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Metric Fluctuations from a NKK theory of gravity in a de Sitter Expansion

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A gauge invariant scalar metric fluctuations formalism from a Noncompact Kaluza-Klein (NKK) theory of gravity is presented in this talk notes. In this analysis we recover the well-known result $\delta\rho/\rho \simeq 2\Phi$ obtained typically in the standard 4D semiclassical approach to inflation and also the spectrum of these fluctuations become dependent of the fifth (space-like) coordinate. This fact allows to establish an interval of values for the wave number associated with the fifth dimension.

Keywords: Scalar metric fluctuations; 5D apparent vacuum; linearized Einstein's equations; noncompact fifth dimension.

En estas notas se presenta un formalismo recientemente introducido por los presentes autores para describir fluctuaciones escalares de la métrica invariantes de norma en el contexto de una teoría de Kaluza-Klein no-compacta penta-dimensional. En este análisis se recupera uno de los resultados obtenidos típicamente bajo un tratamiento 4D semiclásico de inflación para las fluctuaciones en la densidad de energía $\delta\rho/\rho \simeq 2\Phi$. Algo a resaltar es que el espectro para estas fluctuaciones es dependiente de la quinta coordenada. Este hecho nos permite establecer cotas para el número de onda asociado a la quinta dimensión.

Descriptores: Fluctuaciones escalares de la métrica; vacío aparente 5D; ecuaciones de Einstein linearizadas; quinta dimensión no-compacta.

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1. Introduction

These talk notes are based on our recent work [1]. The goal is to study gauge invariant scalar metric fluctuations from a NKK theory of gravity in a de Sitter expansion. As we know the inflationary theory of the universe provides a physical mechanism to generate primordial energy density fluctuations. This is studied in the framework of the relativistic theory of cosmological perturbations. The theory of linearized gravitational perturbations in an expanding universe is used to describe the process of structure formation in the early universe [2] and it is indispensable to relate inflationary scenarios with observational evidences mainly with the Cosmic Microwave Background (CMB) anisotropies. In the most widely accepted inflationary scenarios the dynamics is described by a quantum scalar field φ named inflaton that is splitted into a homogeneous and an inhomogeneous components. Usually the homogeneous one is interpreted as a classical field that drives the expansion, while the second one is responsible of the quantum fluctuations that originate the primordial energy density fluctuations [3].

On the other hand, physical theories in more than four dimensions have played an important role in modern physics including cosmology. The idea of extra dimensions in physics was proposed firstly by Gunnar Nordström in 1914 [4] and subsequently by Kaluza in 1921 [5] and Klein in 1926 [6]. They attempted to unify gravity with electromagnetism by introducing an extra dimension. Since then, the possible existence of extra dimensions got strong motivation and many interesting attempts to incorporate gravity and gauge interactions in an unique scheme have been made. Currently one of the theories with more impact in cosmology is the brane world scenario. In such framework the question about how large could extra dimensions be without getting into conflict with observational evidences, has a lot relevance. However for many researchers a more interesting question is how could this extra dimensions manifest themselves. According to brane world scenario matter should be localized onto an hypersurface (the brane) embedded in a higher dimensional space-time (the bulk) [7]. The main motivation of these models comes from string theories and their extension M-theory, which have suggested another approach to compactify extra dimensions. The proposal of great interest in cosmology is that our universe may be such a brane-like object where the standard model of particles is confined on a brane and only gravity and other exotic matter as some scalar fields (like the dilaton) can propagate in the bulk [8].

Another theory of great relevance and on which the present work is based is the Space-Time-Matter theory or Induced Matter theory. This theory can be thought as a noncompact Kaluza-Klein theory due to the fifth dimension is extended. In the 90's Paul Wesson, J. Ponce de Leon and collaborators, based in the Campbell-Magaard's theorem, showed that it is possible to interpret most properties of matter in 4D as a result of the 5D Riemannian geometry. This formalism allows dependence on the fifth coordinate and does not make assumptions about the topology of the fifth dimension. In other words, they proposed that 4D field equations with sources can be locally embedded in 5D field equations without sources [9]. The Campbell-Magaard's theorem establishes that any analytic N-dimensional Riemannian manifold can be locally embedded in a (N+1)-dimensional Ricci-flat manifold. In the cosmological context there is a class of 5D cosmological models which are reduced to the usual 4D ones by taking a foliation on the extra coordinate. These ideas will be implemented to develop the new formalism presented in this notes.

2. Formalism

We consider the action

$$I = -\int d^4x\, d\psi \sqrt{\left|\frac{{}^{(5)}\bar{g}}{{}^{(5)}\bar{g}_0}\right|}\, \left[\frac{{}^{(5)}\bar{R}}{16\pi G} + {}^{(5)}\mathcal{L}(\varphi,\varphi_{,A})\right], \quad (1)$$

for a scalar field φ, which is minimally coupled to gravity. Since we are aimed to describe a manifold in apparent vacuum the Lagrangian density $\mathcal{L}$ in (1) should be only kinetic in origin

$$^{(5)}\mathcal{L}(\varphi,\varphi_{,A}) = \frac{1}{2} g^{AB}\varphi_{,A}\varphi_{,B}, \quad (2)$$

where A, B can take the values 0,1,2,3,4 and the perturbed line element $dS^2 = g_{AB}dx^A dx^B$ is given by

$$dS^2 = \psi^2\left(1+2\Phi\right)dN^2 - \psi^2\left(1-2\Psi\right)e^{2N}dr^2 - \left(1-Q\right)d\psi^2. \quad (3)$$

Here, the fields Φ, Ψ and Q are functions of the coordinates $[N, \vec{r}(x,y,z), \psi]$, where N, x, y, z are dimensionless and the fifth coordinate ψ has spatial units. Note that $^{(5)}\bar{R}$ in the action (1) is the Ricci scalar evaluated on the background metric $\left(dS^2\right)_b = \bar{g}_{AB}dx^A dx^B$. In our case we consider the background canonical metric

$$\left(dS^2\right)_b = \psi^2 dN^2 - \psi^2 e^{2N}dr^2 - d\psi^2, \quad (4)$$

which is 3D spatially isotropic, homogeneous and flat [10]. Moreover, the metric (4) is globally flat (i.e., $\bar{R}^A_{BCD} = 0$) and describes an apparent vacuum $\bar{G}_{AB} = 0$. The energy-momentum tensor is given by

$$T_{AB} = \varphi_{,A}\varphi_{,B} - \frac{1}{2}g_{AB}\varphi_{,C}\varphi^{,C}, \quad (5)$$

which is obviously symmetric. Hence, using the fact that the metric (3) is also symmetric we obtain that $\Psi = \Phi$ and $Q = 2\Phi$. Thus, the line element (3) now becomes

$$dS^2 = \psi^2\left(1+2\Phi\right)dN^2 - \psi^2\left(1-2\Phi\right)e^{2N}dr^2 - \left(1-2\Phi\right)d\psi^2, \quad (6)$$

where the field $\Phi(N,\vec{r},\psi)$ is the scalar metric perturbation of the background 5D metric (4). For the metric (6), $|^{(5)}\bar{g}| = \psi^8 e^{6N}$ is the absolute value of the determinant for the background metric (4) and $|^{(5)}\bar{g}_0| = \psi_0^8 e^{6N_0}$ is a dimensionalization constant, where ψ_0 and N_0 are constants. Besides, $G = M_p^{-2}$ is the gravitational constant and $M_p = 1.2\ 10^{19}$ GeV is the Planckian mass. In this work we consider $N_0 = 0$, therefore $\left|^{(5)}\bar{g}_0\right| = \psi_0^8$. Here, the index "0" denotes the value at the end of inflation.

On the other hand, the contravariant metric tensor, after a Φ-first order approximation, is given by

$$g^{NN} = \frac{(1-2\Phi)}{\psi^2},$$
$$g^{ij} = -\frac{e^{-2N}(1+2\Phi)}{\psi^2},$$
$$g^{\psi\psi} = -(1+2\Phi), \quad (7)$$

which can be written as $g^{AB} = \bar{g}^{AB} + \delta g^{AB}$, being $\bar{g}^{AB}$ the contravariant background metric tensor and δg^{AB} their corresponding fluctuations. The dynamics for φ and Φ are well described by the Lagrange and Einstein equations, which we shall study in the following subsections.

2.1. 5D Dynamics

The Lagrange equations for the fields φ and Φ are respectively given by

$$\frac{\partial^2\varphi}{\partial N^2} + 3\frac{\partial\varphi}{\partial N} - e^{-2N}\nabla_r^2\varphi - \psi\left(\psi\frac{\partial^2\varphi}{\partial\psi^2} + 4\frac{\partial\varphi}{\partial\psi}\right) - 2\Phi\left[\frac{\partial^2\varphi}{\partial N^2} + 3\frac{\partial\varphi}{\partial N} - e^{-2N}\nabla_r^2\varphi + \psi\left(\psi\frac{\partial^2\varphi}{\partial\psi^2} + 4\frac{\partial\varphi}{\partial\psi}\right)\right] - 2\left(\frac{\partial\varphi}{\partial N}\frac{\partial\Phi}{\partial N} + \psi^2\frac{\partial\Phi}{\partial\psi}\frac{\partial\varphi}{\partial\psi}\right) = 0, \quad (8)$$

$$\left(\frac{\partial\varphi}{\partial N}\right)^2 + e^{-2N}\left(\nabla\varphi\right)^2 + \psi^2\left(\frac{\partial\varphi}{\partial\psi}\right)^2 = 0. \quad (9)$$

Now, we can make the semi classical approximation $\varphi(N,\vec{r},\psi) = \varphi_b(N,\psi) + \delta\varphi(N,\vec{r},\psi)$, such that φ_b is the solution of eq. (8) in absence of the metric fluctuations [i.e.,for $\Phi = \delta\varphi = 0$] and $\delta\varphi$ represents the quantum fluctuations of the inflaton field φ. Hence, the Lagrange equations for φ_b

and $\delta\varphi$ are

$$\frac{\partial^2\varphi_b}{\partial N^2}+3\frac{\partial\varphi_b}{\partial N}-\psi\left[\psi\frac{\partial^2\varphi_b}{\partial\psi^2}+4\frac{\partial\varphi_b}{\partial\psi}\right]=0, \tag{10}$$

$$\frac{\partial^2\delta\varphi}{\partial N^2}+3\frac{\partial\delta\varphi}{\partial N}-e^{-2N}\nabla_r^2\delta\varphi-\psi\left[4\frac{\partial\delta\varphi}{\partial\psi}+\psi\frac{\partial^2\delta\varphi}{\partial\psi^2}\right]-2\frac{\partial\varphi_b}{\partial N}\times\frac{\partial\Phi}{\partial N}-2\psi^2\left[\frac{\partial\varphi_b}{\partial\psi}\frac{\partial\Phi}{\partial\psi}+\left(\frac{\partial^2\varphi_b}{\partial\psi^2}+\frac{4}{\psi}\frac{\partial\varphi_b}{\partial\psi}\right)\Phi\right]=0. \tag{11}$$

Note that for $\Phi=\delta\varphi=0$, the eq. (9) transforms in

$$\left(\frac{\partial\varphi_b}{\partial N}\right)^2+\psi^2\left(\frac{\partial\varphi_b}{\partial\psi}\right)^2=0, \tag{12}$$

which will be useful later.

Considering the linearized field equations $\delta G_{AB}=-8\pi G\delta T_{AB}$, after some algebra we reduce them to the form

$$\frac{\partial^2\Phi}{\partial N^2}+3\frac{\partial\Phi}{\partial N}-e^{-2N}\nabla_r^2\Phi-2\psi^2\frac{\partial^2\Phi}{\partial\psi^2}+\frac{16\pi G}{3}\Phi\left[\left(\frac{\partial\varphi_b}{\partial N}\right)^2+\psi^2\left(\frac{\partial\varphi_b}{\partial\psi}\right)^2\right]=0. \tag{13}$$

From Eq. (12), the Eq. (13) we obtain

$$\frac{\partial^2\Phi}{\partial N^2}+3\frac{\partial\Phi}{\partial N}-e^{-2N}\nabla_r^2\Phi-2\psi^2\frac{\partial^2\Phi}{\partial\psi^2}=0, \tag{14}$$

that is the equation of motion for the 5D scalar metric fluctuations $\Phi(N,\vec{r},\psi)$.

3. Effective 4D de Sitter expansion

In this section we study the effective 4D Φ-dynamics in an effective 4D de Sitter background expansion of the universe, which is considered 3D (spatially) flat, isotropic and homogeneous.

3.1. Ponce de Leon metric

We consider the coordinate transformation [11]

$$t=\psi_0 N,\qquad R=\psi_0 r,\qquad \psi=\psi. \tag{15}$$

Hence, the 5D background metric (4) becomes

$$\left(dS^2\right)_b=\left(\frac{\psi}{\psi_0}\right)^2\left[dt^2-e^{2t/\psi_0}dR^2\right]-d\psi^2, \tag{16}$$

which is the Ponce de Leon metric [12], that describes a 3D spatially flat, isotropic and homogeneous extended (to 5D) Friedmann-Robertson-Walker metric in a de Sitter expansion. Here, t is the cosmic time and $dR^2=dX^2+dY^2+dZ^2$. This Ponce de Leon metric is a special case of the separable models studied by him, and is an example of the intensely studied class of canonical metrics $dS^2=\psi^2 g_{\mu\nu}dX^\mu dX^\nu-d\psi^2$ with $\mu,\nu=0,1,2,3$ [13]. Now we can take a foliation $\psi=\psi_0$ in the metric (16), such that the effective 4D metric results

$$\left(dS^2\right)_b\rightarrow\left(ds^2\right)_b=dt^2-e^{2t/\psi_0}dR^2, \tag{17}$$

that describes a 4D expansion of a 3D spatially flat, isotropic and homogeneous universe that expands with a constant Hubble parameter $H=1/\psi_0$ and a 4D scalar curvature ${}^{(4)}\mathcal{R}=12H^2$. Hence, the effective 4D metric of (6) on hypersurfaces $\psi=1/H$, is

$$dS^2\rightarrow ds^2=(1+2\Phi)\,dt^2-(1-2\Phi)\,e^{2Ht}dR^2. \tag{18}$$

This metric describes the perturbed 4D de Sitter expansion of the universe, where $\Phi(\vec{R},t)$ is gauge-invariant.

3.2. Dynamics of Φ in an effective 4D de Sitter expansion

In order to study the 4D dynamics of the gauge-invariant scalar metric fluctuations $\Phi(\vec{R},t)$ in a background de Sitter expansion we transform the Eq.(14) using the expressions (15) with the foliation $\psi=\psi_0=1/H$, Eq.(14) acquires the form

$$\frac{\partial^2\Phi}{\partial t^2}+3H\frac{\partial\Phi}{\partial t}-e^{-2Ht}\nabla_R^2\Phi-2\left.\frac{\partial^2\Phi}{\partial\psi^2}\right|_{\psi=H^{-1}}=0, \tag{19}$$

where

$$\left.\frac{\partial^2\Phi}{\partial\psi^2}\right|_{\psi=H^{-1}}=k_{\psi_0}^2\Phi.$$

To simplify the structure of this equation we propose the redefined quantum metric fluctuations $\chi(\vec{R},t)=e^{3Ht/2}\Phi(\vec{R},t)$, thus Eq.(19) can be expressed in terms of χ as

$$\ddot{\chi}-e^{-2Ht}\nabla_R^2\chi-\left[\frac{9}{4}H^2+4k_{\psi_0}^2\right]\chi=0. \tag{20}$$

Furthermore the redefined field $\chi(\vec{R},t)$ can be expanded as

$$\chi=\frac{1}{(2\pi)^{3/2}}\int d^3k_R dk_\psi\left[a_{k_Rk_\psi}e^{i\vec{k}_R.\vec{R}}\xi_{k_Rk_\psi}(t)+cc\right]\times\delta(k_\psi-k_{\psi_0}), \tag{21}$$

being $a_{k_Rk_\psi}$ and $a^\dagger_{k_Rk_\psi}$ the annihilation and creation operators respectively, and cc denoting the complex conjugate of the first term in brackets. These operators satisfy the commutator relations

$$\left[a_{k_Rk_\psi},a^\dagger_{k'_Rk'_\psi}\right]=\delta^{(3)}\left(\vec{k}_R-\vec{k}'_R\right)\delta\left(\vec{k}_\psi-\vec{k}'_\psi\right),$$

$$\left[a_{k_Rk_\psi},a_{k'_Rk'_\psi}\right]=\left[a^\dagger_{k_Rk_\psi},a^\dagger_{k'_Rk'_\psi}\right]=0.$$

Hence, the dynamics of the time dependent modes $\xi_{k_Rk_{\psi_0}}(t)$ is given by

$$\ddot{\xi}_{k_Rk_{\psi_0}}(t)+\left[k_R^2e^{-2Ht}-\left(\frac{9}{4}H^2+4k_{\psi_0}^2\right)\right]\times\xi_{k_Rk_{\psi_0}}(t)=0, \tag{22}$$

which has a general solution

$$\xi_{k_R k_{\psi_0}}(t) = A_1 \mathcal{H}_\mu^{(1)}[y(t)] + A_2 \mathcal{H}_\mu^{(2)}[y(t)], \qquad (23)$$

where μ=$1/2\sqrt{9+16k_{\psi_0}^2/H^2}$ and $y(t)$=$(1/H)k_R e^{-Ht}$.

Using the Bunch-Davies vacuum condition [14], we obtain

$$\xi_{k_R k_{\psi_0}}(t) = i\sqrt{\frac{\pi}{4H}}\mathcal{H}_\mu^{(2)}[y(t)], \qquad (24)$$

which are the normalized time dependent modes of the field $\chi(\vec{R},t)$.

3.3. Energy density fluctuations

In order to obtain the energy density fluctuations on the effective 4D FRW metric, we calculate

$$\frac{\delta\rho}{\langle\rho\rangle} = \left.\frac{\delta T_N^N}{\langle T_N^N\rangle}\right|_{t=\psi_0 N, R=\psi_0 r, \psi=1/H}, \qquad (25)$$

being $\delta T_{NN} = -(1/2)\delta g_{NN}\varphi_{,L}\varphi^{,L}$ linearized and where the brackets $\langle ...\rangle$ denote the expectation value on the 3D hypersurface $R(X,Y,Z)$. Using the semiclassical expansion $\varphi(\vec{R},t) = \varphi_b(t) + \delta\varphi(\vec{R},t)$, after some algebra we obtain

$$\frac{\delta\rho}{\langle\rho\rangle} \simeq 2\Phi\left\{1 - \frac{\left\langle(\delta\dot{\varphi})^2 + e^{-2Ht}\left(\nabla_R\delta\varphi\right)^2 + 2V(\delta\varphi)\right\rangle}{\left(\dot{\varphi}_b\right)^2 + 4H^2\left(\varphi_b\right)^2}\right\}$$
$$\simeq 2\Phi, \qquad (26)$$

where we have considered the approximation

$$\frac{\left\langle(\delta\dot{\varphi})^2 + e^{-2Ht}\left(\nabla_R\delta\varphi\right)^2 + 2V(\delta\varphi)\right\rangle}{\left(\dot{\varphi}_b\right)^2 + 4H^2\left(\varphi_b\right)^2} \ll 1, \qquad (27)$$

being $V(\delta\varphi) = V(\varphi) - V(\varphi_b)$

$$V(\delta\varphi) = -\frac{1}{2}\left.\left[g^{\psi\psi}\left(\frac{\partial\varphi}{\partial\psi}\right)^2 - \bar{g}^{\psi\psi}\left(\frac{\partial\varphi_b}{\partial\psi}\right)^2\right]\right|_{\psi=H^{-1}},$$

with

$$V(\varphi_b) = -\frac{1}{2}\bar{g}^{\psi\psi}\left.\left(\frac{\partial\varphi_b}{\partial\psi}\right)^2\right|_{\psi=H^{-1}} = 2H^2\left(\varphi_b\right)^2.$$

It is important to notate that the approximation (27) is valid only during inflation on super Hubble scales (on the infrared sector), on which the inflaton field fluctuations are very "smooth". Finally, we can compute the amplitude of $\Phi(\vec{R},t)$ for a de Sitter expansion on the infrared sector ($k_R \ll e^{Ht}H$) through the expression

$$\left\langle\Phi^2\right\rangle = \frac{e^{-3Ht}}{(2\pi)^3}\int_0^{\epsilon e^{Ht}H} d^3k_R\, \xi_{k_R}\xi_{k_R}^*, \qquad (28)$$

where $\epsilon \simeq 10^{-3}$ is a dimensionless constant. Hence the squared Φ-fluctuations has a power-spectrum $\mathcal{P}(k_R)$ given by

$$\mathcal{P}(k_R) \sim k_R^{3-\sqrt{9+16k_{\psi_0}^2/H^2}}, \qquad (29)$$

which is nearly scale invariant for

$$k_{\psi_0}^2\psi_0^2 = k_{\psi_0}^2/H^2 \ll 1.$$

In other words, the 3D power-spectrum of the gauge-invariant metric fluctuations depends of the wave number k_{ψ_0} related to the fifth coordinate on the hypersurface $\psi = \psi_0 \equiv H^{-1}$. This 3D power spectrum corresponds to the spectral index

$$n_s = 4 - \sqrt{9+16k_{\psi_0}^2/H^2}. \qquad (30)$$

On the other hand it is well known from observational data [15] that $n_s = 0.97 \pm 0.03$. This fact allows to establish that $0 \le k_{\psi_0} < 0.15\, H$, which is the main result of this paper.

4. Final Comments

In this notes, based on our recent work [1], we have studied 4D gauge-invariant metric fluctuations from a NKK theory of gravity. In particular we have examined these fluctuations in an effective 4D de Sitter expansion for the universe using a first-order expansion for the metric tensor. A very important result of this formalism is the confirmation of the well known 4D result $\delta\rho/\rho \simeq 2\Phi$ [2], during inflation. Furthermore, the spectrum of the energy fluctuations depends of the fifth coordinate. This fact allows to establish the interval $k_{\psi_0} < 1.5 \times 10^{-10}$ $\mathrm{M_p}$, where we have used the typical inflationary value $H = 10^{-9}$ $\mathrm{M_p}$. Finally, an advantage of this formalism is that it could be extended to other inflationary and cosmological models where the expansion of the universe is governed by a single scalar field.

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