = (2,0)$:

$$(1_i 2_{i+1} 2_{i+2}) = \frac{\sqrt{[1_i][2_{i+2}]}}{\overline{T_{1_i 2_{i+1} 2_{i+2}}}}\, C_1^{\dagger}(1_i 2_{i+2}). \qquad (10)$$

For a family of paths of length $n = (0,3)$ we find:

$$\begin{aligned} &(1_i 2_{i+2} 2_{i+4} 2_i) = \alpha\, C_2^{\dagger}(1_i 2_{i+2} 2_i) + \xi \\ &= \left(\alpha \frac{\overline{T_{2_{i+2}2_{i+4}2_i}}}{\sqrt{[2_{i+2}][2_i]}} + A\right)(1_i 2_{i+2} 2_{i+4} 2_i) \\ &+ \left(\alpha \frac{\overline{T_{2_{i+2}2_{i+1}2_i}}}{\sqrt{[2_{i+2}][2_i]}} - (-1)^i \frac{\nu_0}{\mu} A\right)(1_i 2_{i+2} 2_{i+1} 2_i). \qquad (11) \end{aligned}$$

Which leads us to values for both constants:

$$\left(\alpha \frac{\overline{T_{2_{i+2}2_{i+4}2_i}}}{\sqrt{[2_{i+2}][2_i]}} + A\right) = 1,$$

$$\left(\alpha \frac{\overline{T_{2_{i+2}2_{i+1}2_i}}}{\sqrt{[2_{i+2}][2_i]}} - (-1)^i \frac{\nu_0}{\mu} A\right) = 0.$$

If we consider the elementary and nonessential path $\eta = (1_3 2_4 1_2 2_3 2_2)$ we find, for the first steps of the decomposition:

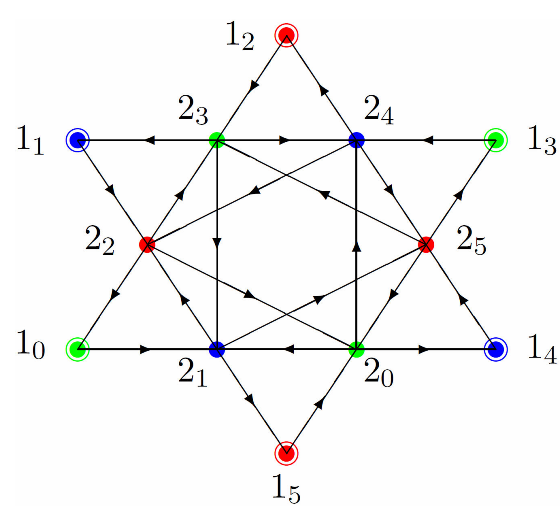

FIGURE 2. The $SU(3)$ E_5 graph.

$$C_2^\dagger C_2(\eta) = \frac{T_{2_4 1_2 2_3}}{\sqrt{[2_4][2_3]}} \left(\frac{\overline{T_{2_4 1_2 2_3}}}{\sqrt{[2_4][2_3]}}\, 1_3 2_4 1_2 2_3 2_2 + \frac{\overline{T_{2_4 2_2 2_3}}}{\sqrt{[2_4][2_3]}}\, 1_3 2_4 2_2 2_3 2_2 \right), \tag{12}$$

$$C_3 C_2^\dagger C_2(\eta) = \frac{T_{2_4 1_2 2_3}}{\sqrt{[2_4][2_3]}} \times \left(\frac{\overline{T_{2_4 1_2 2_3}}}{\sqrt{[2_4][2_3]}} \frac{T_{1_2 2_3 2_2}}{\sqrt{[1_2][2_2]}}\, 1_3 2_4 1_2 2_2 + \frac{\overline{T_{2_4 2_2 2_3}}}{\sqrt{[2_4][2_3]}} \frac{T_{2_2 2_3 2_2}}{\sqrt{[2_2][2_2]}}\, 1_3 2_4 2_2 2_2 \right) = 0. \tag{13}$$

We find in this path that the terms for the path $1_3 2_4 2_2 2_3 2_2$ do not cancel out. This is due to the particular geometry of the $\mathcal{E}_5$ but we can recover the decomposition of the path keeping in mind that there is a ξ path that allows for the desired cancellation, thus:

$$\alpha_1 = \frac{[2_4][2_3]}{|T_{2_4 1_2 2_3}|^2},$$

$$\xi = -\alpha_1 \frac{T_{2_4 1_2 2_3}}{\sqrt{[2_4][2_3]}} \frac{\overline{T_{2_4 2_2 2_3}}}{\sqrt{[2_4][2_3]}}\, 1_3 2_4 2_2 2_3 2_2.$$

A note of caution is in order on the use of the ξ path. In Ref. 9, the geometric necessity for the ξ path is clear: in some paths the application of the creation operator in the first step of the algorithm results in the creation of two paths, one that presents a backtracking in the $i-1$th vertex and the original path. The remaining iterations of the algorithm produce paths that "move" the backtracks towards the end of the path and ensure the cancellation of all paths thus created. Since for these paths the algorithm creates a path in which the backtrack has been moved back one step, one requires a cancelling path with the properties indicated in Trinchero's work, *i.e.* $\xi^{(i)}$ satisfying $c_j(\xi^{(i)}) = 0\,\forall j$ such that $i-1 < j < n-2$. Since our $SU(3)$ graphs are now embedded in a two dimensional lattice there is no natural ordering of vertices, that is, there is no way to claim that a vertex comes before or after another one unambiguously which leads to the problem of the definition of backtracking paths. There is however a way out: to relax the definition of the ξ path in such a way that it is capable of ensuring the required cancellation, that is, that it presents a triangular path in the same position of the original path.

4. Discussion

We have proposed here an algorithm that allows one to calculate how a given path can be obtained as an iterated and ordered application of creation and cap operators on a path belonging to the kernel of the annihilation and cup operators. This means that our algorithm provides a way to obtain a given path by acting on an essential path with a specific combination of creation and cap operators. We not only propose said algorithm but we have shown how it works by performing calculations successfully for a number of selected paths for A type and E type graphs. In future works we shall elucidate the decomposition of the space of paths over $SU(3)$ graphs [5] and we will provide proof for this algorithm and the properties of the ξ path. In spite of lacking a formal proof, our calculations give us confidence that our proposal is a step in the right direction.

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