Relativistic particles with auxiliary variables

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Recibido el 1 de mayo de 2006; aceptado el 1 de noviembre de 2006

We consider the motion of a particle described by an action that is a functional of the one-dimensional metric of the worldline and its first Frenet-Serret [FS] curvature. The metric and the curvature along with the orthogonal [FS] basis which connect them to the embedding functions defining the worldline are introduced as auxiliary variables by adding appropiate constraints. The conserved stress tensor associated with the theory is established in terms of the constraints.

Keywords: Relativistic particle.

Se considera el movimiento de una partícula descrita por una acción que es una funcional de la métrica uni-dimensional de la línea de mundo y su primera curvatura de Frenet-Serret [FS]. La métrica y la curvatura junto con la base ortonormal FS que las conecta con las funciones de inmersión que definen la línea de mundo se introducen como variables auxiliares añadiendo constricciones adecuadas. El tensor de esfuerzos conservado asociado a la teoría se establece en términos de las constricciones.

Descriptores: Partícula relativista.

PACS: 03.30.+p; 04.20.-q; 11.25.-w

1. Introduction

The relativistic particle still is one of the most interesting dynamical systems to investigate if one wishes to try to understand aspects of physics at a fundamental level. Among the many reasons for this, it is pointed out that relativistic particle theory has many features that have higher-dimensional analogues in the relativistic realm while, at the same time, it is a prototype of general relativity [1]. The notion of a relativistic pont-like particle, is an idealization that has guided the development of reparametrization invariant theories, in particular, string theory and its membrane descendants. The massive relativistic particle, with an action proportional to the lenght of its worldline, represents the simplest global geometrical quantity invariant under reparametrizations. A natural extension is to consider higher order geometrical models for particles, described by an action that dependes on the curvature of the worldline [2]. These models can be constructed by means of the geometrical scalars associated with the parametrization of the worldline. While systems depending on the first and second [FS] curvature were intensively studied as toy models for higher dimensional relativistic systems such as rigid strings [3, 4], it has turned out that these systems possess interesting features in their own right; the free particle is the one dimensional analogue for the Nambu-Goto action, the action:

$$S = \int d\xi \sqrt{-\gamma} k^2, \tag{1}$$

is the one dimentional analogue to the Polyakov action.

The geometrical model describing relativistic particles is well known, but it will tipically involve derivatives higher than first and inherit a level of non-linearity. There is, however, a useful stratagem to lower the effective order or to tame this non-linearity introduced by Guven [5] in the study of biological membrane geometry, which involves the introduction of auxiliary fields, not just the induced metric as had already been done [3, 6, 7].

The purpose of this paper is to introduce the aplication of the Guven's technique to the study of the relativistic particles. The paper is divided as follows: In Sec. 2 some aspects of the worldline geometry are considered, both intrinsic and extrinsic. The basic notions of [FS] geometry are introduced. The respective connections are defined along with the covariant derivative under worldline reparameterizations and the covariant derivative under frame normal rotations. In Sec. 3 the general form of the action studied is presented, the constraints are introduced into a total action. Later in this section we show the results for the variations of each variable, and find the relations between the Lagrange multipliers introduced. The conserved stress tensor associated to the theory is determined. In Sec. 4 two examples are shown, one for the free particle model, and the second for the linear correction. In Sec. 5 the case of p-branes is considered. Concluding remarks are presented in Sec. 6.

2. Worldline geometry

The geometry of interest is a time-like curve embedded in Minkosky space (the signature used is the one with only one minus sign). Although this treatment can be extended to curves inmersed in a background different than Minkowsky. The worldline can be described by the parametrizarion: $x^\mu = X^\mu(\xi)$ where ξ is an arbitrary parameter on the worldline and X^μ are the position functions or embedding.

Now let us consider some aspects on the intrinsic geometry. The one-dimentional induced metric:

$$\gamma = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} = \dot{X} \cdot \dot{X}, \tag{2}$$

where the tangent vector is:

$$\dot{X}^\mu = \frac{dX^\mu}{d\xi}. \tag{3}$$

By the way if one uses proper time as the parameter, $\gamma = -1$.

2.1. Frenet-Serret equations

The unidimensional versions of the Gauss-Weingarten equations are the [FS] equations^i:

$$\ddot{X}^\mu = \Gamma \dot{X}^\mu - K^i n_i^\mu, \tag{4}$$

$$\dot{n}^{\mu i} = k^i \dot{X}^\mu + \omega^{ij} n_j^\mu. \tag{5}$$

From these equations we can easily see that the one-dimentional affine connection is:

$$\Gamma = \gamma^{-1} \dot{X} \cdot \ddot{X}. \tag{6}$$

In the description of the extrinsic geometry the normal frame is not fixed. To define any normal vector n_i^μwe only ask that:

$$n^i \cdot n^j = \delta^{ij}, \tag{7}$$

$$n_i \cdot \dot{X} = 0, \tag{8}$$

where normal indices are raised and lowered with the Kronecker delta. The covariant derivative under worldline reparameterizations is defined by:

$$\nabla = \frac{d}{d\xi} - \Gamma, \tag{9}$$

where the one-dimensional affine connection γ vanishes under a proper time parametrization. The extrinsic curvature or first [FS] curvature along the ith normal is:

$$K^i = -n^i \cdot \nabla^2 X = -n^i \cdot \ddot{X}. \tag{10}$$

The point like analogue of the mean extrinsic curvature:

$$k^i = \gamma^{-1} K^i = (-\gamma)^{-1} n^i \cdot \nabla^2 X, \tag{11}$$

is a scalar under reparametrizations.

From the [FS] equations we can also see that the connection associated with the freedom of rotations of the normals is given by:

$$\omega^{ij} = \dot{n}^i \cdot n^j, \tag{12}$$

this connection is used to define the worldline covariant derivative under normal frame rotations:

$$\tilde{\nabla} = \frac{d}{d\xi} - \Gamma - \omega. \tag{13}$$

Finally, the first [FS] curvature, the geodesic curvature:

$$k = \sqrt{k^i k_i}, \tag{14}$$

which is simply the modulus of k^i [8].

3. The actions

In this paper we apply the method to an invariant under reparametrization action.

$$\mathcal{S}[X] = \int_c d\xi \sqrt{-\gamma} L(\gamma, K^i). \tag{15}$$

The approach here will be to do the variations on each of the independent auxiliary variables. To do this consistently we introduce as constraints the structural relationships connecting the auxiliary variables. The constraints introducing the auxiliary variables are:

$$\gamma - \dot{X} \cdot \dot{X} = 0, \tag{16}$$

$$K^i - \dot{X} \cdot \tilde{\nabla} n^i = 0, \tag{17}$$

$$\dot{X} - \nabla X = 0, \tag{18}$$

$$\dot{X} \cdot n^i = 0, \tag{19}$$

$$n^i \cdot n^j - \delta^{ij} = 0. \tag{20}$$

Now the corresponding Lagrange multipliers are introduced into a new total action to implement the constraints:

$$\begin{aligned} S_T &= S_T[\gamma, K^i, n^i, \dot{X}, X, f, \lambda_i, \lambda_{ij}, \Lambda_i, \lambda] \\ &= \mathcal{S}[\gamma, K^i] + \int_c d\xi \sqrt{-\gamma} f(\dot{X} - \nabla X) \\ &+ \int_c d\xi \sqrt{-\gamma} \lambda_i (\dot{X} \cdot n^i) \\ &+ \int_c d\xi \sqrt{-\gamma} \lambda_{ij} (n^i \cdot n^j - \delta^{ij}) \\ &+ \int_c d\xi \sqrt{-\gamma} \Lambda_i (K^i - \dot{X} \cdot \tilde{\nabla} n^i) \\ &+ \int_c d\xi \sqrt{-\gamma} \lambda (\gamma - \dot{X} \cdot \dot{X}). \end{aligned} \tag{21}$$

Note that the new total action is a functional of $\gamma, K^i, n^i, \dot{X}, X, f, \lambda_i, \lambda_{ij}, \Lambda_i, \lambda$, and the original action is treated as a function os the auxiliary variables γ and K^i. Now γ, K^i, n^i, $\dot{X}$, and X are independent variables which can be deformed independently. Note that it is not necesary to track the deformation induced on γ and K^i by a deformation in X, because they are now independet variables.
When the variation of X is done, we get a divergence:

$$\frac{\delta S_T}{\delta X} = \nabla f = 0, \tag{22}$$

so that in equilibrium, f is covariantly conserved on the worldline. This is the conserved stress tensor associated with the theory. When the variation on $\dot{X}$ is done we express f as a linear combination of the [FS] basis:

$$f = (2\lambda + K^i\Lambda_i)\dot{X} - \lambda_i n^i, \tag{23}$$

where it is used the analogue to the Weingarten equation $\tilde{\nabla} n^i = K^i \dot{X}$. This equation itself follows from the constraints on K^i and n^i. The Lagrange multiplier λ_i is fixed when the variation on n^i is considered. For this we use the Gauss equation $\nabla \dot{X} = -K^i n_i$ which itself follows from the Weingarten equation and the constraint $\dot{X} \cdot n^i$. Thus one has:

$$(\tilde{\nabla}\Lambda^i + \lambda^i)\dot{X} + (2\lambda^{ij} - \Lambda^i K^j)n^i = 0, \tag{24}$$

and:

$$\lambda^i = -\tilde{\nabla}\Lambda^i, \tag{25}$$

$$2\lambda^{ij} = \Lambda^i K^j. \tag{26}$$

The variations of K^i and γ gives respectively:

$$\Lambda_i = -L_i, \tag{27}$$

$$\lambda = \frac{T}{2}, \tag{28}$$

where:

$$L_i = \frac{\partial L}{\partial K^i}, \tag{29}$$

$$T \equiv -\frac{2}{\sqrt{-\gamma}} \frac{\partial(\sqrt{-\gamma}\ L)}{\partial \gamma}. \tag{30}$$

So that the conserved stress is:

$$f^\mu = [T - L_i K^i]\dot{X}^\mu - \tilde{\nabla} L^i n_i^\mu. \tag{31}$$

Note that T is only one part of the total conserved stress tensor associated with the theory.

4. Relativistic particles models

In this section the framework developed is applied to obtain the conserved stress tensor for two cases of relativistic particles of physical significance.

4.1. Free particle

The massive free relativistic particle with an action proportional to the lenght of the worldline, represents the simplest global geometrical quantity invariant under reparametrizations, and it is the one-dimensional analogue to the Nambu-Goto action for a relativistic membrane. The action describing this particle is given trivially by:

$$S_1 = -m \int_c d\xi \sqrt{-\dot{X} \cdot \dot{X}}, \tag{32}$$

where:

$$L = -m, \tag{33}$$

$$L_i = 0, \tag{34}$$

$$T = \frac{m}{\gamma}, \tag{35}$$

and the conserved stress tensor is given thus by:

$$f^\mu = -\frac{1}{\sqrt{-\gamma}} p^\mu. \tag{36}$$

Note that the conserved stress is proportional to the momenta, so that the momenta of the particle is conserved. This is what one should naturally expect because it is just a free particle what is being studied.

4.2. Linear correction

This action describes a relativistic massless particle with chilarity given by the constant α:

$$S_2 = \int_c d\xi \sqrt{-\gamma}(-m + \alpha k), \tag{37}$$

where

$$L = -m + \alpha k, \tag{38}$$

$$L_i = \alpha\gamma^{-1}\hat{k}_i, \tag{39}$$

$$T = \gamma^{-1}(m + \alpha k), \tag{40}$$

$$f^\mu = m\gamma^{-1}\dot{x}^\mu - \alpha\gamma^{-1}\tilde{\nabla}\hat{k}_i n^{\mu i}. \tag{41}$$

If $m = 0$:

$$f^\mu = -\alpha\gamma^{-1}\tilde{\nabla}\hat{k}_i \cdot n^{\mu i}. \tag{42}$$

5. P-branes case

It is straightforward to adapt the disscusion to consider higher co-dimensions [9], see [10] for a generalization of auxiliary variables for relativistic membranes for an arbitrary codimension. In this section we consider the case of p-branes.

Simmilarly the constraints introducing the auxiliary variables are given by:

$$e_a = \nabla_a X, \tag{43}$$

$$e_a \cdot n^i = 0, \tag{44}$$

$$n^i \cdot n^j = \delta^{ij}, \tag{45}$$

$$K^i_{ab} = e_a \tilde{\nabla}_b n^i, \tag{46}$$

$$g_{ab} = e_a \cdot e_b. \tag{47}$$

In the same way that in the relativistic particle case the corresponding Lagrange multipliers are introduced to implement the constraints to the total action given by:

$$S_T = S[g_{ab}, K^i_{ab}] + \int d\xi \sqrt{-g} f^a \cdot (e_a - \nabla_a X)$$
$$+ \int d\xi \sqrt{-g}[\lambda^a_i (e_a \cdot n^i) + \lambda_{ij}(n^i \cdot n^j - \delta^{ij})]$$
$$+ \int d\xi \sqrt{-g}[\Lambda^{ab}_i (K^i_{ab} - e_a \tilde{\nabla}_b n^i)$$
$$+ \lambda^{ab}(g_{ab} - e_a \cdot e_b)], \tag{48}$$

and when the variations for each of the independent variables are done we get the expression for the conserved stress tensor:

$$f^a = (T^{ab} - S^{ac}_i K^{bi}_c) e_b - \tilde{\nabla}_b S^{ab}_i \cdot n^i, \tag{49}$$

and it is easily seen that this stress is the equivalent to the given in the one-dimensional case

$$f^{a\mu} \longrightarrow f_p. \tag{50}$$

6. Concluding remarks

In this paper we present the one-dimensional version of the method given in Ref.5, and we showed that this technique help us to avoid non-linearities, and reveals us an structure inherent to any theory of embbeded surfaces. The novelty of this method consist in treating the extrinsic curvature (first [FS] curvature for the particle), like g_{ab}, (γ for the particle) as an auxiliary variable. This treatment can be also extended to charged particles and to null-like particles, this is particles moving on the cone-light [2, 9].

Acknowledgments

Warm thanks to E. Rojas for his detailed reading and criticism through the whole elaboration of this paper. This work was supported by the CONACyT grant CO1-41639

i The Gauss-Weingarten equations appearing in theory of surfaces are: $D_a e_b = \gamma^c_{ab} - K^i_{ab} n_i$, $D_a n^i = K^i_{ab} e^b + \omega^{ij}_a n_j$, and describe completely the extrinsic geometry of the world sheet

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