Schrödinger's Born-Infeld representation, the non Abelian case

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We propose a non-Abelian Born-Infeld theory based on an Abelian theory by Erwin Schrödinger that, as he showed, is equivalent to Born-Infeld theory. Its construction does not require at any stage the square root structure that characterizes the Dirac-Born-Infeld (DBI) action. Various non-Abelian generalizations are possible. We focus our attention, in this work, in one of them. For it, it is shown that Instantons solutions exist.Our formalism could be of interest in connection with string theory and possible extensions of well known physical results in the usual Born-Infeld Abelian case.

Keywords: Born-Infeld; Non-Abelian.

Se propone una teoría no-Abeliana de Born-Infeld basada en una teoría Abeliana de Erwin Schrödinger que, como él lo ha mostrado, es equivalente a la teoría propuesta por Born e Infeld. Su construcción no requiere en ninguna etapa de la estructura de raíz cuadrada que caracteriza la acción Dirac-Born-Infeld (DBI). Varias generalizaciones no Abelianas son posibles; nos centramos en este trabajo en una de ellas. Para esto, se muestra que las soluciones de Instantones existen. Nuestro formalismo puede ser de interés en conexión con teoría de cuerdas y posibles extensiones de resultados físicos bien conocidos en el caso de Born-Infeld Abeliano usual.

Descriptores: Born-Infeld; no-Abeliano.

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Seventy years ago Erwin Schrödinger wrote a paper entitled Contributions to Born's New Theory of the Electomagnetic Field [1]. As is known and he himself pointed out the classical (Dirac-Born-Infeld,DBI) Born's theory [2–6] can be constructed by means of the two vectors $\mathbf{B}$ and $\mathbf{E}$, the magnetic induction and the electric field-strength respectively. The partial derivatives of the Lagrangian with respect to the components of $\mathbf{B}$ and $\mathbf{E}$ define a second pair of vectors correspondingly $\mathbf{H}$, the magnetic field and $-\ \mathbf{D}$, the electric displacement. It was already shown by Born that one can choose four different ways to write a Lagrange function in terms of one of the magnetic vectors with one of the electric vectors. For each of these theories, the Lagrangians have essentially the same structure.

Schrödinger proposed a theory whose structure is entirely different from the above mentioned. He used two complex combinations of $\mathbf{B}$, $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{D}$ as independent variables

$$\mathbf{\Omega} = \mathbf{B} - i\mathbf{D}\ ,\ \mathbf{\Sigma} = \mathbf{E} + i\mathbf{H}, \tag{1}$$

and constructed a Lagrangian in such a way that the complex conjugate of one of these variables is identical with the partial derivative of the Lagrangian with respect to the other one. This he called the condition of conjugateness. The Lagrangian is

$$\mathcal{L} = \frac{\mathbf{\Omega}^2 - \mathbf{\Sigma}^2}{\mathbf{\Omega} \cdot \mathbf{\Sigma}}\ \ . \tag{2}$$

In this Lagrangian the square root structure typical of the Dirac-Born-Infeld action has disappeared. The Lagrangian results are rational and homogeneous of the zeroth degree.

Schrödinger showed that the classical treatment of the Lagrangian (2) is entirely equivalent to Born's theory (DBI), this will be shown below. He then writes *consequently it cannot provide us with any new insight, which could not, virtually, be derived from Born's original treatment as well*. He recognized that for practical calculations, however, the imaginary vectors structure will be hard to handle with. Quoting, once more Schrödinger: *yet for certain theoretical considerations of a general kind I am inclined to consider the present treatment as the standard form on account of its extremal simplicity, the Lagrangian being simply the* $ratio$ *of the two invariants, whereas in Maxwell's theory it was equal to one of them*.

In this work we will present a generalization of Schrödinger's Lagrangian (2) to a non-Abelian gauge theory. The complexification of the fields will provide us with a direct clue to find the non-Abelian framework based on the Abelian Schrödinger's representation. As in the standard Yang-Mills theories, this theory admits instantons solutions. This work, however, is also partially motivated by the fact that in the framework of string theory, the possibility to define a non-Abelian generalization of the standard Dirac-Born-Infeld bosonic and/or supersymmetric actions [7] has been extensively explored beginning in 1990 [8]. If this theory could be constructed, it should represent the world-volume U(N) gauge theory that arises when one has N coincident type II Dp branes. The symmetrized trace prescription proposed by Tseytlin [9] seems to be correct up to terms of the order F^4, but if fails at higher orders [10]. Also in the supersymmetric setup certain terms cannot be expressed in terms of symmetrized traces [11]. Lacking a general rule to construct the non-Abelian Born-Infeld theory, our formulation represents different alternatives to be explored in future works, as it has been the case with previous proposals and also in connection with the generalization of classical results in the standard Born-Infeld Abelian theory.

At this stage we only show one consistent and relatively straightforward way to generalize Schrödinger's classical representation of the DBI action to non-Abelian gauge field theories and analyze some of its above mentioned physical properties. It will be, as in the Abelian case, the ratio of two invariants and does not have the usual form of a square root [12]. The formal structure is simple and provides us with a different starting point to investigate another non-Abelian generalization of the Dirac-Born-Infeld action. We will make a brief comment on other possible ways to generalize Schrödinger's construction.

We will begin by reviewing the main aspects of Schrödinger's proposal. His Lagrangian is then written in terms of the usual tensorial formulation in Electrodynamics. Next we present our non-Abelian proposal, for which, as in the Abelian case the corresponding condition of conjugateness will allow us to identify complex fields and reduce them to the number of the usual physical fields of the corresponding non-Abelian gauge theory. As in the usual gauge theories, the proposed action would have instantons solutions. We conclude with a few remarks.

As already mentioned in the Introduction, Schrödinger's proposal begins by postulating the Lagrangian (2) (the singular case $\mathbf{\Omega}\cdot\mathbf{\Sigma}=0$ is discussed in Ref. 1). The complex combinations (1) are considered as independent variables but such that their complex conjugates denoted by $*$ fulfill

$$\mathbf{\Omega}^* = \frac{\partial\mathcal{L}}{\partial\mathbf{\Sigma}} = -\frac{2\mathbf{\Sigma}}{\mathbf{\Omega}\cdot\mathbf{\Sigma}} - \frac{\mathbf{\Omega}^2-\mathbf{\Sigma}^2}{(\mathbf{\Omega}\cdot\mathbf{\Sigma})^2}\mathbf{\Omega}, \qquad (3)$$

$$\mathbf{\Sigma}^* = \frac{\partial\mathcal{L}}{\partial\mathbf{\Omega}} = \frac{2\mathbf{\Omega}}{\mathbf{\Omega}\cdot\mathbf{\Sigma}} - \frac{\mathbf{\Omega}^2-\mathbf{\Sigma}^2}{(\mathbf{\Omega}\cdot\mathbf{\Sigma})^2}\mathbf{\Sigma},$$

this he called the condition of conjugateness.

Schrödinger then remarks that to get the field-equations corresponding to (2) one should not pay attention to the relation (1), but actually consider $\mathbf{\Omega}$ and $\mathbf{\Sigma}$ as fundamental variables. He then, assumes (as in Born's theory and, as well known, in Maxwell's theory) that the six complex vector $\mathbf{\Omega}$ and $\mathbf{\Sigma}$ is the four-dimensional curl of a potential four-vector, and consequently only its four components are to be varied independently. This is equivalent to assume that the field equations are

$$\nabla\times\mathbf{\Sigma} + \frac{\partial\mathbf{\Omega}}{\partial t} = 0, \quad \nabla\cdot\mathbf{\Omega} = 0. \qquad (4)$$

Then, using the conjugateness condition one can obtain by variation in the usual way

$$\nabla\times\mathbf{\Sigma}^* + \frac{\partial\mathbf{\Omega}^*}{\partial t} = 0, \quad \nabla\cdot\mathbf{\Omega}^* = 0. \qquad (5)$$

It is also shown, using (3), that $\mathcal{L}$ becomes purely imaginary and is also equal to

$$\mathcal{L} = -\frac{\mathbf{\Omega}^{*2}-\mathbf{\Sigma}^{*2}}{\mathbf{\Omega}^*\cdot\mathbf{\Sigma}^*} \ . \qquad (6)$$

The stress energy momentum tensor is calculated and it is proved that there always exists a Lorentz frame in which all the four composing three vectors are parallel in a certain world point. Further simplification is obtained by making use of the fact that the six components of $\mathbf{\Omega}$ and $\mathbf{\Sigma}$ can be multiplied by a factor $e^{i\gamma}$, γ real. It is called the γ -transformation. It does not interfere with the conjugateness condition, for in (3) the right-hand sides take the factor $e^{-i\gamma}$. The numerical values of the Lagrangian (2) remain unmodified as well as those of the stress-energy-momentum tensor components. The application to (4) and (5) with $\gamma = const.$ produces another solution, though with the same energy, momentum and stress densities as before in every world point.

A consequence of (3) is

$$\mathbf{\Omega}^*\cdot\mathbf{\Sigma} + \mathbf{\Sigma}^*\cdot\mathbf{\Omega} = 0. \qquad (7)$$

Making use of the above mentioned Lorentz transformation one can make all components vanish except, say, Ω_1 and Σ_1 and choose γ so as to make Ω_1 real. Through the relation (7) one can write

$$\Omega_1(\Sigma_1+\Sigma_1^*) = 0, \qquad (8)$$

Σ_1 results imaginary and can be put as

$$\Sigma_1 = iA\Omega_1, \qquad (9)$$

where A is a real constant. By substitution in (3) it is easily seen that the only allowed expressions for Ω_1 and Σ_1 are

$$\Omega_1 = \frac{\sqrt{1-A^2}}{A}, \quad \Sigma_1 = i\sqrt{1-A^2}, \qquad (10)$$

A takes values from -1 to $+1$ and the positive sign of the square root should be taken. This is called the "standard field". It is purely magnetic field with permeability A^{-1}, a purely electric field with dielectric constant A^{-1} can be obtained by a $\gamma-$ transformation. This standard field does not require then a further γ-transformation, but only a Lorentz transformation would be necessary to obtain the most general field. The Lagrangian for the standard case results then in

$$\mathcal{L} = -i\frac{1+A^2}{A} \ . \qquad (11)$$

The identity with Born's theory (DBI) is not fully performed in Schrödinger's work, it is mentioned that the condition of conjugateness (3) is equivalent to relations (12) (see below). This can easily be corroborated. The rest of the procedure is, according with the footnote in page 472, as follows; he refers us to Born-Infeld paper [5] and makes two corrections to misprints, there

$$\mathbf{H} = \frac{\partial L}{\partial\mathbf{B}} = \frac{\mathbf{B}-G\mathbf{E}}{\sqrt{1+F-G^2}}, \qquad (12)$$

$$\mathbf{D} = -\frac{\partial L}{\partial\mathbf{E}} = \frac{\mathbf{E}+G\mathbf{B}}{\sqrt{1+F-G^2}},$$

with

$$L = \sqrt{1 + F - G^2} - 1, \quad F = \mathbf{B}^2 - \mathbf{E}^2, \quad G = \mathbf{B}\cdot\mathbf{E}. \quad (13)$$

One then chooses a frame with $\mathbf{B}||\mathbf{E}$. Consequently $\mathbf{H}||\mathbf{D}||\mathbf{B}||\mathbf{E}$. By inserting (13) into (12) one gets

$$\mathbf{H} = A\mathbf{B}, \qquad \mathbf{E} = A\mathbf{D}, \qquad (14)$$

with

$$A = \sqrt{\frac{1 - \mathbf{E}^2}{1 + \mathbf{B}^2}}\ , \qquad (15)$$

A results to be the dielectric constant and also the permeability. Expressing $\mathbf{B}^2$ in terms of $\mathbf{E}^2$ from (15) one gets

$$\mathbf{B}^2 + \mathbf{D}^2 = \frac{1 - A^2}{A^2}, \qquad (16)$$

and, of course

$$\mathbf{H}^2 + \mathbf{E}^2 = 1 - A^2. \qquad (17)$$

These last two equations reduce to Eqs. (10) when $\mathbf{D}$ and $\mathbf{E}$ are abolished by a γ-transformation.

To obtain Schrödinger's Lagrangian in tensorial notation we write explicitly the dual tensor of the electromagnetic field-strength

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\gamma\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix},$$

and

$$G^{\mu\nu} = \begin{pmatrix} 0 & -D_1 & -D_2 & -D_3 \\ D_1 & 0 & -H_3 & H_2 \\ D_2 & H_3 & 0 & -H_1 \\ D_3 & -H_2 & H_1 & 0 \end{pmatrix},$$

where $G^{\mu\nu}$ corresponds to Maxwell Electrodynamics. For a non-linear electrodynamics in general it is calculated by means of

$$G^{\mu\nu} = 2\frac{\partial L}{\partial F_{\mu\nu}} = \frac{\partial L}{\partial I_1}2F^{\mu\nu} - \frac{\partial L}{\partial I_2}\tilde{F}^{\mu\nu}, \qquad (18)$$

where L is the Lagrangian of interest and

$$I_1 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu}, \quad I_2 = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad (19)$$

are the two Lorentz invariants (13). Equation (18) provides the constitutive equations of a theory depending of I_1 and I_2 [13].

We define now

$$\Phi^{\mu\nu} \equiv \tilde{F}^{\mu\nu} - iG^{\mu\nu}, \qquad (20)$$

which results in terms of $\mathbf{\Omega}$ and $\mathbf{\Sigma}$ in

$$\Phi^{\mu\nu} = \begin{pmatrix} 0 & -\Omega_1 & -\Omega_2 & -\Omega_3 \\ \Omega_1 & 0 & \Sigma_3 & -\Sigma_2 \\ \Omega_2 & -\Sigma_3 & 0 & \Sigma_1 \\ \Omega_3 & \Sigma_2 & -\Sigma_1 & 0 \end{pmatrix}.$$

The Lagrangian (2) can now be written as

$$\mathbb{L} = \frac{\frac{1}{2}\Phi^{\mu\nu}\Phi_{\mu\nu}}{\frac{1}{4}\Phi^{\mu\nu}\tilde{\Phi}_{\mu\nu}} \equiv \frac{\mathbb{I}_1}{\mathbb{I}_2}\ . \qquad (21)$$

Now, in analogy with (18), the $\mathcal{G}$ field tensor corresponding to the Lagrangian (21) results in the following:

$$\mathcal{G}^{\alpha\beta} = -\frac{1}{\mathbf{\Omega}\cdot\mathbf{\Sigma}}\begin{pmatrix} 0 & -2\Omega_1 + \mathcal{L}\Sigma_1 & -2\Omega_2 + \mathcal{L}\Sigma_2 & -2\Omega_3 + \mathcal{L}\Sigma_3 \\ 2\Omega_1 - \mathcal{L}\Sigma_1 & 0 & 2\Sigma_3 + \mathcal{L}\Omega_3 & -2\Sigma_2 - \mathcal{L}\Omega_2 \\ 2\Omega_2 - \mathcal{L}\Sigma_2 & -2\Sigma_3 - \mathcal{L}\Omega_3 & 0 & 2\Sigma_1 + \mathcal{L}\Omega_1 \\ 2\Omega_3 - \mathcal{L}\Sigma_3 & 2\Sigma_2 + \mathcal{L}\Omega_2 & -2\Sigma_1 - \mathcal{L}\Omega_1 & 0 \end{pmatrix}, \qquad (22)$$

where $\mathcal{L}$ is the Lagrangian defined in Eq.(2).

By demanding

$$(\tilde{\Phi}^{\alpha\beta})^* = \mathcal{G}^{\alpha\beta}, \qquad (23)$$

one gets exactly the condition of conjugateness (3). So we have really rewritten Shrödinger's proposal and the same conclusions follow from this tensorial notation.

Taking advantage of the previous tensorial notation, we propose now for non-Abelian theories the following Lagrangian

$$Ł = \frac{\frac{1}{2}T_r(\Phi^{\alpha\beta}\Phi_{\alpha\beta})}{\frac{1}{4}T_r(\Phi^{\alpha\beta}\tilde{\Phi}_{\alpha\beta})} \equiv \frac{\mathit{II}_1}{\mathit{II}_2}\ , \qquad (24)$$

where $\Phi^{\alpha\beta} = \Phi^{\alpha\beta,a}\tau_a$, with τ_a the generators of the gauge group. As in the Abelian formulation the square root, which is so characteristic in the DBI theory and the non-Abelian generalization in [9], has disappeared and the Lagrangian is rational and of the zeroth degree. The procedure formally follows in the same manner as in the foregoing Abelian case. One can define a tensor

$$\mathbb{G}^{\alpha\beta,a} = \frac{\partial Ł}{\partial \mathit{II}_1}\frac{\partial \mathit{II}_1}{\partial \Phi_{\alpha\beta,a}} + \frac{\partial Ł}{\partial \mathit{II}_2}\frac{\partial \mathit{II}_2}{\partial \Phi_{\alpha\beta,a}}, \qquad (25)$$

where

$$\Phi^{\alpha\beta,a} \equiv \tilde{F}^{\alpha\beta,a} - i\mathcal{G}^{\alpha\beta,a}, \qquad (26)$$

with $\tilde{F}^{\alpha\beta,a}$ the dual tensor to the usual field strength tensor of the corresponding non-Abelian theory and $\mathcal{G}^{\alpha\beta,a}$ the corresponding tensor to the Lagrangian defined only by the invariant $\mathit{\Pi}_1$, that is the one associated with the Yang-Mills Lagrangian under consideration and $\mathit{\Pi}_2$ its corresponding, so called θ-term. This procedure allows us to find the constitutive equations. They can be defined by means of the use of the tensor $\mathbb{G}^{\alpha\beta,a}$ in (25). In order to get the appropriate number of fields one needs to identify complex fields. This can be done by imposing the condition of conjugateness in this theory, in analogy with the Abelian case we demand [14]

$$(\tilde{\Phi}^{\alpha\beta,a})^* = \mathbb{G}^{\alpha\beta,a}. \tag{27}$$

As it was assumed by Schrödinger himself in his Abelian proposal, we assume also here that the field strength $\Phi^{\alpha\beta,a}$ is constructed as usual, from a potential four-vector and consequently one gets the corresponding field equations. It is straightforward to see that, if we construct the Lagrangian with the complex field strengths (27) it results oppositely equal to (24). So, that Ł becomes purely imaginary as it happens in the Abelian formulation, Eqs. (2),(3),(6).

The non-Abelian theory (24) is a natural extension of Schrödinger's representation of DBI action. There is no ambiguity in ordering the product of the matrices, we take simply the trace of the action of the non-Abelian theory under consideration and divide it by the corresponding $\theta - term$ which is also a trace. Being the denominator a trace, it can, by example, easily be expanded in a series which multiplies the trace in the numerator. Other possibilities can be considered, before taking the trace in the denominator (and numerator). One can take the matrix product in the numerator and find the inverse matrix corresponding to the denominator. Having to multiply these matrices, one must then give a prescription to define the Lagrangian one would one to consider; the trace or the symmetrized trace [9] can be used [15]. This procedure is, however, not so simple as to take the ratio of the invariants we are used to,$\mathit{\Pi}_1$ and$\mathit{\Pi}_2$ in the definition of Ł (24) .

The usual way to demonstrate that instantons solutions to the field equations of non-Abelian theories exist, is to show that solutions to the condition that the field strength is equal to its dual (or minus its dual), correspond to a minimum of the Yang-Mills action of interest. Consequently instantons are solutions of the corresponding field equations [16, 17]. We observe that when $\Phi^{\alpha\beta,a} = \pm\tilde{\Phi}^{\alpha\beta,a}$, the Lagrangian (24) is a constant. Then instantons solutions exist to the associated field equations. In a forthcoming paper we will search for an expansion of the Lagrangian (24), under the condition of conjugateness (27) in terms of the field strengths. As mentioned, the complex fields are identified through the condition (27) and one gets the same number of fields as in the corresponding standard Yang-Mills theory. This expansion could be a first attempt towards a possible comparison with results in string theory [10]. Also, applications considering specific Lie groups will be considered to extend other results that have been studied in the Abelian case, as classical solutions that describe soliton configurations as well as physical effects related to electric fields [18] approaching limiting values.

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