Geometrical thick branes in 5D Weyl gravity

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In our proposal the Riemannian structure of the 5D space-time is replaced by a Weyl integrable one, which allows for variations in the length of vectors during parallel transport. The conformal technique is used to find a solution that respects 4D Poincaré invariance. This solution represents a set of thick branes that do not respect Z_2 symmetry and generalises, in this way, the Randall-Sundrum solution. In our set-up the non symmetric branes are constructed from the scalar field and we have no need of introducing them by hand as delta functions in the action (as in the Randall-Sundrum case). We examine the fluctuations of the metric around the background solution. The equation for the transverse traceless modes of the fluctuations that decouple from the scalar field supports a naturally massless and normalizable 4D graviton. Thus, our model reproduces conventional 4D gravity. The spectrum of the Kaluza-Klein modes turns out to be continuous (as in the Randall-Sundrum case).

Keywords: Localization of 4D gravity; thick branes; 5D Weyl manifold.

En el presente trabajo, la estructura Riemanniana del espacio-tiempo en 5D es reemplazada por una variedad integrable de Weyl que permite variaciones en la longitud de vectores durante el transporte paralelo de los mismos. A continuación se utiliza la técnica conforme para hallar una solución que preserva la invariancia de Poincaré. Dicha solución representa una familia de membranas anchas que violan la simetría Z_2 y generalizan el modelo Randall–Sundrum. En este modelo, las membranas anchas son construidas a partir del campo escalar, evitando, de este modo, la introducción de las membranas delgadas (funciones delta) en la acción, como en el caso Randall–Sundrum. Posteriormente se estudian las fluctuaciones de la métrica en torno a la solución clásica. La ecuación de los modos transversos de traza nula de dichas fluctuaciones, mismas que se desacoplan de las del campo escalar, permiten la existencia de un gravitón en 4D naturalmente normalizable y sin masa. De este modo, nuestro modelo reproduce la gravedad convencional en 4D. El espectro de modos masivos de Kaluza-Klein resulta ser continuo como en el caso Randall–Sundrum.

Descriptores: Localización de la gravedad en 4D; membranas anchas; variedades de Weyl en 5D.

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1. Introduction

During last years it has shown an increasing interest in space times with large extra dimensions since in these models gravity propagates in all dimensions while matter is confined to a 4D submanifold (a 3–brane) with no contradiction with recent gravitational experiments [1, 2]. In the framework of brane scenarios in 5D space time it has been shown a path towards the solution of some relevant problems of high–energy physics. In particular, it was discovered that in such brane scenarios 4D gravity can be realized consistently and it is feasible to live in $4+1$ non–compact dimensions in perfect compatibility with experimental gravity [3,4]. Since then, several generalizations of these scenarios have been constructed with the aid of thick branes [5-9].

In this frame, we begin by studying a 5D Weyl gravity model in which branes arise naturally without introducing them by hand in the action of the theory. We implement the conformal technique to obtain a classical solution that respects 4D Poincaré invariance and represents a localized function with no reflection Z_2–symmetry. By looking at the energy density of the scalar field of this solution we interpret the field configuration as a set of non Z_2–symmetric thick branes. We investigate as well the behaviour of the curvature scalar and make an analysis of the fluctuations of the metric around the classical background solution in order to know whether 4D gravity can be described in our setup. We found that this is the case since the quantum mechanical problem with a potential well which vanishes asymptotically for the transverse traceless sector of the fluctuations of the metric yields a continuum spectrum of KK–states with a zero mode that corresponds to the normalizable, stable 4D graviton.

2. The model

Let us start by considering a pure geometrical Weyl action in five dimensions. This non–Riemannian generalization of the Kaluza–Klein theory is given by

$$S_5^W = \int\limits_{M_5^W} \frac{d^5x\sqrt{|g|}}{16\pi G_5} e^{\frac{3}{2}\omega}[R + 3\tilde{\xi}(\nabla\omega)^2 + 6U(\omega)], \quad (1)$$

where M_5^W is a Weyl manifold specified by the pair (g_{MN}, ω), g_{MN} being the metric and ω a Weyl scalar function. The Weylian Ricci tensor reads

$$R_{MN} = \Gamma^A_{MN,A} - \Gamma^A_{AM,N} + \Gamma^P_{MN}\Gamma^Q_{PQ} - \Gamma^P_{MQ}\Gamma^Q_{NP},$$

where

$$\Gamma^C_{MN} = \{^{\,C}_{MN}\} - \frac{1}{2}\left(\omega_{,M}\delta^C_N + \omega_{,N}\delta^C_M - g_{MN}\omega^{,C}\right)$$

are the affine connections on M_5^W, $\{{}^{\,C}_{MN}\}$ are the Christoffel symbols and $M,N = 0,1,2,3,5$; the constant $\tilde{\xi}$ is an arbitrary coupling parameter, and $U(\omega)$ is a self–interaction potential for the scalar field ω. This action is of pure geometrical nature since the scalar field ω enters in the definition of the affine connections of the Weyl manifold and, thus, cannot be discarded in principle from our consideration. Apart from the self–interaction potential, the action (1) is invariant under Weyl rescalings

$$g'_{MN} \rightarrow \Omega^{-2} g_{MN}, \qquad \omega' \rightarrow \omega + \ln \Omega^2,$$
$$\tilde{\xi}' \rightarrow \tilde{\xi}/(1+\partial_\omega \ln \Omega^2)^2, \tag{2}$$

where Ω^2 is a smooth function on M_5^W. It follows from these relations that the potential must undergo the transformation $U' \rightarrow \Omega^2 U$ in order to keep such an invariance. Thus, $U(w) = \lambda e^{\omega}$, where λ is a constant parameter, is the form of the potential which preserves the scale invariance of the Weyl manifold (1). When this invariance is broken, the Weyl scalar field is transformed into a degree of freedom which models the thick branes.

In order to find the solutions of the theory with 4D Poincaré invariance we consider the following ansatz for the line element

$$ds_5^2 = e^{2A(y)} \eta_{mn} dx^m dx^n + dy^2, \tag{3}$$

where $e^{2A(y)}$ is the warp factor, $m,n = 0,1,2,3$ and y is the extra coordinate.

The 5D stress–energy tensor is given by its 4D and pure 5D components

$$T_{mn} = \frac{1}{8\pi G_5} e^{2A} [3A'' + 6(A')^2] \eta_{mn},$$
$$T_{55} = \frac{6}{8\pi G_5} (A')^2. \tag{4}$$

here the comma denotes derivatives with respect to the fifth coordinate y.

3. The solution

To find a solution we shall use the conformal technique: by means of a conformal transformation we jump from the Weyl frame to the Riemann one, to find solutions to our system, then return to the Weyl frame. We perform the conformal transformation $\widehat{g}_{MN} = e^{\omega} g_{MN}$, mapping the Weylian action (1) into the Riemannian one

$$S_5^R = \int_{M_5^R} \frac{d^5x\sqrt{|\widehat{g}|}}{16\pi G_5} [\widehat{R} + 3\xi(\widehat{\nabla}\omega)^2 + 6\widehat{U}(\omega)], \tag{5}$$

where $\xi = \tilde{\xi} - 1$, $\widehat{U}(\omega) = e^{-\omega} U(\omega)$ and all hatted magnitudes and operators are defined in the Riemann frame. In this frame we have a theory which describes 5D gravity minimally coupled to a scalar field which possesses a self–interaction potential. After this transformation, the line element (3) yields the Riemannian metric

$$\widehat{ds}_5^{\,2} = e^{2\sigma(y)} \eta_{nm} dx^n dx^m + e^{\omega(y)} dy^2, \tag{6}$$

where $2\sigma = 2A + \omega$. Further, by following [8] we introduce the new variables $X = \omega'$ and $Y = 2A'$ and get the following pair of coupled field equations from the action (5)

$$X' + 2YX - \frac{3}{2}X^2 = \frac{1}{\xi}\frac{d\widehat{U}}{d\omega} e^{-\omega},$$
$$Y' + 2Y^2 - \frac{3}{2}XY = \left(\frac{1}{\xi}\frac{d\widehat{U}}{d\omega} + 4\widehat{U}\right) e^{-\omega}. \tag{7}$$

In general, it is not trivial to fully integrate these field equations. Under some assumptions, it is straightforward to construct several particular solutions to them. However, quite often such solutions lead to expressions of the dynamical variables that are too complicated for an analytical treatment in closed form.

As pointed out in Ref. 8, this system of equations can be easily solved if one uses the condition $X = kY$, where k is an arbitrary constant parameter.After imposing these conditions and setting $\xi = 1 - k/4k$, the field equation (9) simplifies to

$$Y' + \frac{4-3k}{2} Y^2 = \frac{4\lambda}{1-k}. \tag{8}$$

This choice accounts to having a self–interaction potential $U = \lambda e^{2\omega}$ in the Weyl frame, which indeed breaks the invariance of the action (1) under Weyl rescalings. It turns out that this restriction leads to a Riemannian potential of the form $\widehat{U} = \lambda e^{(4k\xi/1-k)\omega}$. Thus, under these conditions, both field equations in (7) reduce to a single differential equation

$$Y' + \frac{4-3k}{2} Y^2 = \frac{4\lambda}{1-k} e^{(\frac{4k\xi}{1-k}-1)\omega}. \tag{9}$$

By solving (9) we find the following solution

$$e^{2A(y)} = k_3 \left(e^{ay} + k_1 e^{-ay}\right)^b,$$
$$\omega = \ln\left[k_2 \left(e^{ay} + k_1 e^{-ay}\right)^{kb}\right], \tag{10}$$

where

$$a = \sqrt{\frac{4-3k}{1-k} 2\lambda}, \qquad b = \frac{2}{4-3k}, \tag{11}$$

and k_1, k_2 and k_3 are arbitrary constants.

This represents a solution which does not respect Z_2–symmetry ($y \longrightarrow -y$) due to the presence of the constant parameter k_1. If we look at the particular case when $k_1 = 1$, $k_2 = 2^{-kb}$ and $k_3 = 2^{-b}$ we recover the Z_2–symmetric solution previously found by [8] in the Weyl frame and by [5]– [6] in the Riemann one:

$$e^{2A(y)} = [\cosh(ay)]^b, \qquad \omega = bk \ln[\cosh(ay)]. \tag{12}$$

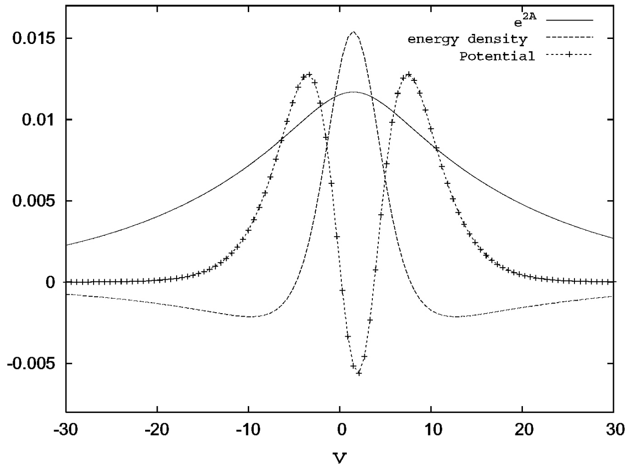

FIGURE 1. The shape of the warp factor e^{2A}, The energy density function μ and the shape of the potential a volcano with finite bottom, all of them reescaled.

4. Physics of the system

Let us turn to study the physics of the found solution in the Weyl frame, we first analyze the behaviour of the warp factor, the energy density of the scalar field, the curvature scalar of the system as well as the metric fluctuations.

The warp factor and the scalar field depends on the values of the constants k_1, k_2, k_3, a, b. Thus, we have a family of solutions depending on the values of the parameters k_1, λ and k. We look for cases where the warp factor is a localized, smooth and well behaved function which in fact models the fifth dimension. By taking into account this fact, we have the following cases of interest:

A) $\lambda > 0$, $k > 4/3$ and $k_1 > 0$. In this case the domain of the fifth coordinate is $-\infty < y < \infty$; thus, we have a non–compact manifold in the extra dimension. It turns out that in this case the warp factor represents a smooth localized function but non Z_2–symmetric characterized by the width parameter $\Delta \sim 1/a$ (see Fig. 1).

It is easy to see that k_1 measures the Z_2–asymmetry of the solution through a shift along the positive axis of the extra coordinate given by the value y_0. Since this constant appears multiplying an exponential function of y, its effect is quite small and, hence, the solution slightly deviates from the Z_2–symmetric one. However, the physical implications of this fact are quite important, namely, the 5D space time is not restricted to be an orbifold geometry, allowing for a more general kind of dimensional reductions when going down to four dimensions.

The energy density of the scalar matter is given by

$$\mu(y) = \frac{-6a^2bk_3}{8\pi G_5}(e^{ay} + k_1 e^{-ay})^{b-2} \times \left[1 - \frac{b}{4k_1}(e^{ay} - k_1 e^{-ay})^2\right]. \quad (13)$$

This function shows two negative minima and a positive maximum between them and, finally, it vanishes asymptotically for $y = \pm\infty$. It is displayed in Fig. 1.

The 5-dimensional curvature scalar reads

$$R_5 = \frac{-16a^2bk_1}{(e^{ay} + k_1 e^{-ay})^2} \times \left[1 + \frac{5b}{16k_1}(e^{ay} - k_1 e^{-ay})^2\right]. \quad (14)$$

We can see that this quantity is always bounded, thus, in contrast with the singular manifold that arises in the Riemann frame, we have a 5D manifold that is regular in the Weyl frame.

B) $k_1 > 0$, $\lambda > 0$ and $1 < k < 4/3$. In this case we have $a \in \Im$, $b > 0$ and we must replace $a \to i\alpha$ in order to have a real warp factor. Moreover, the only possible choice for the parameter k_1 is $k_1 = 1$ (otherwise the solution becomes complex), and we get a Z_2–symmetric function

$$e^{2A(y)} = k_3 \cos^b(\alpha y). \quad (15)$$

Thus, this represents a manifold which is periodic in the extra dimension, so $-\pi \leq \alpha y \leq \pi$, and we have the same compact case that was obtained in Ref. 8.

Other cases of physical interest are contained in A) and B). The remaining possible values of these parameters lead to unphysical situations in which the warp factor and the scalar energy density are singular at certain values of the fifth dimension y and, hence, do not represent localized functions.

5. Fluctuations of the metric

Let us turn to study the metric fluctuations h_{mn} of the metric (3) given by the perturbed line element

$$ds_5^2 = e^{2A(y)}[\eta_{mn} + h_{mn}(x,y)]dx^m dx^n + dy^2. \quad (16)$$

Even if one cannot avoid considering fluctuations of the scalar field when treating fluctuations of the background metric, in Ref. 5 it was shown that the transverse traceless modes of the metric fluctuations decouples from the scalar sector and hence, can be approached analytically.

By following this method, we perform the coordinate transformation $dw = e^{-A}dy$, which leads to a conformally flat metric and to the following wave equation for the transverse traceless modes h^T_{mn} of the metric fluctuations

$$(\partial_w^2 + 3A'\partial_w + \Box^\eta)h^T_{mn} = 0. \quad (17)$$

This equation supports a massless and normalizable 4D graviton given by $h^T_{mn} = C_{mn}e^{imx}$, where C_{mn} are constant parameters and $m^2 = 0$.

In Ref. 3 it was proved useful to recast equation (17) into Schrödinger's equation form. In order to accomplish this, we adopt the following ansatz for the transverse traceless modes of the fluctuations $h^T_{mn} = e^{imx}e^{-3A/2}\Psi_{mn}(w)$ and get

$$[\partial_w^2 - V(w) + m^2]\Psi = 0, \tag{18}$$

where we have dropped the subscripts in Ψ, m is the mass of the KK excitation, and the potential reads

$$V(w) = \frac{3}{2}\partial_w^2 A + \frac{9}{4}(\partial_w A)^2. \tag{19}$$

The shape of the potential $V(r)$ for the case A) is given by the following expression:

$$V(w(y)) = 3a^2bk_1(e^{ay} + k_1e^{-ay})^{b-2} \times \left[1 + \frac{5b}{16k_1}(e^{ay} - k_1e^{-ay})^2\right] \tag{20}$$

By looking at Fig. 1 we see that this potential represents a well with a finite negative minimum at a certain value y_0, which is located between two positive maxima (potential barriers) and then vanishes as $y = \pm\infty$ (a volcano potential with finite bottom).

In the particular case $k = 5/3$ (hence $b = -2$), the coordinate transformation can be successfully inverted and we get $A(w) = -\ln\left[\left(a^2k_3w^2 + 4k_1\right)/k_3\right]/2$ which yields a potential of the form

$$\widehat{V}(r) = \frac{3a^2k_3\left(5a^2k_3w^2 - 8k_1\right)}{4\left(a^2k_3w^2 + 4k_1\right)^2}. \tag{21}$$

In the Schrödinger equation, the spectrum of eigenvalues m^2 parameterizes the spectrum of graviton masses that a 4D observer located at w_0 sees. It turns out that for the zero mode $m^2 = 0$, this equation can be solved. The only normalizable eigenfunction reads

$$\Psi_0 = q\left[c_1^4(w - w_0)^2 + 6\lambda\right]^{-3/4},$$

where q is a normalization constant. This function represents the lowest energy eigenfunction of the Schödinger equation (18) since it has no zeros. This fact allows for the existence of a 4D graviton with no instabilities from transverse traceless modes with $m^2 < 0$. In addition to this massless mode, there exists a tower of higher KK modes with positive $m^2 > 0$.

Thus, we have obtained Weylian thick brane generalizations of the RS model with no reflection symmetry imposed in which the 4D effective theory possesses an energy spectrum quite similar to the spectrum of the thin wall case, in particular, 4D gravity turns out to be localized at a certain value of the fifth dimension.

6. Concluding Remarks

By means of the conformal technique we obtained a 4D Poincaré invariant solution which represents a well behaved localized function and does not respect Z_2–symmetry along the extra dimension. Thus, we have obtained a thick brane generalization of the Randall–Sundrum model. This field configuration does not restrict the 5–dimensional space time to be an orbifold geometry. When we set the parameter $k_1 = 1$, our solution reproduces the Z_2–symmetric solutions previously found in the literature in both the Riemann and the Weyl frame. By looking at the scalar energy density μ of our field configuration, we see that it shows a thick brane with positive energy density centered at y_0 and accompanied by a small amount of negative energy density at each side.

The scalar curvature of the Riemannian manifold turns out to be singular for the found solution, whereas the corresponding quantity in the Weyl integrable geometry presents a regular behaviour along the whole fifth dimension.

By studying the behaviour of the transverse traceless modes of the fluctuations of the metric we recast their equations into a one dimensional Schödinger equation with a quantum mechanical potential that represents a volcano with finite bottom. We solve the Schödinger equation for the massless zero mode $m^2 = 0$ obtaining a single bound state which represents the lowest energy eigenfunction, allowing for the existence of a 4–dimensional graviton with no instabilities from transverse traceless modes with $m^2 < 0$. We also get a huge tower of higher KK states with positive $m^2 > 0$ that are suppressed at y_0, turning into continuum plane wave modes as y approaches spatial infinity [4, 5]. Since all values of m^2 are allowed, the spectrum turns out to be continuum and gapless.

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