

Group averaging and the Ashtekar-Horowitz model

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We investigate refined algebraic quantisation of the constrained Hamiltonian system known as the Ashtekar-Horowitz model. We study two versions of this model which are defined on a two-torus and on a cylinder, respectively. The dimension of the physical Hilbert space depends on the topological structure of the model. In particular, we see that for the compact version of the model the representation of the physical observable algebra is irreducible for generic potentials but decomposes into irreducible subrepresentations for certain special potentials. The superselection sectors are related to singularities in the reduced phase space and to the rate of divergence in the formal group averaging integral. For both versions, there is no tunnelling into the classically forbidden region of the unreduced configuration space.

Keywords: Group averaging; constrained systems; superselection sectors.

En este artículo se investiga la cuantización algebraica refinada del sistema Hamiltoniano con restricciones conocido como el modelo de Ashtekar-Horowitz. Se estudian dos versiones de dicho modelo, las cuales están definidas en un toro de dos dimensiones y en un cilindro, respectivamente. La dimensión del espacio de Hilbert físico depende de la estructura topológica del modelo. En particular, se encuentra que en la versión compacta del modelo la representación del álgebra de los observables físicos es irreducible para ciertos potenciales genéricos pero se descompone en sub-representaciones irreducibles para ciertos potenciales especiales. Los sectores de superselección están relacionados tanto con las singularidades en el espacio fase reducido, así como con el grado de divergencia en las integrales al promediar sobre el grupo. En ambas versiones, no hay tunelamiento en las regiones del espacio no reducido de configuración prohibidas clásicamente.

Descriptores: Promedio sobre el grupo; sistemas con restricciones; sectores de superselección.

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1. Introduction

We study quantisation of the constrained Hamiltonian system known as the Ashtekar-Horowitz model originally introduced in Ref. 1 to model the situation occurring in general relativity in which certain parts of the unreduced configuration space are not in the projection of the constraint hypersurface. In particular, the Hamiltonian constraint

$$\mathcal{H} := \frac{-1}{\sqrt{\det(q)}} \left(\pi^{ab} \pi_{ab} - \frac{1}{2} (\pi_a^a)^2 \right) - \sqrt{\det(q)} {}^{(3)}R \approx 0,$$

is a complicated function of the configuration variables that depends quadratically on the momenta $\pi^{ab}(t, x)$, and we have that the constraint surface in the phase space only projects down to certain subsets of the configuration space [1]. These subsets of the configuration space thus play no part in the classical theory, but they could give rise to tunnelling effects in Dirac-style quantisations [2]. The quantisation discussed in Ref. 1 indeed displayed such effects, containing physical states that have support in the classically forbidden region of the configuration space.

We shall investigate the Ashtekar-Horowitz system within the refined algebraic quantisation programme [3, 4, and references therein]. We analyse two different versions of the model that are defined on a two-torus (Boulware's version [5]) and on a cylinder, respectively. The main new issue of interest for us is that the quantum theory obtained within the refined algebraic quantisation programme entails, in both versions, not just a physical Hilbert space $\mathcal{H}_{\text{phys}}$ but also a

precisely-defined algebra $\hat{\mathcal{A}}_{\text{obs}}$ of physical observables. We wish to study this algebra and in particular ask whether its representation contains superselection sectors in the sense that if $\mathcal{H}_{\text{phys}}$ is invariant under $\hat{\mathcal{A}}_{\text{obs}}$ and if the representation of this algebra decomposes the physical Hilbert space into a direct sum of subspaces, each of them carrying an irreducible representation of the algebra of observables, then any matrix element of an operator in the algebra of observables between states belonging to different subspaces necessarily vanishes. Each of the subspaces just mentioned qualify as Hilbert spaces on their own.

A major piece of technical input in refined algebraic quantisation is the rigging map, which maps a dense subspace of suitably well-behaved states in the unconstrained Hilbert space to distributional states that solve the constraints [3, 4]. In Boulware's version, the integral of matrix elements over the gauge group does not converge in absolute value, which complicates attempts to define a rigging map by group averaging. However, the formal group averaging expression nevertheless suggests a rigging map candidate on which the divergences are renormalised. We show that this candidate is a genuine rigging map and the resulting representation of $\hat{\mathcal{A}}_{\text{obs}}$ is irreducible. In this case the representation of $\hat{\mathcal{A}}_{\text{obs}}$ decomposes into superselection sectors, labelled by the degrees of divergence in the formal rigging map candidate, and the representation within each superselection sector is irreducible. Superselection sectors exist precisely when some vectors in $\mathcal{H}_{\text{phys}}$ are supported on the part of the unreduced configura-

tion space that is associated with singular parts of the reduced phase space. A detailed discussion of Boulware's version of the model can be found in Ref. 6.

We also present a version of the Ashtekar-Horowitz model defined on a cylindrical configuration space. We show that there is a distinguished coordinate in the configuration space in the sense that the phenomena occurring in Boulware's version do not arise when we allow this coordinate to be non-periodic. We then find that in this version the stationary points of generic potentials make a vanishing contribution to the rigging map.

In our quantisation of either system there is no tunnelling of the kind found in Ref. 1 into the classically forbidden region of the unreduced configuration space.

The rest of the paper is as follows. Section 2 introduces the classical aspects of the Ashtekar-Horowitz model in its two versions. Section 3 analyses its quantisation within the refined algebraic quantisation programme. Section 4 presents brief concluding remarks.

2. Classical System

In this section we study classical aspects of the Ashtekar-Horowitz model. The system has a four-dimensional unreduced phase space and a single constraint, quadratic in the momenta. We consider two versions of the model that differ on the configuration space we start with, for the first one we consider a two-torus while for the second one we take a two-dimensional cylinder. We see that the topological structure of the reduced phase space on both versions differs if certain stationary points are allowed.

2.1. Boulware's version

The configuration space of the system is $\mathcal{C} := T^2 \simeq S^1 \times S^1$. We write the points in \mathcal{C} as (x, y) , where $x \in S^1$ and $y \in S^1$, and points in the phase space $\Gamma := T^*\mathcal{C}$ as (x, y, p_x, p_y) , where $p_x \in \mathbb{R}$ and $p_y \in \mathbb{R}$. The action reads

$$S = \int dt (p_x \dot{x} + p_y \dot{y} - \lambda C), \quad (1)$$

where the overdot denotes differentiation with respect to the parameter t and λ is a Lagrange multiplier associated with the constraint

$$C := p_x^2 - R(y), \quad (2)$$

where $R : S^1 \rightarrow \mathbb{R}$ is a smooth function. We assume R to be positive at least somewhere. We also assume that R has at most finitely many stationary points. We further assume that each stationary point of R has a nonvanishing derivative of R of some order. To simplify the discussion of the classical system, we assume that no zero of R is a stationary point.

The constraint surface $\bar{\Gamma}$ is the subset of Γ where $C = 0$. By our assumptions about R , $\bar{\Gamma}$ is the Cartesian product of $S^1 \times \mathbb{R} = \{(x, p_y)\}$ with finitely many disjoint circles in

$S^1 \times \mathbb{R} = \{(y, p_x)\}$. Note that orbits generated by the constraint C on the constraint hypersurface $\bar{\Gamma}$ have constant y and p_x .

As it was shown in Ref. 6, each connected component of the reduced phase space Γ_{red} is a two-dimensional symplectic manifold with certain one-dimensional singular subsets, and the symplectic volume of Γ_{red} is finite and equal to $2\pi \int_{R>0} |R'(y)| / \sqrt{R(y)} dy$. Further, the one-dimensional singular subsets emerge when we allow stationary points of R on Γ_{red} . Thus we see that the classical singularities occur at the stationary points of R : this will become important on comparison to the quantum theory.

2.2. Cylindrical version

The configuration space is given as the cylinder $\mathcal{C} := \mathbb{R} \times S^1$. Points in \mathcal{C} are labelled by (x, y) , where $x \in \mathbb{R}$ and $y \in S^1$, and points in the phase space $\Gamma := T^*\mathcal{C}$ are labelled by (x, y, p_x, p_y) , where p_x and p_y are real-valued. Note that the periodicity condition on the x -coordinate is eliminated as compared to Boulware's version. We use the canonical transformation $(x, p_x) \mapsto (-p_z, z)$ which gives the constraint

$$C := z^2 - R(y). \quad (3)$$

We denote as \mathcal{C}' the space labelled by (z, y) . We will work on the same assumptions for $R(y)$ as in section 2.1.. The constraint surface $\bar{\Gamma}$ is the Cartesian product of the plane $\mathbb{R}^2 = \{(p_z, p_y)\}$ with finitely many disjoint circles in $\mathbb{R} \times S^1 = \{(z, y)\}$. Orbits generated by the constraint C on $\bar{\Gamma}$ have constant z and y .

After gauge fixing on $\bar{\Gamma}$ we obtain a reduced phase space Γ_{red} with a well defined symplectic structure inherited from the phase space Γ , and the singular subsets of section 2.1. are absent. This may be seen from the fact that we could choose from the gauge orbits a unique point by the condition $x = 0$ and consider on Γ_{red} the symplectic form $\Omega_{\text{red}} = dp_y \wedge y$. Hence we obtain a Γ_{red} with an infinite volume and no singularities.

3. Refined algebraic quantisation

In this section we quantise the two versions of the system, following refined algebraic quantisation as reviewed in [3,4]. We see that the quantum theories are essentially different from each other.

3.1. RAQ of Boulware's version

We start by fixing the structure in the auxiliary Hilbert space, and then we construct a rigging map suggested by the group averaging integral.

Our auxiliary Hilbert space \mathcal{H}_{aux} is the space of square integrable functions on \mathcal{C} in the inner product

$$(\phi_1, \phi_2)_{\text{aux}} := \int dx dy \overline{\phi_1(x, y)} \phi_2(x, y), \quad (4)$$

where the overline denotes complex conjugation. We take the quantised version of the classical constraint (2) to be

$$\hat{C} := -\frac{\partial^2}{\partial x^2} - R(y). \quad (5)$$

\hat{C} is essentially self-adjoint on \mathcal{D}_{aux} and exponentiates into the one-parameter family of unitary operators

$$U(t) := e^{-it\hat{C}}, \quad t \in \mathbb{R}. \quad (6)$$

For the test space $\mathcal{D}_{\text{aux}} \subset \mathcal{H}_{\text{aux}}$, we choose the space of functions $f : \mathcal{C} \rightarrow \mathbb{C}$ of the form

$$f(x, y) = \sum_{m \in \mathbb{Z}} e^{imx} f_m(y),$$

where each $f_m : S^1 \rightarrow \mathbb{C}$ is smooth and only finitely many f_m are different from zero for each f . \mathcal{D}_{aux} is clearly a dense linear subspace of \mathcal{H}_{aux} . If $f \in \mathcal{D}_{\text{aux}}$, then

$$(U(t)f)(x, y) = \sum_m e^{-it[m^2 - R(y)]} e^{imx} f_m(y), \quad (7)$$

which shows that $U(t)f \in \mathcal{D}_{\text{aux}}$. \mathcal{D}_{aux} is thus invariant under $U(t)$.

The above structure determines the observable algebra $\hat{\mathcal{A}}_{\text{obs}}$ as the algebra of operators $\hat{\mathcal{O}}$ on \mathcal{H}_{aux} such that the domains of $\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}^\dagger$ include \mathcal{D}_{aux} , $\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}^\dagger$ map \mathcal{D}_{aux} to itself and $\hat{\mathcal{O}}$ commutes with $U(t)$ on \mathcal{D}_{aux} for all t .

Recall now that the final ingredient in refined algebraic quantisation is to specify an antilinear rigging map $\eta : \mathcal{D}_{\text{aux}} \rightarrow \mathcal{D}_{\text{aux}}^*$, where $\mathcal{D}_{\text{aux}}^*$ denotes the algebraic dual, topologised by pointwise convergence. The map η must be real and positive, states in its image must be invariant under the dual action of $U(t)$, and η must intertwine with the representations of $\hat{\mathcal{A}}_{\text{obs}}$ on \mathcal{D}_{aux} and $\mathcal{D}_{\text{aux}}^*$ in the sense that for all $\mathcal{O} \in \hat{\mathcal{A}}_{\text{obs}}$ and $f \in \mathcal{D}_{\text{aux}}$,

$$\eta(\hat{\mathcal{O}}f) = \hat{\mathcal{O}}(\eta f). \quad (8)$$

Given the rigging map, the physical Hilbert space $\mathcal{H}_{\text{phys}}$ is the Cauchy completion of the image of η in the inner product

$$(\eta(f), \eta(g))_{\text{phys}} := \eta(g)[f]. \quad (9)$$

To find a rigging map we consider the group averaging proposal that seeks a rigging map as an implementation of the formal expression

$$\eta : \phi \mapsto \int_{-\infty}^{\infty} dt \phi^\dagger U(t). \quad (10)$$

In our system, however, a saddle-point estimate [7] shows that there are states for which the absolute value of the integrand on the right-hand side of (10) is asymptotically proportional to $|t|^{-1/2}$ as $|t| \rightarrow \infty$, and for such states the integral is not absolutely convergent.

Group averaging (10) suggests, however, a rigging map candidate. Thus we define (see details in [6]) the map $\eta_p : \mathcal{D}_{\text{aux}} \rightarrow \mathcal{D}_{\text{aux}}^*$ by

$$(\eta_p(f))(x, y) = \sum_{mj} \frac{e^{-imx} \overline{f_m(y)}}{|R^{(p)}(y_{p|m|j})|^{1/p}} \delta(y, y_{p|m|j}). \quad (11)$$

Here we considered $y_{p|m|j}$ as solutions to the equation $R(y) = m^2$, where the third index labels the solutions for given m , and the first term denotes the order of the lowest non-vanishing derivative of R at a solution. Hence we allow stationary points of R among the solutions to $R(y) = m^2$. If $f, g \in \mathcal{D}_{\text{aux}}$, (11) gives

$$\eta_p(f)[g] = \sum_{mj} \frac{\overline{f_m(y_{p|m|j})} g_m(y_{p|m|j})}{|R^{(p)}(y_{p|m|j})|^{1/p}}, \quad (12)$$

From (12) it is seen that each η_p has a finite-dimensional, nontrivial image and satisfies the rigging map axioms, with the possible exception of the intertwining property (8). In [6] it was shown that each η_p satisfies also the intertwining property and hence provides a rigging map. Each of the images of these maps provides therefore a physical Hilbert space, denoted respectively by $\mathcal{H}_{\text{phys}}^p$, with the inner product given by (9) and (12). As all the spaces are finite-dimensional, no Cauchy completion is needed.

As the images of any two of the rigging maps have trivial intersection in $\mathcal{D}_{\text{aux}}^*$, we can regard each $\mathcal{H}_{\text{phys}}^p$ as superselection sectors in the *total* Hilbert space

$$\mathcal{H}_{\text{phys}}^{\text{tot}} := \bigoplus_p \mathcal{H}_{\text{phys}}^p. \quad (13)$$

As shown in Ref. 6, the representation of $\hat{\mathcal{A}}_{\text{obs}}$ on each $\mathcal{H}_{\text{phys}}^p$ is irreducible. Further, as all the states in $\mathcal{H}_{\text{phys}}^{\text{tot}}$ have their support in the classically allowed region of \mathcal{C} , there is no tunnelling of the kind found in [1] into the classically forbidden region of \mathcal{C} . If the classically allowed region of \mathcal{C} is not connected, there is however tunnelling between all its components that support states in $\mathcal{H}_{\text{phys}}^{\text{tot}}$.

3.2. RAQ of the cylindrical version

We consider the auxiliary Hilbert space $\mathcal{H}_{\text{aux}} := L^2(\mathcal{C}')$ as the space of square integrable functions on \mathcal{C}' within the inner product adapted from (4) to our case, and we take the quantised version of the classical constraint (3) to be

$$\hat{C} := z^2 - R(y), \quad (14)$$

which is essentially self-adjoint in \mathcal{H}_{aux} . We exponentiate the constraint to obtain a one-parameter unitary representation as in (6). We set $\mathcal{D}_{\text{aux}} \subset \mathcal{H}_{\text{aux}}$ as the space of smooth rapidly decreasing functions in \mathcal{H}_{aux} . The action of the unitary operator $U(t)$ on states $\phi \in \mathcal{D}_{\text{aux}}$ reads

$$(U(t)\phi)(z, y) := e^{-it(z^2 - R(y))} \phi(z, y) \in \mathcal{D}_{\text{aux}}. \quad (15)$$

As discussed before in Sec. 3.1, the algebra of observables $\hat{\mathcal{A}}_{\text{obs}}$ is completely determined by the above structure.

The next step is to specify a rigging map $\eta : \mathcal{D}_{\text{aux}} \rightarrow \mathcal{D}_{\text{aux}}^*$. The map η is defined by its properties as described in Refs. 3 and 4 and can be obtained systematically from expression (10). As in Sec. 3.1, a saddle-point estimate shows that the integral (10) is not absolutely convergent for states in \mathcal{D}_{aux} . However, we use (10) to formally write for $R(y) > 0$ a would-be rigging map η as

$$(\eta(f))(z, y) = \frac{1}{2} \sum_{j=\pm} \frac{\overline{f(z, y)}}{|z|} \delta(z, z_j(y)), \quad (16)$$

where $z_j(y)$ labels the two solutions of the quadratic equation $z_j^2 - R(y) = 0$.

From (16), the action of $\eta(f) \in \mathcal{D}_{\text{aux}}^*$ on a state $g \in \mathcal{D}_{\text{aux}}$ is given by

$$\eta(f)[g] = \pi \sum_j \int_{R(y)>0} dy \frac{\overline{f(z_j, y)} g(z_j, y)}{|z_j|}. \quad (17)$$

As the zeroes of R are by assumption not critical points and the functions $f, g \in \mathcal{D}_{\text{aux}}$ are continuous, an elementary analysis shows that the integral in (17) is convergent in absolute value and hence well defined. We then define the physical inner product as in relation (9). From (17) it is clear that η is real and positive, and solves the constraints. To show that η intertwines with the algebra of observables $\hat{\mathcal{A}}_{\text{obs}}$ we can follow similar developments as those presented in [6]. The map η is thus a genuine rigging map, and the physical Hilbert space $\mathcal{H}_{\text{phys}}$ is the Cauchy completion of the image of η equipped with the inner product (9) obtained by (17). Note that $\mathcal{H}_{\text{phys}}$ is infinite-dimensional contrary to the results obtained in the Sec. 3.1.

As in Sec. 3.1, all the states in $\mathcal{H}_{\text{phys}}$ have their support in the classically allowed region of \mathcal{C}' , there is no tunnelling into the classically forbidden region of \mathcal{C}' .

4. Discussion

We have studied refined algebraic quantisation of two different versions of the Ashtekar-Horowitz model.

In Boulware's version [6, 8], although the system did not allow a rigging map to be defined in terms of an absolutely convergent integral of matrix elements over the gauge

group, the formal group averaging expressions nevertheless suggested a rigging map candidate that is renormalised to consider formally divergent terms due to the vanishing of the derivatives of the function $R(y)$. We showed that for generic potential functions this candidate is indeed a rigging map and the divergences in this rigging map appear to be related to the rate of divergence in the formal group averaging integral. The resulting representation of the observable algebra $\hat{\mathcal{A}}_{\text{obs}}$ on the physical Hilbert space $\mathcal{H}_{\text{phys}}^{\text{tot}}$ decomposed into superselection sectors. The dimension of $\mathcal{H}_{\text{phys}}^{\text{tot}}$ was finite. The system exhibits a striking connection between the singular subsets in the reduced phase space Γ_{red} and the superselection sectors in the quantum theory. Because of the periodicity of the coordinate x on the unreduced configuration space $\mathcal{C} \simeq T^2$, the conjugate momentum p_x gets quantised in integer values. For generic potentials, these integer values entirely miss the singular, measure zero subsets of Γ_{red} , and in this case the quantum theory has no superselection sectors. However, when the potential is such that one or more of the quantised values of p_x hit some of the singular subsets of Γ_{red} , superselection sectors arise in the quantum theory.

Although the compactness of \mathcal{C} simplified some aspects of the analysis, the compactness is as such not essential: The results remain qualitatively similar if the y -direction is unwrapped to the real axis, provided the range of y in which R takes positive values remains bounded. What is essential is the periodicity in the x -direction. As seen in Sec. 2.1, it is the x -periodicity that in the classical theory renders the volume of Γ_{red} finite and creates the singular subsets; in the quantum theory, it is the associated discreteness of p_x that makes the physical Hilbert space finite-dimensional and allows the isolated stationary points of the potential to make nonzero contributions to the rigging map. However, as seen in Sec. 2.2, if x takes values in \mathbb{R} , these phenomena do not arise. The reduced phase space has then infinite volume and no classical singularities. The physical Hilbert space is infinite-dimensional, and the stationary points of R make a vanishing contribution to the rigging map.

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