RESEARCH Revista Mexicana de Física **60** (2014) 371–375 SEPTEMBER-OCTOBER 2014

Higher dimensional Elko theory

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Received 21 March 2014; accepted 18 August 2014

We show that the so-called Elko equation can be derived from a 5-dimensional Dirac equation. We argue that this result can be relevant for dark matter and cosmological scenarios. We generalize our procedure to higher dimensions.

Keywords: Elko equation; Dirac equation; dark matter.

PACS: 04.20.Gz, 04.60.-Ds, 11.30.Ly

It is known that one of the most interesting proposals to explain dark matter [1] is provided by the so called Elko theory [2-3] (see also Refs. [4-12]). This theory describes spin half-integer fermions with dual helicity eigenspinors of the charge conjugation operator. It turns out that the growing interest in this kind of matter is due to the fact that besides being a candidate for dark matter it may also provide of our an alternative explanation of the accelerated expansion our universe [13-15] (see also Refs. [2], [16], and [17] references therein). In the context of inequivalent spin structures on arbitrary curved spacetimes exotic dark spinor fields has been introduced [18]. Moreover, dynamical dispersion relations for Elko dark spinors fields have lead to mass generation proposal [19] and a light Elko signals exploration at acelerators has been developed [20]. It is interesting that Elko spinor fields have been considered as a tool for probing exotic topological spacetime features [21]. Even more recently, a Lagrangian for mass dimension one fermion has been derived [22].

Recently, it has been proposed [23] a 5-dimensional Elko theory in the context of Minkowski branes and it was shown that if a 5-dimensional mass term (see Ref. [24] for details of Kaluza-Klein theory) is introduced there is not the possibility to localize these modes on the corresponding branes. Part of the motivation in these developments arises from the desire to shed some light on the brane world theory that originates from so-called M-theory (see Ref. [25] and references therein), which is a generalization of superstring theory [26]. In this case, the brane is embedded in a higher dimensional space-time. So it appears interesting to explore whether the Elko theory is some how related to M-theory. As a first step in this direction, one would like to explore whether there exists a higher dimensional Elko theory. In this work we show the surprising result that the Elko equation in four dimensions can be obtained from a higher dimensional Dirac equation.

Let us start mentioning that the Elko theory is based on the fundamental equation

$$\left[\gamma^\mu \hat{p}_\mu \delta_a^b + i m_0 \varepsilon_a^b\right] \psi_b = 0. \tag{1}$$

Here, the indices μ, ν etc. run from 0 to 3 and the indices a, b run from 1 to 2. Further the γ-matrices satisfy the Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta^{\mu\nu}, \tag{2}$$

with $\eta^{\mu\nu} = \text{diag}(-1,1,1,1)$. Moreover, $\hat{p}_\mu = -i\partial_\mu$ and ε^{ab} is the completely antisymmetric ε-symbol, with $\varepsilon^{12} = 1 = -\varepsilon^{21}$.

In contrast to the usual Dirac equation [27]

$$\left[\gamma^\mu \hat{p}_\mu + m_0\right] \psi = 0, \tag{3}$$

Eq. (1) requires eight-component complex spinor ψ_b rather than four-component ψ which is the case in Eq. (3). Furthermore, the quantities δ_a^b and ε_a^b in (1) establish that ψ_b is not eigenspinor of the $\gamma^\mu \hat{p}_\mu$ operator as ψ in Eq. (3). In spite of these key differences, one can prove that (1) and (3) imply the Klein-Gordon equations

$$\left[\hat{p}^\mu \hat{p}_\mu + m_0^2\right] \psi_a = 0 \tag{4}$$

and

$$\left[\hat{p}^\mu \hat{p}_\mu + m_0^2\right] \psi = 0, \tag{5}$$

respectively. Therefore, both (1) and (3) can be understood as “the square root of the Klein-Gordon equation.”

Now, consider a 5-dimensional Klein-Gordon equation

$$\left[\hat{p}^{\hat{\mu}} \hat{p}_{\hat{\mu}} + m_0^2\right] \psi = 0, \tag{6}$$

or

$$\left[(\hat{p}_5)^2 + \hat{p}^\mu \hat{p}_\mu + m_0^2\right] \psi = 0, \tag{7}$$

with the indices $\hat{\mu}$, $\hat{\nu}$ etc. assuming the values $0,1,2,3,5$. Here, $\hat{p}^{\hat{\mu}} \hat{p}_{\hat{\mu}} = \eta^{\hat{\mu}\hat{\nu}} \hat{p}_{\hat{\mu}} \hat{p}_{\hat{\nu}}$, with $\eta^{\hat{\mu}\hat{\nu}} = \text{diag}(-1,1,1,1,1)$.

By virtue of (2) one finds that (7) can be written as

$$\left[-(-\hat{p}_5 + \gamma^\mu \hat{p}_\mu)(\hat{p}_5 + \gamma^\nu \hat{p}_\nu) + m_0^2\right] \psi = 0. \tag{8}$$

Thus, introducing the definitions

$$\psi_L \equiv \psi \tag{9}$$

and

$$\psi_R \equiv -\frac{i}{m_0}(\hat{p}_5 + \gamma^\mu \hat{p}_\mu)\psi_L, \tag{10}$$

one sees that (8) leads to

$$(-\hat{p}_5 + \gamma^\mu \hat{p}_\mu)\psi_R + im_0\psi_L = 0, \tag{11}$$

while, the expressions (10) can be rewritten as

$$(\hat{p}_5 + \gamma^\mu \hat{p}_\mu)\psi_L - im_0\psi_R = 0. \tag{12}$$

Let us now assume that $\hat{p}_5\psi_R = 0$ and $\hat{p}_5\psi_L = 0$. (These imitate the cylindrical conditions in the usual dimensional reduction of the Kaluza-Klein theory [24].) With these assumptions, Eqs. (11) and (12) become

$$\gamma^\mu \hat{p}_\mu \psi_R + im_0\psi_L = 0 \tag{13}$$

and

$$\gamma^\mu \hat{p}_\mu \psi_L - im_0\psi_R = 0, \tag{14}$$

respectively. So, if one assumes that the indices a, b run from the labels R and L one learns that (13) and (14) can be obtained from the expression

$$(\gamma^\mu \hat{p}_\mu \delta^b_a + i\varepsilon^b_a m_0)\psi_b = 0, \tag{15}$$

which is precisely formula (1) (see formula (5.14) in Refs. [2]). This means, among other things, that (11) and (12) represent a generalization of Elko equation (1).

Now, it is straightforward to generalize our procedure to higher dimensions. In fact, let us introduce the D-dimensional Klein-Gordon equation

$$\left[\hat{p}^A \hat{p}_A + m_0^2\right]\psi = 0, \tag{16}$$

with the indices A, B etc. running from 0 to $D-1$. We shall assume that the splitting of the indices $A = (\mu, a)$. Thus, considering that $\hat{p}^A\hat{p}_A = \eta^{AB}\hat{p}_A\hat{p}_B$, with $\eta^{AB} = \text{diag}(-1, 1, ..., 1, 1)$ one sees that (16) become

$$\left[\hat{p}^a\hat{p}_a + \hat{p}^\mu\hat{p}_\mu + m_0^2\right]\psi = 0. \tag{17}$$

Using the analogue of (2) for the internal space

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}, \tag{18}$$

and also assuming that

$$\gamma^a\gamma^\mu - \gamma^\mu\gamma^a = 0, \tag{19}$$

one learns that (17) can also be written as

$$\left[-(-\gamma^a\hat{p}_a + \gamma^\mu\hat{p}_\mu)(\gamma^b\hat{p}_b + \gamma^\nu\hat{p}_\nu) + m_0^2\right]\psi = 0. \tag{20}$$

It is not difficult to see that the γ^μ and γ^a matrices can be chosen as $2^{(D-2)/2} \times 2^{(D-2)/2}$ or $2^{(D-3)/2} \times 2^{(D-3)/2}$ matrices, depending if D is an even or odd number, respectively.

Now, let us introduce the definitions

$$\psi_L \equiv \psi \tag{21}$$

and

$$\psi_R = -\frac{i}{m_0}(\gamma^a\hat{p}_a + \gamma^\mu\hat{p}_\mu)\psi_L. \tag{22}$$

These definitions allow us to rewrite (20) as

$$-i(-\gamma^a\hat{p}_a + \gamma^\mu\hat{p}_\mu)\psi_R + m_0\psi_L = 0. \tag{23}$$

The Eqs. (22) and (23) lead to

$$(-\gamma^a\hat{p}_a + \gamma^\mu\hat{p}_\mu)\psi_R + im_0\psi_L = 0 \tag{24}$$

and

$$(\gamma^a\hat{p}_a + \gamma^\mu\hat{p}_\mu)\psi_L - im_0\psi_R = 0. \tag{25}$$

So, once again, if one assumes the dimensional reductions conditions

$$\gamma^a\hat{p}_a\psi_R = 0 \tag{26}$$

and

$$\gamma^a\hat{p}_a\psi_L = 0, \tag{27}$$

one obtains Elko formula (1). Of course, our approach of this higher dimensional generalization of Elko equation resembles the typical procedure of the Dirac equation in the Weyl representation. In this case one starts with the Klein-Gordon equation

$$(\eta^{\mu\nu}\hat{p}_\mu\hat{p}_\nu + m_0^2)\psi = 0, \tag{28}$$

and introduces the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{29}$$

which satisfy

$$\sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij} \tag{30}$$

and

$$\sigma^i\sigma^j - \sigma^j\sigma^i = 2i\varepsilon^{ijk}\sigma_k. \tag{31}$$

Here, $\delta^{ij} = \text{diag}(1, 1, 1)$ is the Kronecker delta and ε^{ijk} is the completely antisymmetric ε-symbol, with $\varepsilon^{123} = 1$. In $(1+3)$-dimensions we have $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and therefore (28) becomes

$$(-\hat{p}_0\hat{p}_0 + \delta^{ij}\hat{p}_i\hat{p}_j + m_0^2)\psi = 0. \tag{32}$$

Using (30) one sees that (32) can be written as

$$(-\sigma^0\hat{p}_0\sigma^0\hat{p}_0 + \sigma^i\sigma^j\hat{p}_i\hat{p}_j + m_0^2)\psi = 0 \tag{33}$$

or

$$(-\sigma^0\hat{p}_0 + \sigma^i\hat{p}_i)(\sigma^0\hat{p}_0 + \sigma^j\hat{p}_j)\psi + m_0^2\psi = 0, \tag{34}$$

where σ^0 is the identity 2×2-matrix.

So, by defining $\psi_L \equiv \psi$ and

$$\psi_R \equiv -\frac{1}{m_0}(\sigma^0\hat{p}_0 + \sigma^j\hat{p}_j)\psi_L, \tag{35}$$

one obtains

$$(\sigma^0\hat{p}_0 + \sigma^j\hat{p}_j)\psi_L + m_0\psi_R = 0 \tag{36}$$

and

$$(\sigma^0 \hat{p}_0 - \sigma^i \hat{p}_i)\psi_R + m_0\psi_L = 0. \tag{37}$$

We recognize in (36) and (37) the Dirac equation in the Weyl representation,

$$\left[\gamma^0_W \hat{p}_0 + \gamma^i_W \hat{p}_i + m_0\right] \psi_W = 0, \tag{38}$$

which in covariant notation becomes

$$\left[\gamma^\mu_W \hat{p}_\mu + m_0\right] \psi_W = 0, \tag{39}$$

where

$$\gamma^0_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \gamma^i_W = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{40}$$

and

$$\psi_W = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}. \tag{41}$$

Here, the W in γ^μ_W and ψ_W means that these quantities are in the Weyl representation. One can verify that

$$\gamma^\mu_W \gamma^\nu_W + \gamma^\nu_W \gamma^\mu_W = -2\eta^{\mu\nu}. \tag{42}$$

It turns out that one can interchange the order of (36) and (37) in the form

$$(-\sigma^0 \hat{p}_0 + \sigma^i \hat{p}_i)\psi_R - m_0\psi_L = 0, \tag{43}$$

and

$$(\sigma^0 \hat{p}_0 + \sigma^j \hat{p}_j)\psi_L + m_0\psi_R = 0. \tag{44}$$

Thus, comparing (43) and (44) with (24) and (25) one observes that the algebra of γ^a and γ^μ given in (2), (18) and (19) is the analogue of the algebra satisfied by σ^0 and σ^i, namely

$$\sigma^0\sigma^0 + \sigma^0\sigma^0 = 2\delta^{00}, \tag{45}$$

$$\sigma^0\sigma^j - \sigma^j\sigma^0 = 0, \tag{46}$$

and

$$\sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}. \tag{47}$$

This means the γ^a plays the role of σ^0, while γ^μ plays the role of σ^i. This analysis may motive to look for a t-time signature, rather than s-space signature. In fact, in the case of t-time signature the formula (18) must be changed by

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab}, \tag{48}$$

At the end, starting with a Dirac equation in spacetime of $(t + s)$-signature and imposing the cylindrical condition in the extra dimensions one must arrive to the Elko equation in $(1 + 3)$-dimensions. In particular it may be interesting to find the analogue of Elko theory in $(2 + 2)$-dimensions (see Refs. 28 to 29)

From the point of view of Kaluza-Klein theory the conditions (26) and (27) correspond to the zero mode of a compactified space. It will be interesting for further research to consider non zero modes of the internal space. In this direction the Ref. [23] dealing with brane-worlds in 5-dimensions may be particularly useful.

It still remains to analyze a kind of Majorana condition for the physical states ψ_a, namely

$$C\psi_a = e^{i\theta}\psi_a, \tag{49}$$

where C denotes a charge conjugation operator and $e^{i\theta}$ is a phase factor. If one chooses $C\psi_a = \psi_a$ one obtains the the self-conjugate Elko spinor or Majorana spinor, while if one requires the condition $C\psi_a = -\psi_a$ one gets the anti-self-conjugate Elko spinor, which is different than the conventional Majorana's choice. But this will require to consider a Clifford algebra in a $(t+s)$-signature in a similar way as one analyzes the Majorana-Weyl spinors in higher dimensions [30].

In 5-dimensions and in the Weyl representation the expression (49) implies

$$-\sigma^2\psi^*_L = \psi_R, \tag{50}$$

or

$$\sigma^2\psi^*_R = \psi_L. \tag{51}$$

Consequently, up to a face, the Majorana spinor obtained from (50) and (51), in the Weyl representation, looks like

$$\psi \longrightarrow \lambda = \begin{pmatrix} -\sigma^2\psi^*_L \\ \psi_L \end{pmatrix} \tag{52}$$

or

$$\psi \longrightarrow \rho = \begin{pmatrix} \psi_R \\ \sigma^2\psi^*_R \end{pmatrix}. \tag{53}$$

It turns out that in the original construction of Elko theory [2-3] the constraint (49) is considered as the starting point. For this reason when the field equation (1) is used one finds that ψ_a contains only 4 complex components. In our case, in writing (15), one sees that originally ψ_a has 8 complex components, but the two approaches must be equivalent after imposing the Majorana condition (49). Roughly speaking, in Refs. 2 to 3 it is followed the route of (49) first and then (1), while in our case the route is (1) (or (15)) first and then (49), but in both cases the number of components associated with ψ_a, satisfying (1) (or (15)) and (49), is 4 complex components. In higher dimensions this analysis is more complicated since besides of imposing (49) one must consider the equations (26) and (27).

It is also worth mentioning some dimensional analysis towards a renormalization of the theory. In general the action

$$S_{(4)} = \int \mathcal{L}_{(4)} d^4x, \tag{54}$$

must be dimensionless. For this reason, since the dimensionality of d^4x is -4, one sees that each term in $\mathcal{L}_{(4)}$ must carry dimensionality $+4$. In this context, one finds that the kinetic term of $\mathcal{L}_{(4)}$ implying the Dirac equation establishes that the spinor field ψ_a has mass dimension $3/2$. Thinking about the Lorentz transformation of type (A, B) this result is obtained considering that the mass dimension of the spinor field is given by $1+A+B$. In fact, for the field of the types $(1/2,0)$ and $(0,1/2)$, the expected mass dimensionality is $3/2$. In Elko theory ψ_a also transforms as $(1/2,0)\oplus(0,1/2)$, but according to Refs. 2 to 3 due to non-locality the dimension of ψ_a is not $3/2$, but 1 (see Ref. [31] for an alternative results). It is worth mentioning that recently a Lagrangian approach for mass dimension one fermions has been proposed [22]. In our case, the situation seems different because our starting point is not the action (54) but

$$S_{(5)} = \int \mathcal{L}_{(5)} d^5x. \tag{55}$$

In this case, in order to have a dimensionless action, $\mathcal{L}_{(5)}$ must contain mass dimension $+5$, and therefore the kinetic term in $\mathcal{L}_{(5)}$, implying the Dirac equation in 5 dimensions, leads to the result that the spinor field ψ_a must carry mass dimension 2, instead of $3/2$ as it is the case in 4 dimensions. We believe that this is an intriguing result that deserves a further research.

Finally, suppose one fix the Clifford algebra (2). One may be interested in exploring the consequences in the Elko theory under the signature change:

$$(-1,1,1,1) \leftrightarrow (1,-1,-1,-1) \tag{56}$$

As it has been emphasized in the the Ref. [32] only if the mass is equal to zero the Dirac equation is invariant under (56). This conclusion may be different in Elko theory because a massless fermion in 5-dimensions may be massive in 4-dimensions. This seems to be an interesting question [33] which may be a subject for further research.

Acknowledgments

I would like to thank professor D. V. Ahluwalia, as well as the two referees, for helpful comments. This work was partially supported by PROFAPI-UAS/2013.

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33. This idea emerged as consequence of some referee's comments.