From conformal Killing vector fields to boost-rotational symmetry

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We discuss a connection between three-dimensional Riemannian manifolds (Σ, γ) admitting a special conformal Killing vector field ξ and static vacuum or non-vacuum spacetimes. Any such (Σ, γ) generates a vacuum spacetime (M, g) but it also generates a spacetime (M, g, Φ), where (g, Φ) satisfies the Einstein-Klein-Gordon massless minimally coupled gravity equations, or the Einstein-Conformal scalar field equations. The resulting spacetimes either admit four Killing vector fields or possess boost and rotational symmetry. We argue that this connection goes beyond the vacuum or Einstein-scalar field system and it should be viewed as a mechanism of generating solutions for the Einstein equations, admitting a hypersurface orthogonal Killing vector field.

Keywords: General relativity; conformal Killing vector field; Einstein equations.

Se discute la conexión entre variedades Riemannianas (Σ, γ) de dimensión tres que admiten un campo vectorial de Killing conforme ξ y espacios- tiempo estáticos asociados a sistemas en el vacío o no-vacío. Cualquiera de estas variedades (Σ, γ) generan un espacio-tiempo (M, g) e igual generan un espacio-tiempo (M, g, Φ), donde (g, Φ) satisfacen las ecuaciones para el campo escalar asociadas a los sistemas de Einstein-Klein-Gordon con acoplamiento mínimo o conforme. Los espacios-tiempo asociados resultantes admiten cuatro campos vectoriales de Killing o una simetría de "boost " y rotacional. Se argumenta como esta conexión va mas allá de los sistemas en el vacío o de los sistemas de campos escalares y esto puede ser visto como un mecanismo para generar soluciones de las ecuaciones de Einstein, que admitan un campo vectorial de Killing ortogonal a una hipersuperficie.

Descriptores: Relatividad general; campo vectorial de Killing conforme; ecuaciones de Einstein.

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1. Introduction

This work is focused on properties and applications to relativistic gravity of three-dimensional Riemannian manifolds (Σ, γ) satisfying the following conditions:

a) The smooth metric γ admits a hypersurface-orthogonal, non-singular conformal Killing vector field ξ,

b) The Ricci tensor of γ is described by $R_{ab} = \lambda(3X_aX_b - \gamma_{ab})$, where $\lambda \neq 0$ and the unique eigenvector X of Ricci is parallel to the field ξ.

c) The expansion $\Theta = D_a\xi^a$ and magnitude $\gamma(\xi, \xi)$ of ξ are determined by the eigenvalues λ of the Ricci tensor.

We shall begin by first discussing the reasoning that lead us to study such manifolds. In this regard, we recall that any smooth spacetime (M, g) admitting a hypersurface-orthogonal timelike Killing vector field ξ' admits a local coordinate chart so that $\xi' = \partial/\partial t$ and the components of g are described by [1]:

$$g = -V^2dt^2 + \gamma_{ab}dx^adx^b, \qquad a, b = 1, 2, 3, \tag{1}$$

where γ_{ab} are the components of the induced positive definite metric on any t = const spacelike hypersurface and $-V^2$ stands for the magnitude of ξ'. Whenever g is a solution of Einstein's vacuum equations, the red-shift factor (or lapse function) V and the components of γ_{ab} satisfy on any t =const:

$$VR_{ab} = D_aD_bV, \tag{2}$$

$$D^aD_aV = 0, \tag{3}$$

where (R_{ab}, D) stand for the Ricci curvature and the covariant derivative operator associated with γ. Moreover, as is well known, and easy verifiable, for any smooth solution (γ, V) of those equations, the York-Cotton tensor $R_{abc}(\gamma)$ satisfies [2]:

$$VR_{abc} = 2R_{ab}D_cV - 2R_{ac}D_bV + \gamma_{ab}R_{cd}D^dV - \gamma_{ac}R_{bd}D^dV. \tag{4}$$

In order to establish the connection between the three manifolds introduced earlier on and relativistic gravity, we view system (2) from a slightly different point of view. We consider a smooth three-manifold Σ equipped with a Riemannian metric γ' and a strictly positive function V' so that (γ', V')

satisfy Eq. (2) and thus also (4). The triplet (Σ, γ', V') gives rise to a static spacetime (M, g) in the following way: on the product manifold $M \equiv \Sigma \times I\!R$, we define the Lorentzian metric $g = -V^2 dt^2 + \gamma$, where γ and V are the lifts of (γ', V') on M by the natural projection of M upon Σ. It is easily seen that this (M, g) is a vacuum spacetime admitting a hypersurface orthogonal timelike Killing vector field ξ', possessing complete orbits. Thus any (Σ, γ, V) as above leads to a static vacuum spacetime (M, g). For simplicity, we hereafter drop the distinction between primed fields (γ', V') defined on Σ and their lifts (γ, V) defined on M. Of course this connection of (Σ, γ, V) to the vacuum (M, g) is rather well known. However, we would like to discuss the role of relation (4) in this type of association. Normally Eq. (2) augmented by a suitable regularity or and boundary conditions, determine the positive function V and the metric γ, and in this event (V, γ) identically satisfies (4). Suppose, however, that one is interested in constructing a solution (γ, V) of (2) so that the Ricci curvature of γ has a special structure, for example being algebraically special on Σ. To be more concrete, let us suppose that $R_{ab}(\gamma) = \lambda(3X_a X_b - \Lambda_{ab})$ for an unknown non-vanishing scalar λ and a smooth field X. If Eq. (2) admits this solution, then the proposed $R_{ab}(\gamma)$ and V ought to obey relation (4). For the proposed Ricci, $R_{abc}(\gamma)$ reduces to $R_{abc} = D_b R_{ca} - D_c R_{ba}$ and it takes the form:

$$R_{abc} = (3X_a X_c - \gamma_{ac}) D_b\lambda - (3X_a X_b - \gamma_{ab}) D_c\lambda$$
$$+3\lambda(X_a D_b X_c + X_c D_b X_a - X_a D_c X_b - X_b D_c X_a). \quad (5)$$

Combining this expression with Eq. (4) yields:

$$V[(3X_a X_c - \gamma_{ac}) D_b\lambda - (3X_a X_b - \gamma_{ab}) D_c\lambda$$
$$+ 3\lambda(X_a D_b X_c + X_c D_b X_a - X_a D_c X_b$$
$$- X_b D_c X_a)] = 3\lambda[2X_a(X_b V_c - X_c V_b)$$
$$+ \gamma_{ab}(X^d V_d X_c - V_c) - \gamma_{ac}(X^d V_d X_b - V_b)]. \quad (6)$$

This set of tensorial relations act as constraints in the following sense: solution (γ, V) of (2) subject to the restriction that $R_{ab}(\gamma) = \lambda(3X_a X_b - \Lambda_{ab})$ would exist provided (X_a, λ, V) and components of γ satisfy those constraints on Σ. The content of the above integrability conditions has been worked out elsewhere [3, 4]. They hold true provided X_a, λ, V and γ obey the following on Σ:

$$D_a X_b + D_b X_a = -\frac{2}{3\lambda}(X^c D_c\lambda)\gamma_{ab}$$
$$+ \frac{1}{3\lambda}(X_a D_b\lambda + X_b D_a\lambda), \quad (7)$$

$$D_a X_b - D_b X_a = \frac{1}{3\lambda}(X_a D_b\lambda - X_b D_a\lambda), \quad (8)$$

$$V Y^a D_a\lambda = -3\lambda Y^a D_a V, \quad (9)$$

where in the last equation Y stands for any smooth vector field perpendicular to X. In terms of a new vector field ξ defined via $\xi_a = \lambda^{-1/3} X_a$, Eq. (7, 8) imply:

$$D_a\xi_b + D_b\xi_a = \frac{2}{3}\Theta\gamma_{ab}, \quad (10)$$

$$D_a(\lambda^{\frac{2}{3}}\xi_b) - D_b(\lambda^{\frac{2}{3}}\xi_a) = 0, \quad (11)$$

and thus $\xi = \lambda^{-1/3} X^a(\partial/\partial x^a)$ satisfies the conformal Killing equation in Σ. Moreover, the expansion Θ and magnitude of ξ are determined by the eigenvalue λ via $\Theta = D_a\xi^a = (-1/\lambda)\xi^a D_a\lambda$ and $\xi^a\xi_a = \lambda^{-2/3}$. On the other hand, (11) implies that the one form $A = \lambda^{2/3}\xi_a\, dx^a$ is closed and thus, as long as Σ is assumed to be simply connected, it is exact. As far as (9) is concerned, utilizing the fact that Y is arbitrary, and as long as $\lambda^{1/3} V \neq 0$, one arrives at:

$$Y^a D_a \log|\lambda^{\frac{1}{3}} V| = 0, \qquad Y^a X_a = 0. \quad (12)$$

Introducing a parameter x varying along the integral curves of ξ, then the above equation implies:

$$|\lambda^{\frac{1}{3}} V| = G^2(x), \quad (13)$$

where $G^2 = G^2(x)$ is a smooth function exhibiting a gradient along the integral curves of ξ. Notice however that (9) is also satisfied identically whenever $\lambda = \lambda(x) \Leftrightarrow V = V(x)$ a relation that will be useful latter on.

In summary, any (γ, V) such that

$$R_{ab}(\gamma) = \lambda(3X_a X_b - \gamma_{ab}),$$

is compatible with (4), provided γ admit a conformal Killing vector ξ parallel to X obeying (10-11) and additionally (λ, V) either obey (13) or $\lambda = \lambda(x)$ and thus $V = V(x)$. Suppose for the moment that all manifolds (Σ, γ) obeying conditions (a-c) with ξ the conformal Killing field of γ are explicitly known. For a specific (Σ, γ), by appealing to (13) one may define a red-shift factor V up to a smooth function G exhibiting a gradient parallel to ξ or simply taking $V = V(x)$. The specification of the arbitrary function $G^2(x)$ or $V(x)$ can be determined by demanding satisfaction of (2). Once V has been determined, the triplet (Σ, γ, V) defines a vacuum spacetime (M, g). This conclusion makes clear our motivations for studying manifolds (Σ, γ) obeying (a-c) mentioned earlier on.

Even though this brief discussion demonstrates the connection between (Σ, γ) and vacuum static spacetimes, actually the manifolds (Σ, γ) are of relevance in constructing non-vacuum spacetimes as well. In order to establish this connection, we shall consider Einstein gravity coupled to a real massless scalar field Φ, and in this work we shall limit ourselves to two particular cases:

α) Φ is minimally coupled to gravity,

β) Φ is conformally coupled to gravity.

For the first case, we recall that the relevant equations are:

$$G_{\mu\nu} = k[\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}\nabla^\sigma\Phi\nabla_\sigma\Phi], \quad (14)$$

$$\nabla^\mu\nabla_\mu\Phi = 0, \quad (15)$$

while for the case that Φ is conformally coupled, the corresponding system is described by:

$$(1-\alpha\Phi^2)R_{\mu\nu} = \alpha(4\nabla_\mu\Phi\nabla_\nu\Phi - 2\Phi\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^\sigma\Phi\nabla_\sigma\Phi), \quad (16)$$

$$\nabla^\mu\nabla_\mu\Phi = 0, \qquad \mu,\nu = 0,1,2,3, \quad (17)$$

where $\alpha = 8\pi G/c^4 6$. For a static spacetime (M,g,Φ) with (g,Φ) satisfying either (14) or the above equations, the metric g admits a hypersurface-orthogonal timelike Killing vector field ξ' such that: $\mathcal{L}_{\xi'}g = 0$, and moreover $\mathcal{L}_{\xi'}\Phi = 0$. Relative to the coordinate gauge of (1), any (g,Φ) obeying (14) satisfies on any $t =$ const hypersurface:

$$R_{ab} = V^{-1}D_aD_bV + kD_a\Phi D_b\Phi, \quad (18)$$

$$D^aD_a\Phi = -V^{-1}D^aVD_a\Phi, \quad (19)$$

$$D^aD_aV = 0, \quad (20)$$

where $(R_{ab}\ ,\ D)$ stand for the Ricci tensor and covariant derivative operator of γ_{ab} respectively. On the other hand, for the case of the conformal coupling, it is convenient to work with the metric $\Lambda = e^{2U}\gamma$ where U is related to V via $V = e^U$. Thus for the conformal system we shall be working with the following representation of g:

$$g := -e^{2U}dt^2 + e^{-2U}\Lambda_{ab}dx^adx^b, \quad (21)$$

and relative to this gauge, the covariant Eqs. (16) yield [3,5]:

$$(1-\Phi^2)(R_{ab}-2D_aUD_bU) = 4D_a\Phi D_b\Phi - 2\Phi D_aD_b\Phi - 2\Phi D_a\Phi D_bU - 2\Phi D_aUD_b\Phi - 2\Lambda_{ab}D^c\Phi D_c\Phi, \quad (22)$$

$$(1-\Phi^2)D^aD_aU = D^a\Phi D_a\Phi + 2\Phi D^a\Phi D_aU, \quad (23)$$

$$D^aD_a\Phi = 0, \quad (24)$$

where currently (R_{ab}, D) stand for the Ricci tensor and covariant derivative computed using the positive definite metric Λ. Following the same reasoning as for the vacuum case, we view the systems (18)-(20) resp (22)-(24) as being defined on a three-manifold Σ equipped with the metric γ case of minimal coupling, or Λ, a case of conformal coupling. Smooth configurations (γ, V, Φ) on Σ satisfying (18)-(20) generate a static spacetime (M, g, Φ) obeying the covariant Eqs. (14), while a triplet (Λ, U, Φ) satisfying (22)-(24) on Σ, generates via (21) a spacetime (M, g, Φ) satisfying the conformal Eqs. (16). Even though currently we are dealing with systems more complex and different from the vacuum system (2), nevertheless we shall apply the same lines of argument as for the vacuum case. On a smooth three-manifold Σ, we are interested in constructing solutions (γ, V, Φ) and (Λ, U, Φ) resp so that the $R_{ab}(\gamma)$, $R_{ab}(\Lambda)$ resp , are algebraically special on Σ. These requirements upon the Ricci require the satisfaction of integrability conditions in addition to (18)-(20) and (22)-(24) respectively . Those integrability conditions can be worked out explicitly (see Ref. 6). However for the present paper we shall not need their explicit forms. Rather we shall need the integrability conditions that will arise by assuming on Σ that the functions (Φ, V) and (Φ, U) resp are functionally related, *i.e.* $F(\Phi, V) = 0$ and $\hat{F}(\Phi, U) = 0$, resp. Under the assumption that the gradient of F and $\hat{F}$ resp is not identically zero, smooth configurations $(\gamma, V, \Phi(V))$ and $(\gamma, U(\Phi), \Phi)$ resp. are compatible with the set (18-20) (22-24) resp only for specific form of F and $\hat{F}$ resp. To construct those functions, let us first consider the system (18-20). Imposing the ansatz $\Phi = \Phi(V)$, Eq. (19), combined with the absence of critical points of V in Σ implies: $\Phi(V) = \alpha \ln V + \beta$ with α, β arbitrary constants. For such $\Phi(V)$ the remaining equations implies that (γ, V) satisfy:

$$V^2R_{ab}(\gamma) = VD_aD_bV + k\alpha^2D_aV\,D_bV, \quad (25)$$

$$D^aD_aV = 0. \quad (26)$$

It is advantageous to conformally deform γ so that the resulting metric Λ possesses zero scalar curvature. Even though there exist an infinite parameter family of conformal factors Ω fulfilling this requirement, our choice of Ω is related to the red-shift factor. We define Λ via: $\Lambda = \Omega^2\gamma = V^{2n}\gamma$, where n is a free parameter. Rewriting (25) and (26) in terms of Λ and associated covariant derivative D, yields:

$$V^2R_{ab}(\Lambda) = (1-n)VD_aD_bV + (k\alpha^2 + 3n - n^2)D_aV\,D_bV, \quad (27)$$

$$VD^aD_aV - nD^aVD_aV = 0. \quad (28)$$

Taking the trace of (27) in view of (28), we arrive at:

$$V^2R(\Lambda) = (k\,\alpha^2 + 4n - 2n^2)D^aVD_aV, \quad (29)$$

and thus, requiring that $R(\Lambda) \equiv 0$, the parameter n is chosen so that: $2n^2 - 4n - k\,\alpha^2 = 0$. This algebraic equation, as long as $k > 0$, admits a positive n_+ and a negative n_- root. Hereafter we shall assume that n has been fixed as one of the two roots, implying via (29) that $R(\Lambda) = 0$.

From the system (27-28) specifies the components of Λ and the function V. Let (Λ, V) be a smooth solution of those equations differentiation of (27), antisymetrization, use of the Ricci identity $D_aD_bX_c - D_bD_aX_c = R_{abc}{}^dX_d$ and the fact that at three dimensions the Riemann tensor is determined by the Ricci curvature, it follows that $R_{abc}(\gamma)$, $R_{ab}(\gamma)$ and V obey:

$$VR_{abc} = (1-n)\Big[\,2(R_{ab}D_cV - R_{ac}D_bV) + \Lambda_{ab}R_{cd}D^dV - \Lambda_{ac}R_{bd}D^dV\Big], \quad (30)$$

a relation which for our purpose is identical to relation (4). Accordingly we shall use it in the same manner as for the vacuum case. We inquire whether (27-28) admits solutions (Λ, V) so that $R_{ab}(\Lambda) = \lambda(3X_aX_b - \Lambda_{ab}), \quad X^aX_a = 1$. Combining this Ricci with (30), and via identical algebraic manipulations as those for the analysis of (4), it follows that Λ admits a hypersurface-orthogonal, conformal Killing vector field ξ which obeys conditions (a-c) introduced earlier on. Moreover, V and the eigenvalue λ of Ricci are either related to each other via $\lambda = \lambda(x) \Leftrightarrow V = V(x)$ or they satisfy the following relation:

$$\lambda^{\frac{1}{3}} = \frac{G^2(x)}{V^{1-n}}, \tag{31}$$

where the positive function $G^2(x)$ has the same meaning as (13), *i.e.* it exhibits a gradient only along the integral curves of the vector field ξ. From this analysis it is clear that manifolds (Σ, Λ) obeying conditions (a-c) are becoming relevant for system (14) as well. For any such (Σ, Λ), the function V is determined via (31) while $G^2(x)$ is specified by requiring Λ and V to obey (27) and (28). Once such $G^2(x)$ has been determined, a static solution of (14) is immediately available. The space-time manifold is defined via $M{=}I\!R{\times}\Sigma$, the metric g is defined by $g = -V^2dt^2 + V^{-2n}\Lambda$, while the field Φ is specified via $\Phi = \alpha \ln V + \beta$.

Let us now shift our considerations to the conformal coupling, *i.e.* the set (22)-(24). Reasoning as for the case of minimal coupling, we look for $(\Lambda, U(\Phi), \Phi)$ satisfying (22)-(24). This requirement fixes the $U(\Phi)$ [3,6]:

$$U(\Phi) = -\ln(1+\Phi) + C_2, \tag{32}$$

and for this $U(\Phi)$ the set (22)-(24) implies that (Λ, Φ) obey:

$$(1-\Phi^2)R_{ab} = 6D_a\Phi D_b\Phi - 2\Phi D_aD_b\Phi - 2\Lambda_{ab}D^c\Phi D_c\Phi, \tag{33}$$

$$D^aD_a\Phi = 0, \tag{34}$$

where R_{ab}, D stand for the Ricci tensor and covariant derivative operators associated with Λ. However, the structure of those equations implies that, for any smooth solution (Λ, Φ), the York-Cotton tensor, Ricci tensor of Λ and Φ obey:

$$\Phi(1-\Phi^2)R_{abc} = 4D_b\Phi R_{ca} - 4D_c\Phi R_{ba} - 2D^d\Phi R_{dc}\Lambda_{ba} + 2D^d\Phi R_{db}\Lambda_{ca}, \tag{35}$$

i.e. a relation functionally identical to the fundamental relation (4). Again we are interested in constructing solutions (Λ, Φ) of (33-34) so that $R_{ab}(\Lambda) = \lambda(3X_aX_b - \Lambda_{ab})$. For this Ricci, the content of this integrability condition (35) requires that Λ admit a hypersurface-orthogonal conformal Killing field ξ obeying (10, 11), and moreover either $\lambda = \lambda(x) \Leftrightarrow \Phi(x)$ or in any region where $\Phi^2 \neq 1$, the field Φ and λ obey:

$$\lambda^{\frac{1}{3}} = \frac{G^2(x)\Phi^2}{1-\Phi^2}, \tag{36}$$

where $G^2(x)$ stands for a non-vanishing function having the same property as the previously discussed cases. Thus again the manifolds (Σ, Λ) obeying conditions (a-c) evidently are of importance here as well. Starting from any such (Σ, Λ), we construct the field Φ by appealing to the above-mentioned relation in conjunction with the dynamical Eqs. (33-34). To the triplet (Σ, Λ, Φ) we associate the spacetime (M, g, Φ), where $M = I\!R \times \Sigma$, while the spacetime metric g is described by:

$$g := -\frac{dt^2}{(1+\Phi)^2} + (1+\Phi)^2\Lambda. \tag{37}$$

The analysis so far demonstrates that any (Σ, γ) obeying conditions (a-c) can be used as a seed to construct spacetimes (M, g, Φ) where (g, Φ) are particular solutions of the Einstein massless scalar field equations. Naturally, the considerations so far lead us to ask: how many such (Σ, γ) exist?

2. On Riemannian Manifolds admitting a conformal Killing field

In this section we shall determine all manifolds (Σ, γ) obeying conditions (a-c). In order to carry out this task we need to build a suitable coordinate chart and here Eqs. (10-11) is the starting point. As a consequence of it we have:

Lemma: *There exists a local coordinate chart* (x, x^1, x^2) *such that* $\xi{=}\partial/\partial x$, $\Theta{=}D^a\xi_a{=}{-}\lambda^{-1}\xi^a D_a\lambda$, *and the components of* γ *can be written in the form:*

$$\gamma = \lambda^{-\frac{2}{3}}\left(dx^2 + \hat{\gamma}_{ij}dx^idx^j\right) = \frac{dx^2}{S^2(x,x^1,x^2)} + \frac{\hat{\gamma}_{ij}(x^1,x^2)}{S^2(x,x^1,x^2)}dx^idx^j, \quad i,j = 1,2, \tag{38}$$

where $x \in (a,b) \in I\!R$ *and* (x^2, x^3) *are arbitrary local coordinates on any* $x = const$ *two surfaces.*

It may be easily verified that, whit respect to this chart, $\xi = \partial/\partial x$ is a conformal Killing field of γ obeying the conditions of the lemma. (The proof of this Lemma is discussed in Ref. 4). Making use of this coordinate we view $R_{ab} = \lambda(3X_aX_b - \gamma_{ab})$ as the dynamical equations determining γ, λ and X. Projecting $R_{ab} = \lambda(3X_aX_b - \gamma_{ab})$ along and perpendicular to $x = $ const coordinate surfaces yields [7]:

$$-R^{(2)} - K^{ij}K_{ij} + K^2 = 4S^3, \tag{39}$$

$$D_iK - D_j\,K^j_i = 0, \tag{40}$$

$$-S\frac{\partial K_{ij}}{\partial x} + 2K_{il}K^l_j - KK_{ij} + \frac{1}{2}R^{(2)}\gamma_{ij} + \frac{1}{S}D_iD_jS - \frac{2}{S^2}D_iSD_jS = -S^3\gamma_{ij}, \tag{41}$$

$$\frac{1}{2}S\frac{\partial\gamma_{ij}}{\partial x} = K_{ij}, \tag{42}$$

where $(R^{(2)},\ D_i)$ stand for the scalar curvature and the covariant derivative of the intrinsic two metric $\gamma_{ij} = S^{-2}\hat{\gamma}_{ij}$, while K_{ij} are the components of the extrinsic curvature of the $x = $ const surfaces described by:

$$K_{ij} = \frac{1}{2}\mathcal{L}_n\gamma_{ij} = \frac{1}{2}S\frac{\partial}{\partial x}\frac{\hat{\gamma}_{ij}}{S^2} = -\frac{\hat{\gamma}_{ij}}{S^2}\frac{\partial S}{\partial x} = -\frac{\partial S}{\partial x}\gamma_{ij}. \tag{43}$$

Rewriting (39-42) in terms of $\hat{\gamma}_{ij}$ yields:

$$S^2\hat{R}^{(2)} + 2S\hat{D}^i\hat{D}_iS - 2\hat{D}^iS\hat{D}_iS + 4S^3 - 2\left(\frac{\partial S}{\partial x}\right)^2 = 0, \quad (44)$$

$$\frac{\partial^2 S}{\partial x \partial x^i} = 0, \qquad i,j = 1,2, \quad (45)$$

$$\left[2S\frac{\partial^2 S}{\partial x^2} - 4\left(\frac{\partial S}{\partial x}\right)^2 + 2S^3 + S^2\hat{R}^{(2)} + 2S\hat{D}^k\hat{D}_kS - 4\hat{D}^kS\hat{D}_kS\right]\hat{\gamma}_{ij} + S\hat{D}_i\hat{D}_jS = 0, \quad (46)$$

$$2\left[S\frac{\partial^2 S}{\partial x^2} - \left(\frac{\partial S}{\partial x}\right)^2 - S^3\right] + S\hat{D}^i\hat{D}_iS - 2\hat{D}^iS\hat{D}_iS = 0 \quad (47)$$

in which the scalar curvature $\hat{R}^{(2)}$ and derivative operator $\hat{D}_i$ are formed using the "intrinsic" metric $\hat{\gamma}_{ij} = \hat{\gamma}_{ij}(x^1, x^2)$ on each $x =$ const 2-surface. Despite the fact that this system is a coupled system of partial differential equations, to our pleasant surprise, it can be integrated explicitly and the momentum constraint is the starting point. The general solution of (45) is described by:

$$S(x, x^1, x^2) = f(x) + \sigma(x^1, x^2), \quad (48)$$

where f and σ are smooth functions of their arguments. This form of $S(x, x^1, x^2)$ suggests a classification of the metrics $\hat{\gamma}$ satisfying (44)-(47) in accordance with one of the following choices:

$$a)\ S = S(x) = f(x), \quad (49)$$

$$b)\ S = S(x^1, x^2) = \sigma(x^1, x^2), \quad (50)$$

$$c)\ S = S(x, x^1, x^2) = f(x) + \sigma(x^1, x^2). \quad (51)$$

Since, on the other hand, S is related to the Ricci eigenvalue λ, and in turn λ determines the expansion Θ of ξ, conditions (49-51) are restrictions upon the behavior of Θ along the integral curves of ξ. Any solution to (44-47) with $S = S(x)$ implies that γ admits a conformal Killing field ξ so that the gradient of the expansion Θ is parallel to ξ. On the other hand, any solution of (44)-(47) subject to $S = S(x^1, x^2)$ implies that γ admit ξ as a hypersurface-orthogonal Killing vector field, while for any solution of (44)-(47) with $S = S(x, x^1, x^2)$, the gradient of Θ is no longer parallel to ξ. Below we shall only highlight the integration procedure of (44)-(47). For the choice $S = S(x)$, equations, (44) implies that any $x =$ const surface is a space of constant curvature $R^{(2)} = 2B$ with B a real constant. The remaining equations imply that $S(x)$ satisfies:

$$\left(\frac{dS}{dx}\right)^2 - 2S^3 - BS^2 = 0. \quad (52)$$

The integration of (44)-(47) eventually yields the following classes of three metrics. For the case $B > 0$:

$$\gamma_1 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2 g^+, \qquad r > 2M > 0, \quad (53)$$

$$\gamma_2 = \frac{dr^2}{1 + \frac{2M}{r}} + r^2 g^+, \qquad r \in (0, \infty), \qquad M > 0, \quad (54)$$

where g^+ stands for a two metric of positive Gaussian curvature. For the case where $B < 0$, the integration yields:

$$\gamma_3 = \frac{dr^2}{\frac{2M}{r} - 1} + r^2 g^-, \qquad 0 < r < 2M. \quad (55)$$

where g^- stands for a two metric of negative Gaussian curvature. And finally for, $B = 0$ the result is:

$$\gamma_4 = r dr^2 + r^2 g^0 \qquad r \in (0, \infty), \quad (56)$$

with g^0 a flat two metric. By construction, $(\gamma_1 - \gamma_4)$ possess a degenerate traceless Ricci where $\lambda = S^3$ and its unique eigenvector X is parallel to the conformal Killing field ξ. If we assume that the $r =$ const spaces to be connected, simply connected and geodesically complete, the metrics $(\gamma_1 - \gamma_4)$ can be considered to be defined on the product manifolds $(I\!R \times S^2, \gamma_1)$, $(I\!R \times S^2, \gamma_2)$, $(I\!R \times H^2, \gamma_3)$, and $(I\!R^3, \gamma_4)$. Every one of those manifolds generates solutions to the vacuum, Einstein-Klein-Gordon and conformal system respectively. In the concluding section, we shall comment on the local and global properties of the resulting spacetimes.

For the choice $S = S(x^1, x^2)$, the integration of (44)-(47) is rather lengthy. We have been able to complete the integration by introducing a coordinate y related to $S(x^1, x^2)$ via $y = S(x^1, x^2)$ and using the freedom in the x^2 coordinate to diagonalize the intrinsic metric of each $x =$ const two spaces. Leaving technicalities aside, the resulting metric Λ is described by (see Ref. 4 for details):

$$\Lambda = \frac{1}{y^2}\left[dx^2 + \frac{dy^2}{y^2(2y + C)} + y^2(2y + C)dz^2\right], \quad (57)$$

where C is a real constant and the range of y is restricted to suitable domains. This family of metrics possesses a degenerate Ricci with the eigenvalue $\lambda = S^3 = y^3$ and admits $\xi = \partial/\partial x$ and $\xi_1 = \partial/\partial z$ as commuting Killing vector fields. Moreover, $R^{ab}R_{ab} = 6\lambda^2 = 6y^6$, implying that curvature singularities take place as $y \to \infty$. Accordingly the singularities in the components of the metric occurring at $y = C/2$ and at $y = 0$ ought to be mere coordinate singularities. Setting $b(y) \equiv (2y + C)y^2$, then Λ possesses the Euclidean signature provided the domain where y takes its values is restricted. For the case where $C = 2c^2$, with c real, Λ possesses a right signature provided either $-c^2 < y < 0$, or $0 < y < \infty$. On the other hand, for the case where $C = -2c^2$, Λ has the right signature provided $c^2 < y < \infty$. Finally there exist a "degenerate" case arising by choosing

$c \equiv 0$, and here $0 < y < \infty$. Amongst those possibilities below, we shall briefly discuss a few properties of Λ under the assumption that $C = 2c^2$ and $-c^2 < y < 0$. For this choice, at first we show extendability of Λ at $y \to -c^2$ and $y \to 0^-$. For this it is convenient to define a new coordinate $y^*(y)$ via:

$$dy^*(y) \equiv \frac{dy}{\sqrt{2}(y+c^2)^{\frac{1}{2}}|y|}, \qquad y \in (-c^2, 0). \tag{58}$$

In terms of y^* we have

$$\lim_{y\to -c^2} \Lambda = \frac{1}{c^4}\left[dx^2 + dy^{*2} + y^{*2}c^8 dz^2\right]. \tag{59}$$

Applying the criterion of "elementary flatness", it follows that Λ is regular at $y^* = 0$ provided z is periodically identified, *i.e.*

$$0 \le z \le 2\pi\frac{1}{c^4}, \tag{60}$$

and thus the singularity of Λ as $y \to -c^2$ behaves as the origin (r, θ) of the Euclidean 2-plane and Λ is extendible up to and including the point $y = -c^2$. As long as z is restricted in the domain specified by (60), the $x = \text{const}$ two spaces will be regular two spaces, and $\partial/\partial z$ is an axial Killing vector field. At the other extremum, i.e. as $y \to 0^-$, it would be sufficient to analyze the behavior of the induced metric Λ_2 on any $z = \text{const}$ slice. In terms of the coordinate y^*, the induced metric Λ_2 takes the form: $\Lambda_2 = [1/y^2(y^*)]\left[dx^2 + dy^{*2}\right]$. Extendability of Λ_2 as $y\to 0^-$, can be accomplished by passing to a suitable set of isothermal coordinates $(\overline{x}(x, y^*)\ \overline{y}(x, y^*))$, and details of this extension is discussed in Ref. 4. In summary, and as long as (60) holds true, Λ is one parameter family of metrics admitting two commuting Killing fields with one of them axial. The fact that Λ admits an axial Killing vector field would be of crucial importance in understanding the structure of the resulting spacetimes and issue that will be discussed further below.

Finally the integration of (44)-(47) for the case where $S = S(x, x^1, x^2)$ has been discussed in detail in the appendix I of Ref. 3 (see also [4]). It yields the following family of metrics:

$$C = \frac{1}{S^2}\left(\frac{dx^2}{a(x)} + \frac{dy^2}{b(y)} + b(y)dz^2\right), \tag{61}$$

where:

$$\begin{aligned} S(x,y) &= x + y, \\ a(x) &= 2x^3 + \frac{l}{2}x^2 + mx + n, \\ b(y) &= 2y^3 - \frac{l}{2}y^2 + my - n, \end{aligned} \tag{62}$$

where l, m, n are arbitrary for the moment free parameters and the ranges of the (x, y, z) are restricted so that C possesses the right signature. This family of metrics admit $\xi = \partial/\partial x$ as a hypersurface-orthogonal conformal Killing vector field and additionally $\xi_z = \partial/\partial z$ as a Killing vector field. The Ricci tensor of (61) has the form $R_{ab} = \lambda(3X_aX_b - \Lambda_{ab})$ with $\lambda := (x+y)^3$ and $X = X^a(\partial/\partial x^a) = (x+y)a^{1/2}(x)(\partial/\partial x)$. Via the linear transformation: $x = Ac_0\overline{x} + c_1, y = Ac_0\overline{y} - c_1, z = \overline{z}$ with A, c_0 and c_1 non vanishing constants, the functions $a(\overline{x})$ and $b(\overline{y})$ can be put in the form:

$$\begin{aligned} a(x) &= -2Amx^3 + x^2 - 1, \\ b(y) &= -2Amy^3 - y^2 + 1, \end{aligned} \tag{63}$$

where A and m are arbitrary real parameters and for convenience we have dropped the over bar from the (x, y) coordinates. Due to the property that $a(x) = -b(-x)$, the roots of $a(x) = 0$ are related to the roots of $b(y) = 0$ and vice-versa. Moreover, $a(x) = 0$ may admit three real distinct roots $x_1 < x_2 < x_3$ whenever $A^2m^2 < 1/27$, two multiple roots $x_1 = x_2 < x_3$ whenever $A^2m^2 = 1/27$, or a single real root x_1 case of $A^2m^2 > 1/27$. Accordingly there exist a number of domains where the C metrics possess the right signature. Here we shall consider properties of the family of C metrics subject to the condition that the coordinates (x, y) are restricted by: $x_2 < x < x_3 \Leftrightarrow y_2 < y < y_3$. At first we shall discuss extendability of C as the root of $a(x)$ and $b(y)$ are approached. Let us for the moment consider the induced metric C_2 on any $x \equiv c$, $c \in (x_2, x_3)$ surface:

$$C_2 = \frac{1}{(x+y)^2}\left[\frac{dy^2}{b(y)} + b(y)dz^2\right]. \tag{64}$$

Setting for the moment

$$b(y) = 1 - y^2 - 2Amy^3 = -2Am(y-y_1)(y-y_2)(y-y_3),$$

where $y_1 < y_2 < y_3$ are the roots of $b(y) = 0$, it follows that:

$$\begin{aligned} \left.\frac{db(y)}{dy}\right|_{y_3} &= -2Am(y_3 - y_1)(y_3 - y_2), \\ \left.\frac{db(y)}{dy}\right|_{y_2} &= 2Am(y_2 - y_1)(y_3 - y_2). \end{aligned} \tag{65}$$

Via identical reasoning as for the case of the metric Λ, we introduce a new coordinate $y^*(y)$ defined by:

$$dy^*(y) = \frac{dy}{\sqrt{b(y)}}, \qquad y \in (y_2, y_3), \tag{66}$$

thus casting C_2 in the form:

$$C_2 = \frac{1}{(x + y(y^*))^2}[dy^{*2} + b(y(y^*))dz^2], \tag{67}$$

from which it follows that:

$$\lim_{y\to y_2} C_2 = \frac{1}{(x+y_2)^2}\left[dy^{*2} + K^2(y_2)y^{*2}dz^2\right], \tag{68}$$

where $K^2(y_2) = 2Am(y_2 - y_1)(y_3 - y_2)$. Regularity of C_2 as $y^* \to 0$ is established by periodically identifying the z coordinate

$$0 \le z \le \frac{2\pi}{[2Am(y_2 - y_1)(y_3 - y_2)]^{\frac{1}{2}}} = \frac{2\pi}{K(y_2)}. \tag{69}$$

Even though this restriction guarantees extendability of C_2 up to and including the $y = y_2$, problems arise at the extendability of $y \to y_3$. At this limit, C_2 is reduced to :

$$\lim_{y \to y_3} C_2 = \frac{1}{(x+y_3)^2}\left[dy^{*2} + K(y_3) y^{*2} dz^2\right], \qquad (70)$$

where $K(y_3) = 2a(y_3 - y_1)(y_3 - y_2)$. Since however the range of z has been fixed by (69), and since the roots of $b(y)$ are distinct, it follows that C_2 exhibits an irremovable conical-like singularity occurring at $y = y_3$. This singularity in the gauge of (61) would appear as one-dimensional singular line. In this respect, the C-metrics exhibit different behavior than the Λ-metrics considered earlier, and the implications of this difference will be clarified in the next section. Extendability of the $C-$metric at the zero of the $x+y$ factor is discussed in Ref. 4.

3. Discussion

In the last section, we have established the existence of manifolds (Σ, γ) obeying the conditions (a-c) of the introduction section, and in this section we shall briefly discuss properties of the resulting spacetimes. Using the manifolds $(I\!R \times S^2, \gamma_1)$, $(I\!R \times S^2, \gamma_2)$, $(I\!R \times H^2, \gamma_3)$, and $(I\!R^3, \gamma_4)$ defined earlier on, we constructed vacuum spacetimes (M, g_i), $i = 1-4$ belonging to the class A in the Ehlers-Kundt [8] classification, *i.e.* all (M, g_i) $i = 1-4$ possess a degenerate four-Ricci. Moreover, the corresponding (M, g, Φ) with (g, Φ) satisfying the minimal or conformal equations are a generalization of the family of vacuum spacetimes (M, g_i) $i = 1-4$ belonging to class A in the Ehlers-Kundt classification. All constructed spacetimes admit four linearly independent Killing vector fields, and additional properties are discussed in Ref. 4. For the remaining part, we shall discuss a few properties of the vacuum spacetimes generated using the metrics (Λ, C) derived and discussed in the last section. These metrics can be used in two distinct ways: either one may maximally extend them consistently with conditions (a-c) and $R_{ab} = \lambda(3X_aX_b - \lambda_{ab})$ and subsequently use the extended manifold to construct the spacetime, or one way use the incomplete manifolds of the last section to construct a (geodesically incomplete) spacetime but subsequently extend this spacetime consistently with the relevant field equations. Here we shall follow the second avenue, and we shall briefly discuss the properties of the vacuum spacetime (M, g) obtained by considering the Λ family defined by (57) and taking as Σ the manifold covered by the single chart (x, y, z), $-c^2 < y < 0$, $0 \le z \le 2\pi(1/c^4)$ while x takes its values over some interval of $I\!R$. As we have seen, the eigenvalue λ and V are related via (13), which currently takes the form:

$$V = V(x,y) = \frac{G^2(x)}{\lambda^{\frac{1}{3}}} = \frac{G^2(x)}{S(y)} = \frac{G^2(x)}{y}. \qquad (71)$$

A straightforward computation in the coordinate gauge of (57) shows:

$$D_aD_bV = D_aD_b\left(\frac{G^2(x)}{y}\right) = S^3V(3X_aX_b - \Lambda_{ab}) + S(G^2_{xx} + CG^2)X_aX_b, \qquad (72)$$

and thus satisfaction of (2) requires $G^2(x)$ to obey:

$$G^2_{xx} + CG^2 = 0. \qquad (73)$$

For the case where $C{=}2c^2$, we choose $G^2(x){=}\sin(\sqrt{2}cx)$, and the resulting spacetime metric denoted hereafter as Λ_4 is described by:

$$\Lambda_4 := -\frac{2c^2\sin^2(\sqrt{2}|c|x)}{y^2}dt^2 + \frac{1}{y^2}\left[dx^2 + \frac{dy^2}{2(y+c^2)y^2} + 2(y+c^2)y^2dz^2\right],$$
$$-\infty < t < \infty, \quad 0 < x < \frac{\pi}{\sqrt{2}c}$$
$$-c^2 < y < 0, \quad 0 < z \le \frac{2\pi}{A}, \qquad (74)$$

where the range of the $x-$coordinate has been restricted so that the Killing field $\xi = \partial/\partial t$ is timelike. It follows from (74) that $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 48y^6$, and thus the curvature of Λ_4 is regular over the domain specified by the coordinate chart specified in (74). Moreover, the behavior of the curvature suggests extendability across the singularities of Λ_4 occurring as $y \to 0$ and $y \to -c^2$, and also extendability across the "Killing horizon" occurring at the point where $V^2(x,y) = 0$. Since Λ_4 is a vacuum, static, axially symmetric metric, it can be cast into the Weyl form. Leaving aside the details, (74) can be written in the form:

$$\Lambda_4 = -e^{2\lambda}dt^2 + e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2e^{-2\lambda}d\varphi^2, \qquad (75)$$

where $\lambda = \lambda_1 + \lambda_2$ and

$$\lambda_1(r,z) = \frac{1}{2}\log(z + 1 + \sqrt{r^2 + (z+1)^2}),$$
$$\lambda_2(r,z) = \frac{1}{2}\log(1 - z + \sqrt{r^2 + (z-1)^2}) \qquad (76)$$

are the Newtonian potentials of two semi-infinite line segments of uniform linear density $\rho = 1/2$, extending from $z = 1$ up to $+\infty$ (case of λ_2) and $z = -1$ up to $-\infty$ (case of λ_1). The function $\nu(r,z)$ is defined via:

$$e^{2(\nu-\lambda)} = \frac{1}{32c^6}\frac{\left(\sqrt{r^2+(z+1)^2} + \sqrt{r^2+(z-1)^2} + 2\right)^2}{\sqrt{r^2+(z+1)^2}\sqrt{r^2+(z-1)^2}} \qquad (77)$$

Relative to this Weyl chart, Λ_4 is well defined on the entire (r,z) plane except at the set points of the z-axis, where the potentials λ_1 and λ_2 are singular. Moreover, by application of the elementary flatness criterion, it follows that Λ_4 is regular along the part of the z-axis free of singularities of the

Newtonian potential, *i.e.* for $-1 < z < 1$. Thus in the Weyl formalism, Λ_4 is generated by the Newtonian potentials of two semi-infinite non-overlapping line segments. Further properties of (75) can be revealed by going over to Bondi coordinates (τ, x, y, ζ) defined via [9]:

$$\tanh\left(\frac{t}{\beta}\right) = \frac{\tau}{x}, \tag{78}$$

$$1 + z = \frac{1}{2\beta}(x^2 - \tau^2 - y^2 - \zeta^2), \tag{79}$$

$$r = \frac{1}{\beta}(y^2 + \zeta^2)^{\frac{1}{2}}(x^2 - \tau^2)^{\frac{1}{2}}, \tag{80}$$

$$\tan\varphi = \frac{y}{\zeta}, \tag{81}$$

with β an arbitrary parameter. After a little algebra it follows that Λ_4 transforms into the form:

$$\Lambda_4 = \frac{\beta}{2c^2}e^{2\lambda_2}\frac{(xd\tau - \tau dx)^2}{x^2 - \tau^2} + \frac{2c^2}{\beta}e^{-2\lambda_2}\frac{(\zeta dy - yd\zeta)^2}{y^2 + \zeta^2} + \frac{1}{16c^6}e^{2(\nu_2 - \lambda_2)}\left[\frac{(xdx - \tau d\tau)^2}{x^2 - \tau^2} + \frac{(ydy + \zeta d\zeta)^2}{y^2 + \zeta^2}\right], \tag{82}$$

where:

$$e^{2\lambda_2} = 1 - z + \sqrt{r^2 + (z-1)^2} = \frac{1}{2\beta}(A - B + 4 + \sqrt{(A+B)^2 + 8(A-B) + 16}), \tag{83}$$

$$e^{2(\nu_2 - \lambda_2)} = \frac{1}{16c^2}\left(\frac{\sqrt{r^2+(z+1)^2} + \sqrt{r^2+(z-1)^2} + 2}{\sqrt{r^2+(z-1)^2}}\right) = F(A, B), \tag{84}$$

$$F(A,B) = \frac{1}{16c^2}\frac{\sqrt{(A+B)^2 + 8(A-B) + 16} + B + A + 4\beta}{\sqrt{(A+B)^2 + 8(A-B) + 16}},$$

and $A := y^2 + \zeta^2$, $B = x^2 - \tau^2$. The transformation (77-81) has eliminated the effects due to λ_1 at the expense of making Λ_4 manifestly time dependent. In the (τ, x, y, ζ) chart, Λ_4 exhibits local boost and rotational symmetry [10]. Boost symmetry is generated by the timelike Killing field $\xi_t = x(\partial/\partial\tau) - \tau(\partial/\partial x)$, while rotations along the $x-$axis is generated by the spacelike Killing field $\xi_\varphi = y(\partial/\partial z) + z(\partial/\partial y)$. In the new chart, g is well defined on $x > \tau$ and $y > 0$, $\zeta > 0$, except on the part of the rotation axis where the singularity of the function λ_2 has been mapped. The timelike Killing vector field ξ_t becomes null on $x = \pm\tau$, defining two branches of the acceleration horizon. The spacetime (M, g) can be analytically extended through the acceleration horizon, and in fact for all values of $-\infty < x < \infty$, $-\infty < t < \infty$. In the extended manifold, there appear two singularities along the rotation axis, symmetrically placed with respect to the origin and along the rotation axis, and moreover four branches of $\mathbb{R} \times \mathbb{R}^2$ acceleration horizons intersecting on the (y, ζ) plane. Thus the seemingly simple-looking metric Λ_4 expressed in the chart of (72) exhibits a remarkably rich structure once it is maximally extended. To get more insights into the structure of Λ_4, we shall briefly discuss the properties of the vacuum spacetime generated, using the family of C-metric analyzed in the previous section. After algebra (see [3, 4]), it leads to the spacetime metric denoted by C_4:

$$C_4 := \frac{1}{A^2(x+y)^2} \times \left[-a(x)dt^2 + \frac{dx^2}{a(x)} + \frac{dy^2}{b(y)} + b(y)dz^2\right], \tag{85}$$

where $a(x)$ and $b(y)$ are described by (62). The ranges of the (x, y, z) coordinates have been specified earlier on, while $t \in (-\infty, \infty)$. Since $\partial/\partial z$ is an axial Killing vector field, the metrics (85) can be cast into the Weyl form. Currently however the generating potential $\lambda = \lambda_1 + \lambda_2$ is a superposition of potential λ_1, due to a semi-infinite line segment of uniform linear density $\rho = 1/2$ along the negative $z-$axis, and the Newtonian potential λ_2 of a finite line segment of uniform density placed along the positive axis [11, 12]. As a consequence of the conical singularity of the $C-$metric discussed earlier on, there exists a strut singularity either between those line segments or starting from the one end of the line segment and extending to infinity. Moreover, by employing the Bondi transformation, the effects of the potential λ_1 can be transformed away and the resulting metric exhibits boost and rotational symmetry [11, 12]. In contrast to the case of Λ_4, currently only a finite part of the rotation axis is singular. The analytical extension of the spacetime (M, C_4) has been discussed in various places in the current literature [13, 14]. The extended manifold can be interpreted as describing two black holes accelerating in opposite directions with the acceleration mechanism supplied either by the strut lying along the common symmetry of the holes, or by the struts pulling the holes in opposite directions along the axis of symmetry. The family of C_4-metrics can be considered as a special case of the family of the Λ_4-metrics in the following sense: while the Λ_4 family in the Weyl chart is generated by the superposition of two semi-infinite line segments, by allowing one of them to become of finite extension the C_4 family of metrics is recovered. However no sources for the metric Λ_4 are currently known. Due to space and time limitations, we shall not discuss any further properties of the extended spacetime (M, Λ_4), nor shall we make a further comparison between (M, Λ_4) and the corresponding (M, C_4). It should to be mentioned that an important issue left uncovered concerns the existence, and properties, of the future $\mathcal{J}^+$ and past null infinity $\mathcal{J}^+$ of the extended spacetimes. Those issues, combined with an analysis of the extended non-vacuum spacetimes (M, g, Φ) generated by three manifolds (Σ, Λ) and (Σ, C), are discussed in details in Ref. 4.

We shall finish by recapitulating the main ideas of the work. Starting from simple considerations, *i.e.* by an inquiry regarding the significance of the fundamental relation (4), we have constructed five classes of 3-dimensional

Riemannian manifolds (Σ, γ) satisfying conditions (a-c). In turn those manifolds generate families of non-trivial vacuum and non-vacuum spacetimes. We may add that this generation technique can be extended to the case where the manifolds (Σ, γ) are allowed to become Lorentzians and in that event the generated spacetimes admit a spacelike or a null Killing vector field (see Ref. 15). Moreover, the three manifolds (Σ, γ) can be used to construct other non-vacuum spacetimes as well. Our choice to work with the Einstein-Klein-Gordon massless minimally coupled gravity equations or the Einstein-Conformal scalar field equations was a matter of convenience. Suppose for instance that, instead of the vacuum system or equations (14-15) and (16-17) resp., we have started with the system $G_{\mu\nu} = T_{\mu\nu}(g, \Phi_i)$, where $T_{\mu\nu}(g, \Phi_i)$ stands for the total energy momentum tensor involving N-fields denoted generically by $(\Phi_1, \Phi_2 \ldots \Phi_N)$. Static configurations $(g, \Phi_1, \Phi_2, \ldots, \Phi_N)$ in the gauge of (1) would satisfy: $VR_{ab}(\gamma) = S_{ab}(\gamma, \Phi_1, \Phi_2, \ldots, \Phi_N)$, $VR(\gamma) = S(\gamma, \Phi_1, \Phi_2, \ldots, \Phi_N)$. for a suitable three tensor S_{ab} Via a conformal deformation, we can pass to a metric $\overline{\gamma} = \Omega^2\gamma$ so that $\overline{R}(\overline{\gamma}) = 0$. Requiring that the effective eqs for $(\overline{\gamma} = \Omega^2\gamma, \overline{\Phi}_i = \Omega^{n_i}\Phi_i, i = 1, 2, \ldots N)$ admit a degenerate three Ricci *i.e.*, $\overline{R}_{ab}(\overline{\gamma}) = \overline{\lambda}(3X_aX_b - \overline{\gamma}_{ab})$ one arrives at an integrability condition analogous to relation (6). Implementation of this procedure for specific field configurations will be discussed in a future work. We may only note here that, in view of the interpretation of the C_4-metric, the existence of accelerating by a strut hairy black holes is an intriguing possibility. Such a possibility can be analyzed by appealing to the main ideas of this work, and currently this question is under active investigation.

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