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Towards an inflationary scenario in noncommutative quantum cosmology

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In this work, we apply a previous proposal to study noncommutative cosmology and apply it to inflation, we analyze an FRW cosmological background with a scalar field, via the WDW equation. In this scenario noconmmutativity is introduced in the gravitational field as well as in the scalar field through a deformation of minisuperspace and are able to find an exact the noncommutative wave function.

Keywords: Quantum cosmology; noncommutative quantum mechanics.

En este trabajo aplicamos propuestas previas al estudio de la cosmología no conmutativa aplicando esto a inflación, analizamos un modelo cosmológico FRW con un campo escalar a través de la ecuación WDW. En este escenario la no conmutatividad es introducida entre el campo gravitacional y el campo escalar por medio de la deformación de las relaciones de conmutación en el mini-superespacio y nos es posible encontrar soluciones exactas para la función de onda.

Descriptores: Cosmología cuántica; mecánica cuántica noconmutativa.

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The simplest approach in the study of the early universe is quantum cosmology (QC), in this simplified scheme the gravitational and matter variables have been reduced to finite degrees of freedom (these models where extensively studied by means of Hamiltonian methods in the 1970's, for reviews see [1, 2]). For homogenous cosmological models the metric depends only on time, this permits to integrate out the space dependence and obtain a model with a finite dimensional configuration space, *minisuperspace*, whose variables are the 3-metric components. This approach is used, because a full quantum theory of gravity has not been constructed, although a few candidates exist (String Theory and Loop Quantum Gravity being the more successful), an in this approach we can canonically quantize the models, yielding a Klein-Gordon type equation.

On another front, in the last few years there have been several attempts to study the possible effects of noncommutativity in the cosmological scenario. In particular in Ref. 3 the authors in a cunning way avoid the difficult technicalities of analyzing noncommutative cosmological models, when these are derived from a noncommutative theory of gravity [4]. Their proposal is to introduce the effects of noncommutativity in quantum cosmology, by a deformation of minisuperspace, and is achieved due to a moyal deformation of the Wheeler-DeWitt (WDW) equation, similar to the case of the noncommutative quantum mechanics [5, 6]. Some work has been done in this direction, for example in Ref. 7 the authors study the implications of noncommutative geometry in minisuperspace variables for an FRW universe with a conformally coupled scalar field, using the bohmian formalism of quantum trajectories [8], also in Ref. 9 a noncommutative deformation of a scalar field coupled to scalar-tensor type gravity was considered.

The aim of this paper is to construct a noncommutative inflationary scenario in which the effects of noncommutativity affect the gravitational as well as the matter sector. The noncommutativity introduced here is along the lines of [3]. Some work has been done in this topic; in Ref. 10 the effects of noncommutativity during inflation are explored, but noncommutativity is only incorporated to the scalar field neglecting the gravitational sector.

Let us start by analyzing the quantum inflationary model, for this we use the line element for a homogeneous and isotropic universe, the so called Friedmann-Robertson-Walker (FRW) metric, in the form

$$ds^2 = -\,\mathrm{N}^2(t)dt^2 + e^{2\alpha(t)}\left[\frac{dr^2}{1-\mathrm{k}r^2} + r^2(d\vartheta^2 + sin^2\vartheta d\varphi^2)\right], \quad (1)$$

where $a(t) = e^{\alpha(t)}$ is the scale factor, $\mathrm{N(t)}$ is the lapse function, and k is the curvature constant that takes the values $0, +1, -1$, which correspond to a flat, closed and open universe, respectively.

The effective action we are to work on is [11]

$$\mathrm{S_{tot}} = S_g + S_\phi = \int dx^4\sqrt{-g}\left[R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right], \quad (2)$$

where ϕ is a scalar field endowed with a scalar potential $\mathrm{V}(\phi) = \mathrm{V_0 e}^{-(\lambda/\sqrt{12})\phi}$, this is the simplest inflationary potential and can be solved analytically.

The Lagrangian for a FRW cosmological model is

$$\mathcal{L} = \mathrm{e}^{3\alpha}\left[6\frac{\dot\alpha^2}{\mathrm{N}} - \frac{1}{2}\frac{\dot\phi^2}{\mathrm{N}} + \mathrm{N}\left(\mathrm{V}(\phi) - 6\kappa\mathrm{e}^{-2\alpha}\right)\right], \quad (3)$$

for simplicity we consider a flat universe ($\mathrm{k} = 0$), yielding the canonical momenta

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = 12 e^{3\alpha} \frac{\dot{\alpha}}{N}, \qquad \dot{\alpha} = \frac{N}{12} e^{-3\alpha} \Pi_\alpha,$$
$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -e^{3\alpha} \frac{\dot{\phi}}{N}, \qquad \dot{\phi} = -N e^{-3\alpha} \Pi_\phi. \quad (4)$$

We are now in position to write the corresponding canonical Hamiltonian $\mathcal{H}$

$$\mathcal{H} = \frac{1}{24} e^{-3\alpha} \left[\Pi_\alpha^2 - 12\Pi_\phi^2 - 24 e^{6\alpha} V(\phi) \right]. \quad (5)$$

The WDW equation for this model is achieved by the usual identification, $\Pi_{q^\mu}=-i\partial_{q^\mu}$ in Eq. (5); here $q^\mu = (\alpha, \phi)$. In this way the total Hamiltonian can be written under a particular factor ordering, as

$$\mathcal{H}\Psi = \frac{1}{24} e^{-3\alpha} \left[-\frac{\partial^2 \Psi}{\partial \alpha^2} + \frac{\partial^2 \Psi}{\partial \tilde{\phi}} - 24 e^{6\alpha} V(\tilde{\phi}) \right] \Psi = 0, \quad (6)$$

with $\tilde{\phi} = \phi/\sqrt{12}$, where Ψ is called the wave function of the universe, and $V(\tilde{\phi}) = V_0 e^{-\lambda\tilde{\phi}}$ is the corresponding escalar potential. From Ref. 12, we know that the inflationary scenario is obtained when $\lambda < \sqrt{2}$.

In order to be able to find analytical solutions for the noncommutative case we do a transformation of the minisuperspace variables. Making the following change of variables

$$x = -6\alpha + \lambda\tilde{\phi}, \qquad y = -\alpha + \frac{6}{\lambda}\tilde{\phi}, \quad (7)$$

the resulting WDW equation is

$$\frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{\lambda^2}\frac{\partial^2 \Psi}{\partial y^2} - \frac{24 V_0}{\lambda^2 - 6^2} e^{-x} \Psi = 0, \quad (8)$$

and by separation variables, using $\Psi = X(x)Y(\tilde{y})$ with $\tilde{y} = \lambda y$, we obtain the set of differential equation for the functions X and Y

$$\frac{d^2 X}{dx^2} + \left(\left(\frac{\eta}{2}\right)^2 - \beta e^{-x} \right) X = 0,$$
$$\frac{d^2 Y}{d\tilde{y}^2} + \left(\frac{\eta}{2}\right)^2 Y = 0, \quad (9)$$

here we have defined $\beta = 24V_0/(\lambda^2 - 6^2)$ and η is a separation constant. The solutions for these equations are given in term of complex order Bessel functions. For $|\lambda| < 6$

$$X(x) = J_{i\eta}\left(\pm 2\sqrt{\beta} e^{-x/2}\right) + J_{-i\eta}\left(\pm 2\sqrt{\beta} e^{-x/2}\right),$$
$$Y(y) = A_0 e^{i\frac{\eta}{2}\lambda y} + A_1 e^{-i\frac{\eta}{2}\lambda y}, \quad (10)$$

and for other values of λ

$$X(x) = I_{i\eta}\left(\pm 2\sqrt{\beta} e^{-x/2}\right) + K_{i\eta}\left(\pm 2\sqrt{\beta} e^{-x/2}\right),$$
$$Y(y) = A_0 e^{i\frac{\eta}{2}\lambda y} + A_1 e^{-i\frac{\eta}{2}\lambda y}, \quad (11)$$

$I_{\pm i\eta}$ and $K_{\pm i\eta}$ are modified Bessel Functions, from which we can construct the complete wave function.

To extract a normalizable wave function we need to construct wave packets to form a Gaussian state.
In order to see if the same physical information is encoded in the new variables, we search for the classical solutions using the semiclassical analysis (WKB-like method) [11, 13]. For this one considers the ansatz on the wave function

$$\Psi(\alpha, \phi) = e^{-S}, \quad (12)$$

and the following usual conditions

$$\left(\frac{\partial S}{\partial x}\right)^2 \gg \left|\frac{\partial^2 S}{\partial x^2}\right|, \qquad \left(\frac{\partial S}{\partial y}\right)^2 \gg \left|\frac{\partial^2 S}{\partial y^2}\right|, \quad (13)$$

in this way, the Einstein-Hamilton-Jacobi equation (EHJ) is obtained from Eq.(8). In the coordinates (x, y), this equation is read as

$$\left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2 - \beta e^{-x} = 0, \quad (14)$$

where we have used $\Pi_x = \partial S/\partial x, \Pi_y = \partial S/\partial y$. By choosing $S = S_x S_y$ we obtain the following solutions

$$S_x = \pm \frac{2}{\sqrt{\beta}\mu} e^{-x/2}, \ \ S_y = \mu = \mathrm{cte}. \quad (15)$$

Solving for the momenta of the original variables from the new variables

$$\Pi_\alpha = \pm \frac{6}{\sqrt{\beta}} e^{-x/2} = \pm \frac{6}{\sqrt{\beta}} e^{3\alpha - \frac{\lambda}{2}\phi},$$
$$\Pi_\phi = \mp \frac{\lambda}{\sqrt{\beta}} e^{-x/2} = \pm \frac{\lambda}{\sqrt{\beta}} e^{3\alpha - \frac{\lambda}{2}\phi}, \quad (16)$$

the classical behavior is found by solving the relationship between (16) and Eqs. (4), obtaining the scale factor as a function of the scalar field

$$a = a_0 e^{\frac{1}{2\lambda}\tilde{\phi}}, \quad (17)$$

and also, the corresponding time behavior,

$$a = a_0 \tau^{\frac{2}{\lambda^2}}, \ \ \tilde{\phi} = \frac{2}{\lambda} \ln\left(\frac{\lambda^2}{4\sqrt{3\beta}}\tau + \tilde{\phi}_0\right), \quad (18)$$

from which we obtain the known result, that for an inflationary scenario, scale factor has an increasing power law power behavior when $\lambda < \sqrt{2}$.

Now that we have constructed our quantum inflationary model we can introduce a noncommutative deformation. We start with the commutative WDW equation, Eq. (8), which is defined in the minisuperspace variables x, y. We will, as in Ref. 3 do a noncommutative deformation of the minisuperspace

$$[x, y] = i\theta, \quad (19)$$

this is equivalent to a deformation in the original variables α and ϕ

$$[\alpha, \phi] = i\overline{\theta} \qquad \text{with} \qquad \overline{\theta} = \frac{\theta\lambda}{6^2 - \lambda^2}, \tag{20}$$

we can see, that the highly noncommutative case is reached in the limit $\lambda \sim |6|$, this is an interesting fact, because even if the parameter θ is small, the effects of noncommutativity can be large if the parameters in the potential are fined tuned, this could be related to the UV/IR mixing that appears in noncommutative field theory.

This noncommutativity can be formulated in terms of noncommutative minisuperspace functions with the moyal product of functions

$$f(x,y) \star g(x,y) = f(x,y) e^{i\frac{\theta}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x}\right)} g(x,y). \tag{21}$$

Then the noncommutative Wheeler-DeWitt (NCWDW) equation can be written as

$$\frac{1}{24}\mathrm{e}^{\frac{6\mathrm{x}-\lambda^2\mathrm{y}}{6-\lambda^2}} \star \left(-\Pi_\mathrm{x} + \Pi_\mathrm{y} - \frac{24\mathrm{V}_0}{\lambda^2 - 6^2}\mathrm{e}^{-\mathrm{x}}\right) \star \Psi(\mathrm{x},\mathrm{y}). \tag{22}$$

We know from noncommutative quantum mechanics [5], that the symplectic structure is modified. Now it is possible to return to the original commutative variables and usual commutation relations if we introduce the following change of variables

$$x \to x + \frac{\theta}{2}\Pi_y \qquad \text{and} \qquad y \to y - \frac{\theta}{2}\Pi_x. \tag{23}$$

The efects of the moyal star product are reflected in the WDW equation, only on the potential

$$V(x,y) \star \Psi(x,y) = V(x + \frac{\theta}{2}\Pi_y, y - \frac{\theta}{2}\Pi_x)\Psi(x,y), \tag{24}$$

after taking this in to account and using the usual substitutions $\Pi_{\mathrm{q}^\mu} = -\mathrm{i}\partial_{\mathrm{q}^\mu}$ we arrive to

$$\left(\frac{\partial^2}{\partial \mathrm{x}^2} - \frac{\partial^2}{\partial \overline{\mathrm{y}}^2} - \beta \mathrm{e}^{-\left(\mathrm{x} + \frac{\mathrm{i}}{2}\theta\frac{\partial}{\partial \overline{\mathrm{y}}}\right)}\right)\Psi = 0, \tag{25}$$

where $\overline{y} = \lambda y$ and $\beta = 24V_0/(\lambda^2 - 36)$. By using the anzats $\Psi(x, \overline{y}) = \chi(x)\psi(\overline{y})$, the equation is separable, this gives the differential equation

$$\frac{d^2\chi(x)}{dx^2} + \left(\nu^2 - \beta e^{\frac{\lambda\nu}{2}\theta} e^{-x}\right)\chi(x), \tag{26}$$

which has solutions for $\lambda < 6$

$$\chi(x) = I_{i\nu}\left(\sqrt{|\beta|}e^{-\frac{x}{2} + \frac{\nu\lambda\theta}{4}}\right) + K_{i\nu}\left(\sqrt{|\beta|}e^{-\frac{x}{2} + \frac{\nu\lambda\theta}{4}}\right), \tag{27}$$

finally get the noncommutative wave function

$$\Psi(x,y) = e^{i\nu\lambda y} \times \left[I_{i\nu}\left(\sqrt{|\beta|}e^{-\frac{x}{2} + \frac{\nu\lambda\theta}{4}}\right) + K_{i\nu}\left(\sqrt{|\beta|}e^{-\frac{x}{2} + \frac{\nu\lambda\theta}{4}}\right)\right], \tag{28}$$

this of course in the minisuperspace variables x and y. This is the noncommutative wave function, because we have taken $\lambda < 6$ and in the commutative case for $\lambda < \sqrt{2}$ is the inflationary limit, we believe that this wave function can describe a noncommutative universe in an inflationary epoch. But in order to prove this statement the classical noncommutative behavior has to be constructed In this paper the first steps of a new noncommutative formulation for inflation is proposed. The noncommutative deformations affects the gravitational sector as well as the matter sector, previous studies in this direction had neglected the effects of noncommutativity in the graviational degrees of freedom. The noncommutativity is constructed as in Ref. 3, so the deformations are done at the minisuperspace, from this construction the NCWDW equation is solved, giving the noncommutative wave functions. The cosmological implications of this model are under research and will be reported elsewhere.

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