Explicit classical solutions and comments on Higher-Derivative Klein-Gordon equation in (1+1)-D

A. Cherman
Fundacão Planetário da Cidade do Rio de Janeiro,
Rua Vice-Governador Rubens Berardo, 100, 22451-070, Rio de Janeiro, RJ, Brasil,
and Centro Brasileiro de Pesquisas Físicas – CBPF,
Rua Dr. Xavier Sigaud 150, 22290-180, Rio de Janeiro, RJ, Brasil,
e-mail: alexandre.cherman@planetario.rio.rj.gov.br

L.G. Ferreira Filho
FAT — UERJ/Resende,
Estrada Resende-Riachuelo s/n, 27523-000, Resende, RJ, Brasil,
e-mail: gonzaga@fat.uerj.br

L.L. Santos Guedes
Fundacão Planetário da Cidade do Rio de Janeiro,
e-mail: leandro.guedes@planetario.rio.rj.gov.br

J.A. Helayël-Neto
Centro Brasileiro de Pesquisas Físicas – CBPF,
e-mail: helayel@cbpf.br

Recibido el 30 de marzo de 2012; aceptado el 6 de junio de 2012

We present a simple and general method to solve higher-power D'Alembert equation in (1+1) dimensions for a set of completely general initial conditions. Explicit solutions are written down for the cases of $\Box^2$ and $\Box^3$ and the singularity structure is manifest through the derivative terms of some of the initial-condition functions. The explicit solutions found out may be useful if one wishes to device external sources which may suppress the non-physical modes introduced by the higher derivatives.

Keywords: Higher-derivative theories; 2D theories; ghosts.

PACS: 03.50.Kk; 03.65.Pm; 03.65.Fd

1. Introduction

Usually, Lagrangeans are built from fields and their first derivatives; but, in principle, one could propose actions based on the presence of higher-derivative terms (such mathematical investigations are believed to have started with Ostrogradsky [1]). With the birth of Quantum Mechanics, Podolsky *et al* [2-4] suggested higher derivatives as a mean to "generalize Maxwell-Lorentz's theory". In the first few years of quantum relativistic field theory (*circa* 1950), higher derivatives were viewed as a fundamental tool to resolve the ultraviolet problem (*e.g.* [5,6]). Pais and Uhlenbeck [7] showed that such solutions were not bound from below; later, Heisenberg [8] showed that negative energies could be eliminated, but this would result in negative-norm states, known as ghost states.

Ghost states were extensively discussed by Gavriedlies, Kuo and Lee [9]. A big step towards the reassesment of theories involving higher derivatives was made by Narnhofer and Thirring [10], in their study of the dipole. Their ideas were carried on by Englert, Karkowski and Rayski [11].

An interesting discussion about the weak field limit of a higher-dimensional Lagrangian in the gravitational context is presented by Giambiagi *et al* [12].

Higher derivatives appeared in gravitation (*e.g.* [13-16]) and within the two-dimensional picture (*e.g.* [17-20]). Higher derivatives were related to supergravity by Namazie [21] and Krasnikov *et al* [22]. Most recently, Ovrut *et al* [23-26] have studied gravitational theories of higher order in a superstring scenario. As for branes, higher derivative terms were considered by Nojiri and Odintsov [27], Neupane [28], Mukohyama [29] and Parry *et al* [30]. Nojiri [31] and Berredo-Peixoto and Shapiro [32] also made contributions to the study of gravitation with higher derivative terms. Nojiri *et al* [33] have studied particular solutions concerning black holes.

Our main goal is to study higher order Lagrangeans *per se*, investigating general analytical solutions for a mathematical equation and relating them to the physical reality. For example, R. Farias [34] has shown the necessary and suficient conditions for the existence of a Lagrangean, in field theory, with higher order derivatives. Pagani, Tecchiolli and Zerbini [35] focused on another aspect of the problem: the stability of such Lagrangeans. The canonical formalism was studied by Nakamura and Hamamoto [36]; its simmetries were addressed by Borneas and Damian [37] and again by Damian [38]. Regularization was revisited by Bakeyev and Slavnov [39] and again by Bakeyev [40].

In this paper we take the (1+1) dimensional limit and build general analytical solutions for higher order Lagrangeans, represented by the higher order D'Alembert equation ($\Box^2$ and $\Box^3$). In Sec. 2, we briefly discuss the general solution to the homogeneous D'Alembert equation with $\Box^2$. Next, in Sec. 3, we introduce an external source and the complete solution to the $\Box^2$-equation is exhibited. Sec. 4 is devoted to the presentation of the general solution to the homogeneous $\Box^3$-equation. The plots of specific solutions are collected in Sec. 5. Finally, in Sec. 6, we set up our Final Discussions.

2. The Homogeneous $\Box^2$-Equation

In order to solve the second order D'Alembert equation, $\Box^2\Phi(x;t) = 0$ in (1+1) dimensions, we first take the following coordinate transformation: $\xi = x - vt$ (right-movers) and $\eta = x+vt$ (left-movers), where ξ and η are the light-cone coordinates.

Taking $v = 1$, we must solve, then, the following equation:

$$\frac{\partial^4\tilde{\Phi}(\xi,\eta)}{\partial\xi^2\partial\eta^2} = 0, \tag{1}$$

The general solution is

$$\tilde{\Phi}(\xi,\eta) = f(\xi) + g(\eta) + \xi h(\eta) + \eta r(\xi),$$

with the following initial conditions:

$$F(x) = \Phi(x;0), \quad G(x) = \frac{d}{dt}\Phi(x;0),$$
$$H(x) = \frac{d^2}{dt^2}\Phi(x;0), \quad R(x) = \frac{d^3}{dt^3}\Phi(x;0).$$

The solution of this equation is:

$$\begin{aligned}\Phi(t;x) = \tilde{\Phi}(\xi;\eta) &= \frac{1}{2}F(\xi) + \frac{1}{2}F(\eta)\\ &- \frac{1}{8}(\xi-\eta)F'(\xi) + \frac{1}{8}(\xi-\eta)F'(\eta)\\ &+ \frac{1}{8}(\xi-\eta)G(\xi) + \frac{1}{8}(\xi-\eta)G(\eta)\\ &+ \frac{3}{4}\int_\xi^\eta dyG(y) - \frac{1}{8}(\xi-\eta)\int_\xi^\eta dyH(y)\\ &+ \frac{1}{8}(\eta-\xi)\Big[\Gamma(\eta)+\Gamma(\xi)\Big] - \frac{1}{4}\int_\xi^\eta \Gamma(y)dy,\end{aligned} \tag{2}$$

where

$$\Gamma(z) = \int \Omega(z)dz \ \text{ and } \ \Omega(z) = \int R(z)dz.$$

In order to better compare such result with the usual wave propagation (described by the D'Alembert equation), we take $H(x) = R(x) = 0$. By doing so, our "general" solution is now written:

$$\begin{aligned}\Phi(x;t) &= \frac{1}{2}F(x-t) + \frac{1}{2}F(x+t) - \frac{1}{4}tF'(x-t)\\ &+ \frac{1}{4}tF'(x+t) + \frac{1}{4}tG(x-t)\\ &+ \frac{1}{4}tG(x+t) + \frac{3}{4}\int_{x-t}^{x+t} dyG(y)\end{aligned} \tag{3}$$

The initial conditions F and G could be any continuous functions, and we take $G(x) = \Phi'(x,0) = 0$ and $F(x) = \Phi(x,0) = e^{-x^2}$ as an example:

$$\begin{aligned}\Phi(x,t) = \frac{1}{2}\Bigg\{&e^{-(x-t)^2} + e^{-(x+t)^2}\\ &-xt\Big[e^{-(x-t)^2} - e^{-(x+t)^2}\Big] + t^2\Big[e^{-(x-t)^2} - e^{-(x+t)^2}\Big]\Bigg\}.\end{aligned} \tag{4}$$

This solution heavily contrasts with the usual D'Alembert equation solution, which under the same initial conditions would read:

$$\Phi(x,t) = \frac{1}{2}e^{-(x-t)^2} + \frac{1}{2}e^{-(x+t)^2}. \tag{5}$$

The second and third terms of the right side of Eq. 4 shows a clear pathological behavior for large values of x and t. Such behavior could be corrected by external sources, as suggested by [10].

3. Solutions for External Sources

In solving the second order inhomogeneous D'Alembert equation, we use a method that is very straightforward and simple, heavily depending on the previous calculated solution of the homogeneous equation shown in the previous section.

As an example of this method, we show the solution of the inhomogeneous D'Alembert equation, which written in the light-cone variables and unitary propagation velocity, reads

$$\frac{\partial^2\tilde{\Phi}}{\partial\xi\partial\eta} = \tilde{J}(\xi,\eta).$$

The general solution of this equation can be written as a superposition of two different functions,

$$\tilde{\Phi}(\eta;\xi) = \tilde{\Phi}_h(\eta;\xi) + \tilde{\Phi}_J(\eta;\xi), \tag{6}$$

where $\tilde{\Phi}_h(\eta;\xi)$ is the usual homogeneous solution and $\tilde{\Phi}_J(\eta;\xi)$ is the particular solution and depends on the external source J. This latter function can be generally written as

$$\tilde{\Phi}_J(\xi;\eta) = \int_0^\xi d\alpha \int_0^\eta d\beta \tilde{J}(\alpha;\beta). \tag{7}$$

The usual initial conditions still apply:

$$\Phi(0;x) = F(x), \qquad \dot{\Phi}(0;x) = G(x).$$

We can easily rearrange Eq. (6) and, together with Eq. (7), write:

$$\begin{aligned}\Phi_h(t;x) =& \Phi(t;x) - \Phi_J(t;x)\\ \Phi_h(0;x) =& F(x) - \int_0^x d\alpha \int_0^x d\beta \tilde{J}(\alpha;\beta)\\ \Phi_h(0;x) =& \mathcal{F}(x).\end{aligned}$$

Analogously,

$$\dot{\Phi}_h(0;x) = G(x) + \int_0^x d\beta \tilde{J}(x;\beta) - \int_0^x d\alpha \tilde{J}(\alpha;x) = \mathcal{G}(x).$$

The problem of solving the inhomogeneous D'Alembert equation is then reduced to the usual problem of solving the homogeneous D'Alembert equation with new initial conditions $\mathcal{F}$ and $\mathcal{G}$. The general solution turns out to be:

$$\Phi(x;t) = \frac{1}{2}\mathcal{F}(x-t) + \frac{1}{2}\mathcal{F}(x+t) + \frac{1}{2}\int_{x-t}^{x+t} dy \mathcal{G}(y) + \int_0^{x-t} d\alpha \int_0^{x+t} d\beta \tilde{J}(\alpha;\beta). \quad (8)$$

Such method is easily extended to the second order case, where the differential equation, written in the light-cone variables and with unitary propagation velocity, reads

$$\frac{\partial^4 \tilde{\Phi}}{\partial \xi^2 \partial \eta^2} = \tilde{J}(\xi,\eta).$$

The particular solution for the inhomogeneous part of this equation is

$$\tilde{\Phi}_J(\xi,\eta) = \int_0^\xi d\alpha \int_0^\alpha d\beta \int_0^\eta d\gamma \int_0^\gamma d\delta \tilde{J}(\beta,\delta).$$

The general solution can be written based on Eq. (2) and the particular solution above:

$$\begin{aligned}\Phi(t;x) =& \tilde{\Phi}(\xi;\eta) = \frac{1}{2}\mathcal{F}(\xi) + \frac{1}{2}\mathcal{F}(\eta) - \frac{1}{8}(\xi-\eta)\mathcal{F}'(\xi)\\ &+ \frac{1}{8}(\xi-\eta)\mathcal{F}'(\eta) + \frac{1}{8}(\xi-\eta)\mathcal{G}(\xi) + \frac{1}{8}(\xi-\eta)\mathcal{G}(\eta)\\ &+ \frac{1}{8}(\xi-\eta)\mathcal{G}(\eta) + \frac{3}{4}\int_\xi^\eta dy \mathcal{G}(y)\\ &- \frac{1}{8}(\xi-\eta)\int_\xi^\eta dy \mathcal{H}(y) + \frac{1}{8}(\eta-\xi)\Big[\Gamma(\eta)+\Gamma(\xi)\Big]\\ &- \frac{1}{4}\int_\xi^\eta \Gamma(y)dy + \int_0^\xi d\alpha \int_0^\alpha d\beta \int_0^\eta d\gamma \int_0^\gamma d\delta \tilde{J}(\beta,\delta), \quad (9)\end{aligned}$$

where

$$\Gamma(z) = \int \Omega(z)dz, \qquad \Omega(z) = \int \mathcal{R}(z)dz$$

and

$$\begin{aligned}\mathcal{F}(x) =& F(x) - \int_0^x d\alpha \int_0^\alpha d\beta \int_0^x d\gamma \int_0^\gamma d\delta \tilde{J}(\beta,\delta)\\ \mathcal{G}(x) =& G(x) + \int_0^x d\beta \int_0^x d\gamma \int_0^\gamma d\delta \tilde{J}(\beta,\delta)\\ &- \int_0^x d\alpha \int_0^\alpha d\beta \int_0^x d\delta \tilde{J}(\beta,\delta)\\ \mathcal{H}(x) =& H(x) - \int_0^x d\gamma \int_0^\gamma d\delta \tilde{J}(x,\delta)\\ &+ 2\int_0^x d\beta \int_0^x d\delta \tilde{J}(\beta,\delta) - \int_0^x d\alpha \int_0^\alpha d\beta \tilde{J}(\beta,x)\\ \mathcal{R}(x) =& R(x) - \int_0^x d\gamma \int_0^\gamma d\delta \frac{\partial}{\partial x}\tilde{J}(x,\delta) - 3\int_0^x d\delta \tilde{J}(x,\delta)\\ &+ 3\int_0^x d\beta \tilde{J}(\beta,x) - \int_0^x d\alpha \int_0^\alpha d\beta \frac{\partial}{\partial x}\tilde{J}(\beta,x).\end{aligned}$$

In order to better compare this general solution with Eq. (4), we take $F(x) = \Phi(x,0) = e^{-x^2}$ and $G(x) = H(x) = R(x) = 0$. The ensuing solution was computed with the help of the MAPLE software and is too large to be reproduced here.

4. The Homogeneous $\Box^3$-Equation

The differential equation, written in the light-cone variables and with unitary propagation velocity, now reads

$$\frac{\partial^6 \tilde{\Phi}}{\partial \xi^3 \partial \eta^3} = 0,$$

with initial conditions given by

$$\begin{aligned}F(x) &= \Phi(x;0) & G(x) &= \frac{d}{dt}\Phi(x;0)\\ H(x) &= \frac{d^2}{dt^2}\Phi(x;0) & R(x) &= \frac{d^3}{dt^3}\Phi(x;0)\\ S(x) &= \frac{d^4}{dt^4}\Phi(x;0) & U(x) &= \frac{d^5}{dt^5}\Phi(x;0).\end{aligned}$$

The general solution can be written as

$$\tilde{\Phi}(\xi,\eta) = f(\xi) + g(\eta) + \xi h(\eta) + \eta r(\xi) + \xi^2 s(\eta) + \eta^2 u(\xi).$$

Solving for the six unknown functions, f, g, h, r, s and u, we get, for $\tilde{\Phi}$:

$$\begin{aligned}
\tilde{\Phi}(\xi,\eta) = -\frac{1}{128}\Bigg\{ & -100F(\xi) - 28F(\eta) + 48\xi F'(\xi) - 36\eta F'(\xi) - 12\xi F'(\eta) - 12\xi^2 F''(\xi) - 2\eta^2 F''(\xi) + 16\xi\eta F''(\xi) \\
& - 2\xi^2 F''(\eta) + 2\xi^2(\eta-\xi)F'''(\xi) + 16\xi G(\xi) + 4\eta G(\xi) - 20\xi G(\eta) + 16\xi(\eta-\xi)G'(\xi) + 2\eta^2 G(\xi) - 2\xi^2 G'(\eta) \\
& + 2\xi^2(\eta-\xi)G''(\xi) + 8\xi^2 H(\xi) - 16\xi\eta H(\xi) + 4\eta^2 H(\xi) + 4\xi^2 H(\eta) - 4\xi^2(\eta-\xi)H'(\xi) - 4\xi^2(\eta-\xi)R(\xi) \\
& - 12(\eta-\xi)\int_a^\xi zF'''(z)dz - 4\xi(\eta-\xi)\int_a^\xi zF''''(z)dz + 2(\eta-\xi)\int_a^\xi z^2F''''(z)dz - 68\int_\xi^\eta G(y)dy \\
& - 20(\eta-\xi)\int_a^\xi zG''(z)dz - 4\xi(\eta-\xi)\int_a^\xi zG'''(z)dz - 2(\eta-\xi)\int_a^\xi z^2G'''(z)dz + 8\xi\int_\xi^\eta H(y)dy \\
& + 40(\eta-\xi)\int_a^\xi H(z)dz + 8(\eta-\xi)\int_a^\xi zH'(z)dz + 8\xi(\eta-\xi)\int_a^\xi zH''(z)dz - 4(\eta-\xi)\int_a^\xi z^2H''(z)dz \\
& + 4\xi^2\int_a^\eta R(y)dy - 4\eta^2\int_a^\xi R(y)dy - 16\xi(\eta-\xi)\int_a^\xi R(y)dy + 24(\eta-\xi)\int_a^\xi zR(z)dz + 8\xi(\eta-\xi)\int_a^\xi zR'(z)dz \\
& - 4(\eta-\xi)\int_a^\xi z^2R'(z)dz + 2\xi(3\xi\eta - 2\xi^2 - \eta^2)\int_a^\xi S(z)dz + 2(3\xi^2 - 4\xi\eta + \eta^2)\int_a^\xi zS(z)dz \\
& + 2(\eta-\xi)\int_a^\xi z^2S(z)dz + \xi^2(\eta-\xi)^2\int_a^\xi U(z)dz - 2\xi(\eta-\xi)^2\int_a^\xi zU(z)dz + (\eta-\xi)^2\int_a^\xi z^2U(z)dz \\
& + 12\int_\xi^\eta\int_a^y zF'''(z)dzdy - 2\int_\xi^\eta\int_a^y z^2F''''(z)dzdy + 4\xi\int_\xi^\eta\int_a^y zF''''(z)dzdy + 20\int_\xi^\eta\int_a^y zG''(z)dzdy \\
& - 2\int_\xi^\eta\int_a^y z^2G'''(z)dzdy + 4\xi\int_\xi^\eta\int_a^y zG'''(z)dzdy - 40\int_\xi^\eta\int_a^y H(z)dzdy - 8\int_\xi^\eta\int_a^y zH'(z)dzdy \\
& + 4\int_\xi^\eta\int_a^y z^2H''(z)dzdy - 8\xi\int_\xi^\eta\int_a^y zH'(z)dzdy + 24\xi\int_\xi^\eta\int_a^y R(z)dzdy - 24\int_\xi^\eta\int_a^y zR(z)dzdy \\
& - 8(\eta-\xi)\int_a^\xi\int_a^y R(z)dzdy - 8\xi\int_\xi^\eta\int_a^y zR'(z)dzdy + 4\int_\xi^\eta\int_a^y z^2R'(z)dzdy - 4\xi(\eta-\xi)\int_a^\xi\int_a^y S(z)dzdy \\
& - 2\xi^2\int_\xi^\eta\int_a^y S(z)dzdy + 4\xi\int_\xi^\eta\int_a^y zS(z)dzdy + 4(\eta-\xi)\int_a^\xi\int_a^y zS(z)dzdy - 2\int_\xi^\eta\int_a^y z^2S(z)dzdy \\
& + 2\xi^2(\eta-\xi)\int_a^\xi\int_a^y U(z)dzdy - 4\xi(\eta-\xi)\int_a^\xi\int_a^y zU(z)dzdy + 2(\eta-\xi)\int_a^\xi\int_a^y z^2U(z)dzdy
\end{aligned}$$

$$+ 8\int_{\xi}^{\eta}\int_{a}^{y}\int_{a}^{z} R(w)dwdzdy + 4\xi\int_{\xi}^{\eta}\int_{a}^{y}\int_{a}^{z} S(w)dwdzdy - 4\int_{\xi}^{\eta}\int_{a}^{y}\int_{a}^{z} wS(w)dwdzdy$$

$$- 2\int_{\xi}^{\eta}\int_{a}^{y}\int_{a}^{z} w^2U(w)dwdzdy + 4\xi\int_{\xi}^{\eta}\int_{a}^{y}\int_{a}^{z} wU(w)dwdzdy - 2\xi^2\int_{\xi}^{\eta}\int_{a}^{y}\int_{a}^{z} U(w)dwdzdy\Bigg\}. \tag{10}$$

There is an apparent dependence on a new parameter, a. However, we actually show that, by differentiating the above expression with respect to a, we get a trivial result. This can be easily demonstrated with the help of MAPLE, for example. We diferentiated Eq. (10) with respect to a obtaining a vanishing result, proving it is a-independent.

Such a parameter appeared also in the solution of the second order D'Alembertian, but it was easily removed through a few algebraic steps. The sheer size of Eq. (10) makes this procedure a little more complicated and we opted for a numerical proof.

In order to better compare with previous results from Sec. 2, we take $G(x) = H(x) = R(x) = U(x) = S(x) = 0$ and $F(x) = \Phi(x,0) = e^{-x^2}$, ending up with:

$$\Phi(x,t) = \frac{1}{8}\Bigg\{4e^{-(x-t)^2} + 4e^{-(x+t)^2} + 5xt\Big[-e^{-(x-t)^2} + e^{-(x+t)^2}\Big] + 2x^2t^2\Big[e^{-(x-t)^2} + e^{-(x+t)^2}\Big] + 4t^2\Big[e^{-(x-t)^2} + e^{-(x+t)^2}\Big] + 4xt^3\Big[-e^{-(x-t)^2} + e^{-(x+t)^2}\Big] + 2t^4\Big[e^{-(x-t)^2} + e^{-(x+t)^2}\Big]\Bigg\}. \tag{11}$$

By inspecting the explicit homogeneous solutions worked out for $\Box^2$ and $\Box^3$, Eqs. (2) and (10) respectively, we see that there appear derivative terms of the initial conditions (F', in the case of $\Box^2$; F', F'', F''', G', G'' and H', in the case of $\Box^3$).

These terms clearly suggest, already at the level of the classical solutions, how these higher-derivative equations are in trouble with the particle interpretation at the second-quantised level. Actually, if we think of a classical pulse at x_0, described by

$$\Phi(0;x) = F(x) = \delta(x - x_0),$$

the evolved solution, $\Phi(t;x)$, will be plagued with the illness of the derivatives of the Dirac delta function.

Even classically, the derivatives of $\delta(x)$ cannot be interpreted as locally propagating excitations. Quantum-mechanically, as already known, higher derivatives potentially spoil the particle interpretation in that they introduce negative-norm states ("ghosts") in the spectrum, which appear as a consequence of a $\delta'(k^2)$ present in the spectral function.

5. Some Plottings

All the following graphs were made with MAPLE.

We first plot the temporal evolution of a signal described by the usual D'Alembert equation; using the initial conditions described in Sec. 2, we plot Eq. (5) in several distinct moments:

We obviously get the usual double-pulse propagation. Doing the same thing for the second order D'Alembertian, Eq. (4), we get:

The pathological behavior is evident as we see the signal propagating both ways and getting stronger as time goes by. The same behavior shows up when plotting the third order D'Alembertian, Eq. (11):

Once again, the pathological behavior is evident.

As mentioned in Sec. 3, such odd behavior could be, in principle, eliminated through the use of an external source.

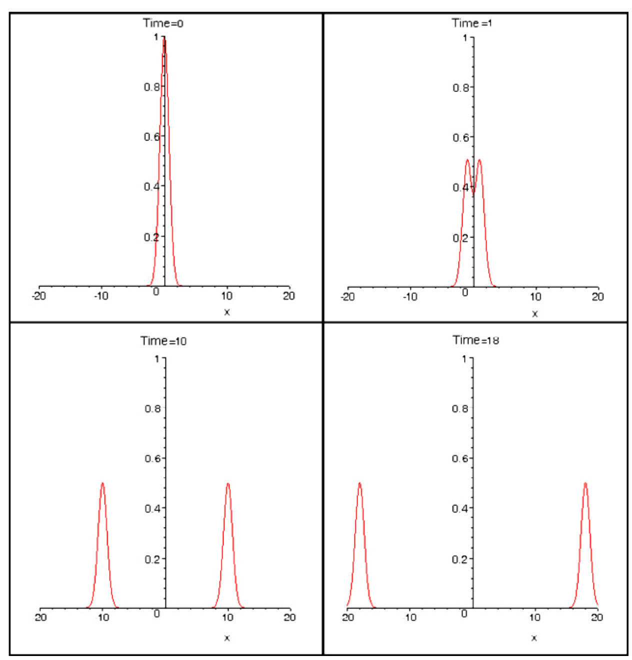

FIGURE 1. Temporal evolution of Eq. (5).

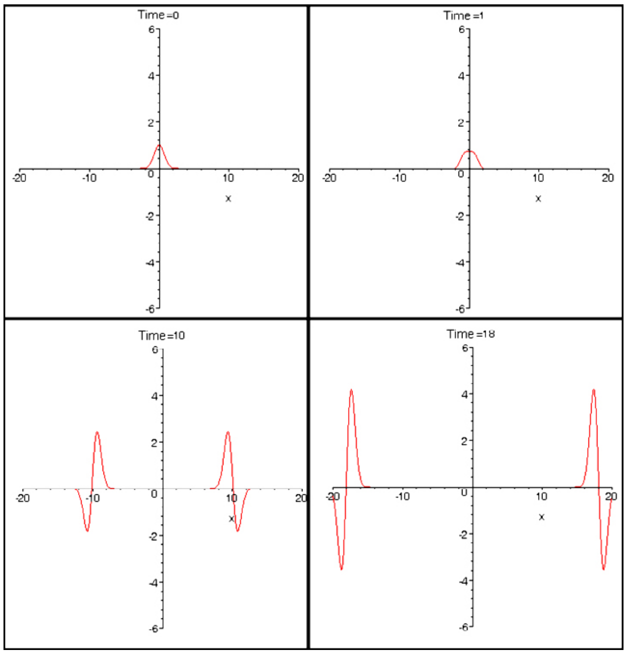

FIGURE 2. Temporal evolution of Eq. (4)

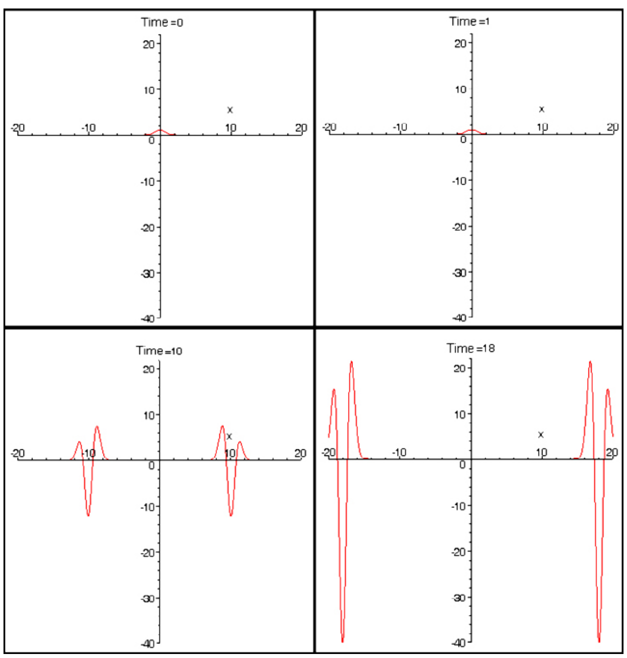

FIGURE 3. Temporal evolution of Eq. (11)

Such a general solution has been written, Eq. (9), but so far we have not been able to pinpoint a useful source or family of sources.

6. Final discussions

We have proposed a method to reduce the higher-order massless D'Alembert equation in (1+1) dimensions to a system of linear higher-order ordinary differential equations. The initial conditions may be completely general and the method works out if we adopt the light-cone coordinates. We have given here the explicit solutions for $\Box^n\Phi = 0$ with $n = 2, 3$.

The main idea behind our proposal was to have an explicit form for the general solution in order to have a clear evaluation of the pathologies that show up already at the classical approach. For example, the derivatives (or higher derivatives) of the functions describing the initial conditions already show us the drawbacks of a higher-order D'Alembertian equation to describe the propagation of sharp signals or localised particles. This becomes clear in our expression of $\Box^3\Phi = 0$, where a second derivative of F completely spoils the propagation of a pulse described at $t = 0$ by $\Phi(0; x) = \delta(x)$.

Also of interest is the case of an external source, J. The solution we have found may shed some light on the question related to the importance of this source to suppress the spurious behavior of the quantum excitations whenever the higher-derivative theory is quantised.

Now, we wish to go a step further and investigate the set of higher-order Dirac-like equations in (1+1)-D and to understand the interplay between algebraic chirality and the structure of left- and right-movers in the presence of higher derivatives.

This issue is under consideration and we shall report on it in a forthcoming publication.

1. M. Ostrogradsky, *Mémoire de l'Académie des Sciences de St. Petersburg* **6** (1850) 385.
2. B. Podolski, *Physical Review* **62** (1942) 68.
3. B. Podolski, and C. Kikuchi, *Physical Review* **65** (1944) 228.
4. B. Podolski, and P. Schwed, *Review of Modern Physics* **20** (1948) 40.
5. W. Thirring, *Physical Review* **77** (1950) 570.
6. H. J. Bhabha, *Physical Review* **77** (1950) 665.
7. A. Pais, and G. E. Uhlenbeck, *Physical Review* **79** (1950) 145.
8. W. Heisenberg, *Nuclear Physics* **4** (1957) 532.

9. A. Gravielides, T. K. Kuo, and S. Y. Lee, *Physical Review D* **13** (1976) 2912.
10. H. Narnhofer, and W. Thirring, *Physics Letters* **76B** (1978) 428.
11. B. -G. Englert, J. Karkowski, and J. M. Rayski Jr., *Physics Letters* **83B** (1979) 399.
12. J. J. Giambiagi, J. A. Nieto, and O. Obregón, *CBPF-Notas de Física* **53** (1994) and *Nuovo Cimento* **B110** (1995) 101-108.
13. A. Gravielides, T. K. Kuo, and S. Y. Lee, *Physical Review D* **12** (1975) 1829.
14. P. Havas, *General Relativity and Gravitation* **8** (1977) 631.
15. K. S. Stelle, *General Relativity and Gravitation* **9** (1978) 353.
16. C. Jankiewicz, *Acta Physica Polonica* **B12** (1981) 859.
17. S. N. Solodukhin, *Physical Review D* **51** (1995) 591.
18. S. Mignemi, and H.-J. Schmidt, *Classical and Quantum Gravity* **12** (1995) 849.
19. S. Naftulin, and S. D. Odintsov, *Modern Physics Letters* **A10** (1995) 2071.
20. A. Hindawi, B. A. Ovrut, and D. Waldram, *Nuclear Physics* **B471** (1996) 409.
21. M. A. Namazie, *Journal of Physics A* **13** (1980) 713.
22. N. V. Krasnikov, A. B. Kyiatkin, an E. R.d Poppitz, *Physics Letters B* **222** (1989) 66.
23. B. A. Ovrut, *Talk presented at STRING'95* (University of Southern California, EUA, 1995). hep-th/9506028.
24. A. Hindawi, B. A. Ovrut, and D. Waldram, *Progress of Theoretical Physics Supplement* **123** (1996) 397.
25. A. Hindawi, B. A. Ovrut, and D. Waldram, *Nuclear Physics* **B476** (1996) 175.
26. K. Forger, B. A. Ovrut, S. J. Theisen, and D. Waldram, *Physics Letters* **B388** (1996) 1539.
27. S. Nojiri, and S. D. Odintsov, *JHEP* **0007** (2000) 49.
28. I. P. Neupane, *JHEP* **0009** (2000) 40.
29. S. Mukohyama, *Physical Review D* **65** (2002) 084036.
30. M. Parry, S. Pichler, and D. Deeg, (2005). hep-ph/0502048.
31. S. Nojiri, *Gravitation and Cosmology* **9** (2003) 71.
32. G. Berredo-Peixoto, and I. L. Shapiro, *Physical Review D* **71**(2005) 064005.
33. Nojiri, S.; Odintsov, S. D. and Ogushi, S. *Physical Review D* **65** (2002) 023521.
34. J. Ronald Farias, *Hadronic Journal* **8** (1985) 227.
35. E. Pagani, G. Tecchiolli, and S. Zerbini, *Letters in Mathematical Physics* **14** (1987) 311.
36. T. Nakamura, and S. Hamamoto, *Progress of Theoretical Physics* **95** (1996) 469.
37. M. Borneas, and I. Damian, *International Journal of Theoretical Physics* **38** (1999) 2241.
38. I. Damian, *International Journal of Theoretical Physics* **39** (2000) 2141.
39. T. D. Bakeyev, and A. A. Slavnov, *Modern Physics Letters* **A11** (1996) 1539.
40. T. D. Bakeyev, *Theoretical and Mathematical Physics* **122** (2000) 355.