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Study of plasma displacement and $\beta_p + l_i/2$ by the simplest Grad-Shafranov equation solution for circular cross section tokamak

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In this work we present the plasma displacement and $\beta_p + l_i/2$ by the Simplest Grad-Shafranov Equation (GSE) solution using Solov'ev assumption for circular cross section HT-7 tokamak. Using diamagnetic and compensation loop, combining with poloidal magnetic probe array signals, plasma displacement and $\beta_p + l_i/2$ are measured. In this paper, theoretical and experimental results in determining plasma displacement and $\beta_p + l_i/2$ are presented. We have seen that the calculated plasma displacement and the calculated $\beta_p + l_i/2$ depend on the kind of discharge or plasma current.

Keywords: Magnetohydrodynamics; two-fluid and multi-fluid plasmas.

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1. Introduction

In a Tokamak, external magnetic measurements have been applied to determine the important information on plasma shapes, the safety factor, the sum of the average poloidal beta β_p and internal inductance l_i [1]. There are methods for extraction of plasma parameters from external magnetic measurements. Swain and Neilson [2] presented an efficient method to reconstruct the plasma shapes and line integrals of the boundary poloidal magnetic field from external magnetic measurements. In their method, the plasma current distribution is approximated by using a few filament currents. In Luxon and Brown's approach [3], the plasma current is modeled using distributed sources. The non-linear Grad–Shafranov equation (GSE) is solved repeatedly to search the best-fit current density profile. As we know, analytical solutions of the Grad–Shafranov [4,5] equation are very useful for theoretical studies of plasma equilibrium, transport, and magnetohydrodynamic (MHD) stability. The well-known Solov'ev quilibrium [6] has been extensively used for such studies, and also as a benchmark of numerical codes that attempt to find more general solutions.

However, the Solov'ev equilibrium solutions typically studied [6-7] are over constrained, either in shape (elliptical) or in plasma current (which is commonly determined by the choice of poloidal beta, β_p).

The existing exact solutions have arisen from a variety of allowed current density profiles or a variety functional of source functions. A simple analytical solution to the inhomogeneous GSE is presented by Zheng *et al*. [8] ,which corresponds to source functions linear in ψ. For this case, six parameters must be determined. The shape of the plasma can be described by four parameters for which rectangular fixed boundary conditions are selected. The shape of the current profile is essentially flat, and the two existing free parameters can be chosen by the plasma current I_p and the poloidal beta β_p. But Rahimirad [9] presented the simplest solution of GSE which is the well-known Solov'ev equilibrium [6], which has five parameters. In this paper we have selected the simplest solution of Grad-Shafranov equation [9], which is the well-known Solov'ev equilibrium [6], so we have six parameters, which can be determined by circular fixed boundary conditions and by the plasma current of the Tokamak.

In this paper, we analyze the possibility of using a specific choice of current distribution for interpretation of magnetic measurements and we directly calculate the poloidal beta β_p and internal inductance l_i form solution of GSE [10-12] and discrete magnetic coils [13]. We solve GSE by considering linear source functions and circular fixed boundary conditions for circular cross section HT-7 tokamak [13] with a plasma minor radius that it is variable. This solution has three quantities (plasma current I_p, plasma minor radius a and $\beta_p + l_i/2$ as input data. The quantities are measured by a Rogowski coil, a Saddle coil and an array of discrete magnetic coils [13], respectively. Then, according to the definition of the poloidal beta β_p and internal inductance l_i [11,14] we substituted the poloidal flux function that obtained by our solution into it. Finally, we calculate the time evolution of the plasma parameters for a typical discharges of circular cross section HT-7 tokamak (see Table I).

We present the plasma displacement and $\beta_p + l_i/2$ by the Simplest Grad–Shafranov Equation (GSE) solution using Solov'ev assumption for circular cross section HT-7 tokamak. Using diamagnetic and compensation loop, combining with poloidal magnetic probe array signals, plasma displacement and $\beta_p + l_i/2$ are measured. In this paper, theoretical and experimental results in determining plasma displacement and $\beta_p + l_i/2$ are presented. We have seen that the calculated plasma displacement and the calculated $\beta_p + l_i/2$ depend on the kind of discharge or plasma current.

TABLE I. Parameter of the HT-7 tokamak.

Parameter	Value
Major Radius	1.22 m
Minor Radius	0.27 m
Toroidal Field	1- 2.5 T
Plasma Current	100-250 kA
Discharge Time	$\sim$ 300 s
Electron Density	$1-6\times 10^{19}$ m^{-3}

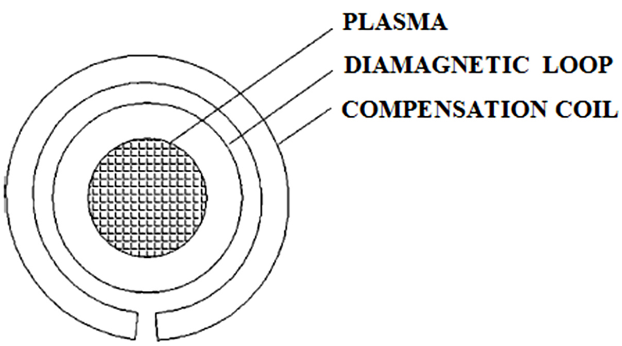

FIGURE 1. The diamagnetic and compensating loops.

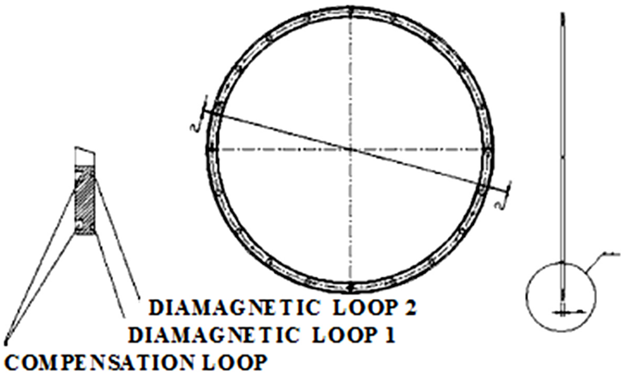

FIGURE 2. The structure of the magnetic arrays.

2. Extended Grad-Shafranov equation

Maxwell's equations together with the force balance equation from MHD equations, in the cylindrical coordinates (R,Z) reduce to the two-dimensional, nonlinear, elliptic Grad-Shafranov equation [7]: As in the linear case, the procedure to derive the Grad-Shafranov equation can be followed obtaining an extended Grad-Shafranov equation [10]

$$\Delta^*\psi = \mu_0 R J_\phi = -\mu_0(\gamma-1)R^2\frac{du(\psi)}{d\psi} - F(\psi)\frac{dF(\psi)}{d\psi}, \quad (1)$$

where Δ^* is

$$\Delta^* = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{\partial^2}{\partial z^2} \quad (2)$$

The internal energy in this extended Grad-Shafranov equation is a function of ψ. The $u(\psi)$ and $F(\psi)$ are two free functions, while μ_0 and J are the vacuum permeability and plasma current density respectively.

3. Grad-Shafranov equation solution

In an axisymmetric system a convenient representation of the magnetic field is:

$$B = F\nabla\phi + \nabla\psi\times\nabla\phi, \quad (3)$$

where ϕ is the ignorable angle in the cylindrical coordinate system (R,ϕ,Z), and $F$and ψ are axisymmetric scalar functions. The function $F$is a flux function, associated with the poloidal current in the system, while ψ is the poloidal flux divided by 2π. The internal energy in this Grad-Shafranov Eq. (1) is a function of ψ. Here J_ϕ is the toroidal plasma current density; u is the internal energy, which is a function of ψ. Note that the toroidal field is not determined by the Grad-Shafranov equation. Its choice will only rescale the values of the safety factor q. It will not change the shape of the flux surfaces, or that of the profiles of internal energy, current density, q, etc.

The simplest solution to Eq. (1) can be found by assuming that

$$\mu_0(\gamma-1)\frac{\partial u}{\partial\psi} = -A_1, \quad F\frac{\partial F}{\partial\psi} = A_2, \quad (4)$$

where A_1 and A_2 are constant. This obviously reduces the set of possible current density profile shapes to $J_\phi \propto RA_1 - A_2/R$.

With the restrictions given by (4), the GSE is reduced to the form:

$$\Delta^*\psi = R^2A_1 - A_2, \quad (5)$$

whose solution is [8]

$$\psi = \psi_0 - \frac{A_1}{8}R^4 - \frac{A_2}{2}Z^2, \quad (6)$$

where, ψ_0 is a solution of the homogeneous equation

$$\Delta^*\psi = 0. \quad (7)$$

If the plasma is assumed to be up–down symmetric, its shape can be described by four parameters. The equatorial innermost and outermost points, R_i and R_0, and the coordinates of the highest point, (R_t, Z_t) or equivalently, the major radius $R_m = (R_i + R_0)/2$, $R_m \neq R_0$, the minor radius $a = (R_0 - R_i)/2$, the elongation $\kappa_0 = Z_t/a$, and triangularity $\delta = (R_0 - R_t)/2$. From (5), the simplest solution is given by [8,10]

$$\psi = c_1 + c_2R^2 + c_3(R^4 - 4R^2Z^2) + c_4(R^2\ln(R) - Z^2) - \frac{A_1}{8}R^4 - \frac{A_2}{2}Z^2. \quad (8)$$

For the determination these six coefficients, it is necessary to have six equations. We assume that the internal energy vanishes at the boundary, hence $\psi(R,Z)|_b = 0$ [8,9].

With Eq. (8), the boundary conditions $R = R_0 \pm a$, $Z = 0$ and $R = R_t$, $Z = Z_t$ gives the following equations:

$$\psi(R_i,0) = c_1 + c_2 R_i^2 + c_3 R_i^4 + c_4 R_i^2 \ln(R_i) - \frac{A_1}{8} R_i^4 = \psi_{180}, \tag{9}$$

$$\psi(R_0,0) = c_1 + c_2 R_0^2 + c_3 R_0^4 + c_4 R_0^2 \ln(R_0) - \frac{A_1}{8} R_0^4 = \psi_0, \tag{10}$$

$$\psi(R_t,Z_t) = c_1 + c_2 R_t^2 + c_3 (R_t^4 - 4R_t^2 Z_t^2) + c_4 (R_t^2 \ln(R_t) - Z_t^2) - \frac{A_1}{8} R_t^4 - \frac{A_2}{2} Z_t^2 = \psi_{90}, \tag{11}$$

We also assume that the plasma is enclosed in a perfectly conducting toroidal boundary with circular cross section, with radius a, so the normal component of the magnetic field.

$$\frac{1}{R}\frac{d\psi(R_t,z_t)}{dR} = 2c_2 + 4c_3(R_t^2 - 2Z_t^2) + c_4(2\ln(R_t)+1) - \frac{A_1}{2} R_t^2 = B_Z(R_t,Z_t), \tag{12}$$

The plasma current can be clearly measured by Rogowski coil [13], so the plasma current can be written.

$$2\pi\mu_0 I_p = \int\int \left(R A_1 + \frac{A_2}{2}\right) dR dZ, \tag{13}$$

It is simpler to first solve for a plasma with unit current and unit major radius and use the scaling relations described above to find the final desired equilibrium. However, even in this simplest case only numerical solutions to Eqs. (9-14) have been found. The coefficients can be computed numerically, given a desired plasma description.

We also selected the constraint $\beta_p + l_i/2$ [11], because the parameter can be experimentally deduced using discrete magnetic probes [13], for circular cross section HT-7 tokamak [13,15-19],

$$\beta_p + l_i/2 = \frac{\left(\oint dl\right)^2}{(2\pi\mu_0 I_p)^2 \int\int R dR dZ} \times \left(2.5 A_1 \int\int \psi(R,Z) R dR dZ + 0.5 A_2 \int\int \frac{\psi(R,Z)}{R} dR dZ\right) \tag{14}$$

4. Plasma displacement and β_P+ l_i/2 by the GSE solution

We have obtained the six coefficient by solving six algebraic equation for circular cross section HT-7 tokamak [20] (see Table I). It has triangularity $\delta = 0$, elongation $\kappa = 0$, a major radius of R_m = 1.22 m, minor radius of a= 0.27 m, $R_i = R_0 - a$, $R_o = R_0 + a$, $R_t = R_0$, $Z_t = a$, defined by one poloidal water-cooling limiter, one toroidal water-cooling belt limiter at the high field side and a new set of actively cooled toroidal double-ring graphite limiters at the bottom and the top of the vacuum vessel, and pulses up to 240 s of long plasma have been achieved with new graphite limiters in the HT-7 in 2004 [20].

$$c_1 = -0.004A_1 - 0.034A_2 + 0.23B_Z - 0.359\psi_0 + 2.359\psi_{180} - 1.51\psi_{90}, \tag{15}$$

$$c_2 = 0.034A_1 - 0.084A_2 + 3.823B_Z - 8.359\psi_0 + 12.309\psi_{180} - 6.351\psi_{90}, \tag{16}$$

$$c_3 = 0.024A_1 + 0.024A_2 - 3.553B_Z + 20.359\psi_0 - 30.212\psi_{180} + 3.351\psi_{90}, \tag{17}$$

$$A_1 = 165.345\mu_0 I_P - 3.10A_2, \tag{18}$$

$$A_2 = 4105.245\{0.0034B_Z + 0.0061\mu_0 I_P + 0.1049\psi_0 + 0.0646\psi_{180} + 0.151\psi_{90} \pm ((-0.0034B_Z - 0.0061\mu_0 I_P + 0.1049\psi_0 + 0.0646\psi_{180} + 0.151\psi_{90})^2 + 0.0001\mu_0 I_P(0.003B_Z + 0.0361\mu_0 I_P - 1.304\left(\beta_p + \frac{l_i}{2}\right)\mu_0 I_P + 0.534\psi_0 + 0.446\psi_{180} + 1.151\psi_{90}))^{0.5}\}, \tag{19}$$

The parameter A_2 has two values, but one value that can be obtained by experimental data is acceptable. The acceptable sign is minus. By using the coefficient in Eq. (8), we can acquire the magnetic flux surface [10,21]. The plasma position and the theoretical HD ΔH_{theo} can be obtained as

$$\Delta H_{\text{theo}} = R_{\text{axis}} - R_o \tag{20}$$

Where the $R_{\text{axis}}$is determined by $d\psi(R_{\text{axis}},0)/dR = 0$ and R_0 is the geometrical centre of the plasma column. We imposed that $\Delta H_{\text{exp}} = \Delta H_{\text{theo}}$ and try to find with the supposition that in the region of equilibrium state the a can be assumed constant, the value of R_0, leaving as free parameter. So we determined $R_0 = 1.22$ m. The ΔH_{exp} is the plasma HD, which can be obtained experimentally using four magnetic probes as described by [13,21].

5. Plasma displacement and $\beta_P + l_i/2$ by the magnetic probes

Because of dependence of the plasma position and plasma current distribution to magnetic field distribution around the plasma, therefore magnetic pickup coils give us information about the plasma position. Therefore, the second relation for the plasma position [22] is

$$\Delta R = \frac{a^2}{4R_0} \times \left\{ \left(\frac{b^2}{a^2} - 1 \right) - 2\ln\frac{a}{b} \right\} + \frac{\pi b^2}{2\mu_0 I_p} \left\{ \Delta B_\theta \left(1 - \frac{a^2}{b^2} \right) - \Delta B_r \left(1 + \frac{a^2}{b^2} \right) \right\} \quad (21)$$

Where we use the quasi-cylindrical coordinate (r, θ, ϕ). Equations (20) and (21) are accurate for HT-7 Tokamak [10,13,20], (see Table I). The magnetic probe array is used to measure the poloidal magnetic field on HT-7 Tokamak [13,20]. There are 12 magnetic probes distributed equally in the circle-cross poloidal direction. Each probe can measure magnetic field in two directions, the poloidal direction and the radial direction. The plasma current is measured by the Rogowski coil [13,20].

The sum of the poloidal beta and half the plasma internal inductance, $\beta_p + l_i/2$, can be measured as follow [13,16-19] (Fig. 1-2, Ref. 13)

$$\beta_p + \frac{l_i}{2} = 1 + \ln\frac{a}{b} + \frac{\pi R_0}{\mu_0 I_0} \left(\langle B_\theta \rangle + \langle B_n \rangle \right), \quad (22)$$

where

$$\langle B_\theta \rangle = B_\theta(\theta = 0) - B_\theta(\theta = \pi), \quad (23)$$

$$\langle B_n \rangle = B_n\left(\theta = \frac{\pi}{2}\right) - B_n\left(\theta = \frac{3\pi}{2}\right), \quad (24)$$

We measured these local magnetic fields with magnetic probes [13] at above angles.

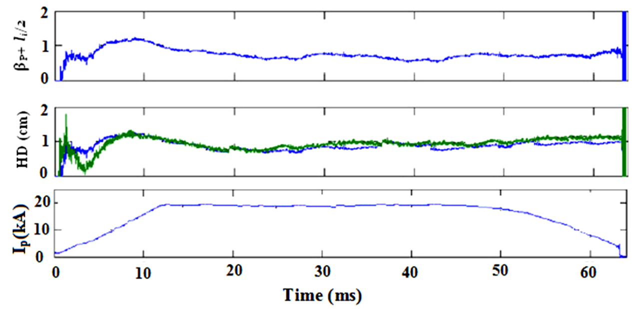

FIGURE 3. The calculated time evolution of the plasma displacement and $\beta_p + l_i/2$ for circular cross section HT-7 tokamak (green measured by discrete coils and blue calculated).

For circular cross section HT-7 tokamak (green measured by discrete coils and blue calculated).

We presented a method for the determination of the plasma displacement and $\beta_p + l_i/2$ for circular cross section HT-7 tokamak. The calculated time evolution of the poloidal beta and internal inductance is shown in Fig. 3.

From the Fig. 3, it can be seen that the calculated plasma displacement and $\beta_p + l_i/2$ inductance depend on the kind of the discharge or plasma current.

6. Conclusions

We present the plasma displacement and $\beta_p + l_i/2$ by the Simplest Grad–Shafranov Equation (GSE) solution using Solov'ev assumption for circular cross section HT-7 tokamak. Using diamagnetic and compensation loop, combining with poloidal magnetic probe array signals, plasma displacement and $\beta_p + l_i/2$ are measured. In this paper, theoretical and experimental results in determining plasma displacement and $\beta_p + l_i/2$ are presented. We have seen that the calculated plasma displacement and the calculated $\beta_p + l_i/2$ depend on the kind of discharge or plasma current.

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