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Numerical evolution of a scalar field soliton

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In this paper, we present the results of the numerical evolution of scalar field solitons in asymptotically flat space-times. First we introduce the model we are working with and then we briefly mention the complications that arise in this particular numerical simulation, presenting some details of code we used and some examples of the results.

Keywords: Numerical relativity; classical black holes and solitons.

En este árticulo, presentamos los resultados de la evolución numérica de solitones de campo escalar en espacios-tiempos asintóticamente planos. En primer lugar introducimos el modelo con el que estamos trabajando y posteriormente mencionamos las complicaciones que surgen en esta simulación numérica, mostrando algunos detalles del código y ejemplos de los resultados.

Descriptores: Relatividad numérica; agujeros negros clásicos y solitones.

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1. Introduction

This article reviews the results of studies of scalar field solitons that have been originally reported in Refs. 18 and 25. We will show the dynamic interplay here between the theoretically motivated studies and the numerical evolution technology. The fact is that, while numerical simulations are often the only available methodology for attacking certain theoretical issues, the availability of some models of purely theoretical interest serve at the same time as the arena where the numerical techniques can be inspired, developed and tested. One such example arises with the study of the issues related to hairy black holes: the topic of black hole hair is often thought of as being of interest only for mathematical physics, but the present work will underline its usefulness in the development of tools for numerical relativity.

The no-hair conjecture for stationary black holes (the supposition that all such black holes are completely determined by the value of the globally conserved charges defined in the asymptotic region such as mass, charge and angular momentum) is these days considered to be clearly wrong, since the list of counterexamples becomes larger by the day: Einstein-Yang-Mills [1], Einstein-Skyrme [2], Einstein-Yang-Mills-dilaton [3], Einstein-Yang-Mills-Higgs [4], Einstein-non-abelian-Procca [4] fields. In some sub-communities, the idea seems to be holding out that a modified version that applies only to stable black holes could remain valid, despite the fact that for some of the examples above some claims of stable non-trivial solutions exist in the literature.

A regime where it seemed for a while that there was hope for a restricted form of the conjecture was the scalar field arena. Here we had the original no-hair theorems of Bekenstein [5], covering the case of minimally coupled scalar fields with convex potentials; other theorems dealing with the case of minimally coupled fields with arbitrary potentials were obtained in Refs. 6 and 7. The so-called Bronnikov-Melnikov-Bocharova-Bekenstein (BMBB) black hole "solution" [8], which corresponds to a spherical symmetric extremal black hole with a scalar field conformally coupled to gravity, seemed to represent a discrete example of scalar hair, as it was shown [9] that there are no other static, spherically symmetric Black Hole solutions in this theory. Later on, it was shown that this configuration, which presents a divergence of the scalar field at the horizon, cannot be considered as a regular black hole solution because the energy momentum tensor is ill-defined at the horizon [10]. Finally it has been shown that, if one requires that the scalar field be bounded throughout the static region, then, there are no solutions at all [11].

For more general cases of non-minimal coupling, there are results [12, 13] showing that under the assumption that a certain "conformal factor" does not vanish or blow up, there are no nontrivial black hole solutions. Next, there is a result by Ref. 14 that does not rely on such an assumption, and which considers the existence of static, spherically symmetric black hole solutions in theories in which the sign of the non-minimal coupling constant is negative (the only case not covered by other theorems). There it is shown that, under certain suppositions concerning the form of the energy-momentum flux, there are no nontrivial solutions. In Ref. 15 it is argued that these suppositions are not fully justifiable and numerical evidence is given against the existence of this type of black hole that does not rely on these assumptions.

It is therefore a rather unexpected development that hairy black hole solutions have now been found in both theories with minimal [16] as well as non-minimal [17] coupled scalar fields, simply by considering asymptotically anti de-Sitter, rather than asymptotically flat, boundary conditions. Moreover these papers have strong indications that, under certain conditions, the new solutions are stable.

In Ref. 18, some of us analyzed the situation regarding the asymptotically anti de Sitter case, in the light of existing results for the asymptotically flat case, discussed the points where the differences are relevant, and give a simple explanation of some of the features of the new solutions and point to some surprising conjectures that can be directly infered from this understanding. The method used in this part is a generalization of one that was successfully employed in deriving a general characterization of hairy black holes in a wide range of theories [19].

Later, a new family of scalar-hairy black holes (BH) and their corresponding solitons (scalarons) were found within an Einstein-Higgs theory with a non-positive semi-definite scalar field potential $V(\phi)$ [20]. This kind of potential violates the weak-energy condition (WEC) and therefore invalidates the applicability of the no-scalar-hair theorems [6,7,21]. These configurations are interesting in several respects. On one hand, they constitute an example that obstructs the extension of no-hair theorems to potentials of this type. On the other hand, they can be useful for testing some of the predictions of the recent isolated-horizon formalism [22]. In fact, these configurations can be shown to be unstable with respect to radial-linear perturbations, and therefore can be seen as *bound states* of non-hairy black holes and scalarons (cf. [23] in the context of colored BH). The simple perturbation analysis, however, does not provide any definite answer about the final fate of these configurations. Nevertheless, heuristic analysis based on energetic arguments does provide some clues about their fate. Presumably, the plain Schwarzschild BH constitutes the lower energy-mass bound (the "ground state") of possible BH configurations with fixed boundary conditions, which correspond to fixed horizon area A_h and asymptotic flatness. Therefore, among all BH configurations within the theory, the Schwarzschild BH is the energetically preferred one.

In this paper, we present a brief description of the numerical code we developed [24] and the results obtained in Ref. 25 where we performed a fully non-linear numerical evolution of the scalar solitons, preparing the way for a future analysis of the scalar-hairy black holes. The philosophy of our analysis is similar to that of Straumman and Zhou for the case of "colored solitons" (solitons in Einstein-Yang-Mills theory) [26]. The initial conditions correspond to unstable scalar solitons in a globally regular space-time. Two different sets of initial perturbations will be considered: one that leads to the formation of a Schwarzschild BH accompanied by a small amount of radiated scalar field, and another one which corresponds to an "exploding" configuration, where a global phase transition is triggered through the formation of an outward moving domain wall.

2. Model and numerical methodology

We shall consider a model of a scalar field minimally coupled to gravity and with a non-trivial self-interaction potential. The model is described by the Lagrangian (we shall use units such that $G = c = 1$):

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{16\pi}R - \frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi - V(\phi)\right]. \tag{1}$$

We choose the following asymmetric scalar-field potential leading to the desired asymptotically flat solutions:

$$V(\phi) = \frac{\sigma}{4}(\phi-a)^2\left[(\phi-a)^2 - \frac{4(\eta_1+\eta_2)}{3}(\phi-a) + 2\eta_1\eta_2\right], \tag{2}$$

with σ, η_1, η_2 and a constant parameters. For this class of potentials one can easily show that, if $\eta_1 > 2\eta_2 > 0$, $\phi = a$ corresponds to a local minimum, $\phi = a + \eta_1$ to the global minimum, and $\phi = a + \eta_2$ to a local maximum. The key feature of this potential for the asymptotically flat and static solutions to exist is that the local minimum at $\phi = a$ is also a zero of $V(\phi)$ [20]. The factor σ in front of the potential fixes the scale, so one can always take $\sigma = 1$ and just re-scale everything for a different σ afterward. For the simulations discussed here, we shall take the following values for the parameters: $\sigma = 1$, $\eta_1 = 0.5$, $\eta_2 = 0.1$ and $a = 0$.

The field equations following from the Lagrangian (1) are the Einstein's field equations and the Klein-Gordon (KG) equation:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \qquad \Box\phi = \frac{\partial V(\phi)}{\partial\phi}, \tag{3}$$

The stress-energy tensor for the scalar field is

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi + V(\phi)\right]. \tag{4}$$

In order to perform a numerical analysis of the problem at hand we shall use a 3+1 approach based on the standard ADM equations [27,28]. Moreover, we shall assume that the shift vanishes. The evolution equations for the 3-metric (γ_{ij}) and the extrinsic curvature (K_{ij}) are

$$\partial_t\gamma_{ij} = -2\alpha K_{ij}\ , \tag{5}$$

$$\partial_t K_{ij} = -\mathcal{D}_i\mathcal{D}_j\alpha + \alpha\left(R_{ij} + KK_{ij} - 2K_{il}K^l_j - 8\pi M_{ij}\right)\ , \tag{6}$$

and the Hamiltonian and momentum constraints are

$$\mathcal{H} := R + K^2 - K_{ij}K^{ij} - 16\pi\rho = 0, \tag{7}$$

$$\mathcal{M}_i := \mathcal{D}_l\left(K^l{}_i - K\delta^l{}_i\right) - 8\pi J_i = 0, \tag{8}$$

with α the lapse function, $\mathcal{D}_i$ and R_{ij} the covariant derivative and Ricci tensor associated with γ_{ij}, $R := \mathrm{tr}R_{ij}$, $K := \mathrm{tr}K_{ij}$, and where the matter sources are defined in terms of the stress-energy tensor as

$$\rho = n_\mu n_\nu T^{\mu\nu}, \tag{9}$$

$$J_i = -n_\mu T_i^\mu, \tag{10}$$

$$S_{ij} = T_{ij}, \tag{11}$$

$$M_{ij} = S_{ij} + \frac{1}{2}\gamma_{ij}\left(\rho - S\right), \tag{12}$$

with n^μ the unit normal to the spatial hypersurfaces.

We shall focus on the dynamics of a spherically symmetric space-time described by the metric

$$ds^2 = -\alpha^2 dt^2 + A dr^2 + B r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{13}$$

The spherical symmetry implies that all dynamical functions depend only on r and t.

The analysis of this problem in the numerical regime is complicated by the need for the code to deal with solutions that have a regular origin which is normally covered by the spherical coordinates which cease to be good coordinates at that point and by the simultaneous requirement that the code, be able to handle the occurrence of apparent horizons. A code of this type including this kind of regularization was presented in Ref. 24. Our regularization method is similar in spirit, if not in detail, to that presented by Arbona and Bona in Ref. 29, the main difference being that the approach of Arbona and Bona was tied to the use of the Bona-Masso evolution system [30–34], while our algorithm is much more general.

3. Evolutions

The development of the code allows us to answer the physically interesting questions regarding the evolution of a perturbed soliton.

The initial conditions used to study the evolution are the static soliton solutions computed in Ref. 20, plus/minus a small Gaussian perturbation in $\partial_t\phi$. Depending on the sign of the initial perturbation, the solitons either collapse to a Schwarzschild black hole or else "explode" into an outward moving domain wall.

3.1. Collapse to a black hole

Figure 1 shows the evolution of the metric functions A. In the plots, solid lines correspond to initial and final configurations, and dotted lines to intermediate stages (the separation in time between these lines is $\Delta t = 25$). At first, there are small oscillations around the initial value, but later on the radial metric starts to grow in a form characteristic of the slice stretching associated with BH spacetimes. The analysis of the behavior of the lapse function shows the characteristic collapse of the lapse indicative of the approach to a singularity.

The corresponding evolution of the scalar field can be seen in Fig. 2. Notice how the scalar field moves toward the local minimum at $\phi = 0$ everywhere. At late times, the evo-

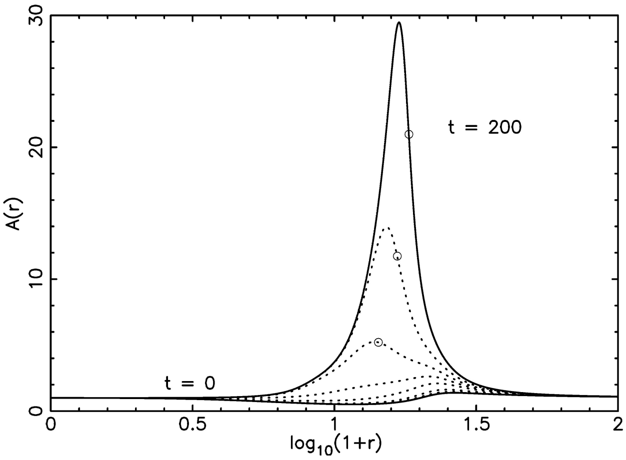

FIGURE 1. Evolution of the metric function A for the negative perturbation. Notice how at late times the metric function grows, indicating slice stretching. The circles show the location of the apparent horizon.

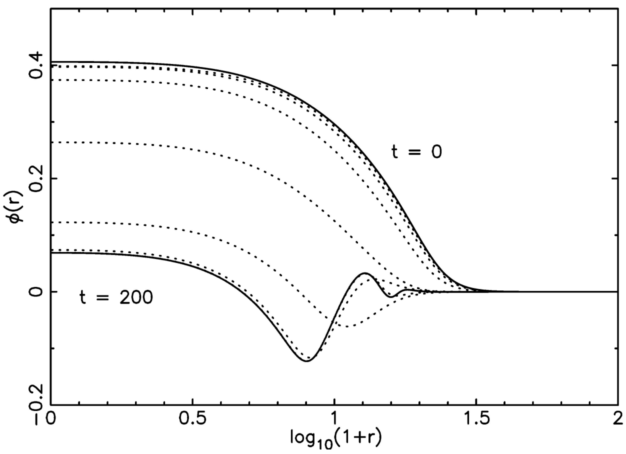

FIGURE 2. Evolution of the scalar field ϕ for negative perturbation. At late times the scalar field has values below 0.1 everywhere, which implies that we are in the region where the potential is positive.

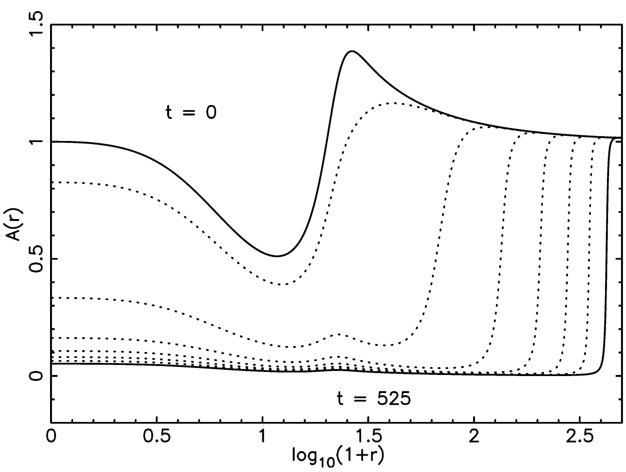

FIGURE 3. Evolution of the metric function A for the positive perturbation.

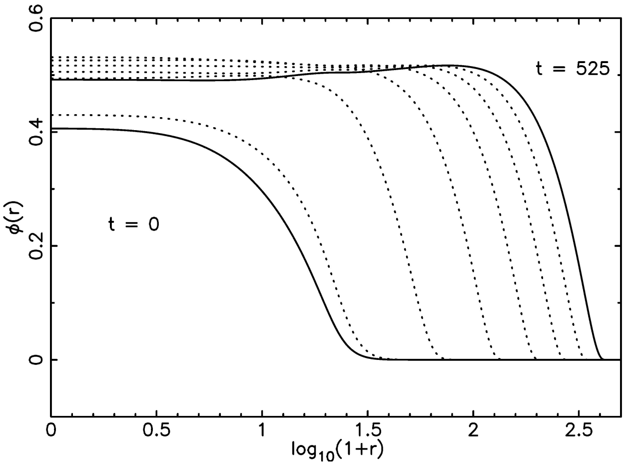

FIGURE 4. Evolution of the scalar field ϕ for the positive perturbation. The scalar field is now moving toward the true minimum of the potential at $\phi = 0.5$ everywhere.

lution of the inner regions is frozen due to the collapse of the lapse there. By that time, however, the scalar field has values below 0.1 everywhere, which implies that we are in the region where the potential is positive.

3.2. Explosion

Figure 3 shows the evolution of the metric function A for this case (lines are now separated in time by $\Delta t = 75$). It is evident that the dynamical evolution is completely different from the case described in the previous section. In the first place, there is no indication of the slice stretching effect. Moreover, in the evolution of the radial metric A, it is clear that there is a wall moving outward. The wall moves essentially at a uniform speed, even if this is not evident in the log plot (this speed is approximately 1 in our units, which coincides with the speed of light in the outer regions). Inside this wall, the radial metric is collapsing to zero. The angular metric is also collapsing to zero in this region, but not as rapidly.

The evolution of the scalar field can be seen in Fig. 4. In contrast to the results of the previous section, in this case the scalar field is moving toward the true minimum of the potential at $\phi = 0.5$. Since this minimum corresponds to a negative value of the potential, the inside of the wall resembles an anti-de-Sitter spacetime, except for the fact that the scalar field is not uniform. Still, one would expect the formation of a big-crunch type singularity in this region in a finite proper time [35, 36]. However, because of the singularity avoiding properties of harmonic slicing, this singularity would only be reached after an infinite coordinate time.

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