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Aspects of cosmic magnetism

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In this paper, after a brief review of the current observational evidences regarding the phenomenon of cosmic magnetism, we discuss the problems associated with the generation of electromagnetic fields by conducting fluid flows. In particularl, we examine the electromagnetic field generated by a conducting fluid flow in the so-called magnetohydrodynamical (MHD) regime in the non-relativistic and general relativistic limit. Our efforts are directed at the status of Cowlings theorem for the two limits. We show that electromagnetic fields generated by conducting fluids in an arbitrary spacetime (M, g) are influenced by the conducting and kinematical variables defining the fluid flow, but also influenced by the curvature and topology of the underlying spacetime. For the particular case of spatially homogeneous and isotropic backgrounds or stationary-axially symmetric circular spacetimes, we show that the dynamical equations describing electromagnetics fields generated by particular conducting flows reduce to a form structurally similar to the non-relativistic limit. Despite this siplification, the issue whether axially symmetric conducting fluid flows can maintain an axisymmetric magnetic field against Ohmic dissipation is still open.

Keywords: Magnetohydrodynamics; compact objects; gravitation.

En este articulo damos una breve revision de las evidencias observacionales relacionadas con el fenomeno del magnetismo cosmico. Se discute el problema asociado con la generacion de un campo magnetico por el flujo de un fluido conductor en el regimen magnetohidrodinamico en el caso no relativista y en relatividad general. El esfuerzo se enfoca en establecer la validez del teorema de Cowling para los dos limites. Los campos electromagneticos generados por fluidos conductores en un espacio tiempo arbitrario (M, g) son influenciados por las variables de conduccion y variables cinematicas que definen el flujo del fluido, asi como por la curvatura y topologia de del espacio tiempo. Para el caso particular de un fondo homogeneo e isotropico o espacios tiempo estacionarios con simetria axial y circular, se muestra que las ecuaciones dinamicas que describen los campos electromagneticos generados por flujos conductores se reducen a una forma estructuralmente similar al limite no relativista. Aun asi, queda abierta la cuestion de si flujos de fluidos conductores axial simetricos pueden mantener un campo magnetico axialsimetrico a pesar de la disipacion ohmica.

Descriptores: Magnetohidrodinámica; objeto compactos; gravitación.

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1. Introduction

This paper analyzes some issues raised by century-old observations that gradually established the existence of electromagnetic fields coherent over scales ranging from planetary up to and including supercluster scales. In particular, we discuss electromagnetic fields generated by a conducting fluid flow in a general relativistic setting. In this paper we restrict our attention to the so-called magnetohydrodynamical (MHD) regime and assume a simple form for the conduction current. However, before we enter into an analysis of that problem, we shall briefly summarize observational facts regarding the phenomenon of cosmic magnetism.

The detection of large-scale electromagnetic fields spanned a period of a century, but it proceeded at an accelerated pace. Up to the arrival of the 20th century, the only large-scale electromagnetic field known to humanity was the Earth‘s field, which is dominated by a magnetic component $\mathbf{B}$. In 1908, Hale, using spectroscopic techniques, was able to measure the magnetic field of the sunspots, the first extraterrestrial field ever to be detected. In 1947, Babcock [1] announced that the 78-Virginis star, an A-type star, posseses a $\mathbf{B}$-field of the order of 500-G. Shortly afterwards, Hiltner and Hall in 1949 reported the detection of a small degree of linear polarization of light emitted by a group of stars [2, 3]. Davis and Greenstein [4] attributed the observed polarization as due to a galactic $\mathbf{B}$-field and their suggestion marked the first empirical evidence that our Milky way may possesses a magnetic field. A similar idea, and for different reasons, was put forward by Fermi [5] in 1949. Motivated by the discovery of cosmic rays and earlier investigations by Alfven [6], Fermi postulated that if our Milky Way is threated by a $\mathbf{B}$-field, of a few μG but coherent over Kpc scale, then such a field would trap cosmic rays within the spiral arm. As it turned out, the Milky Way indeed possesses a magnetic field; however, it took more than a quarter of a century of intense observational efforts to establish its existence. The efforts of the scientific community in the post-1950 period were largely focused on understanding the solar field. In this regard, the development of the magnetograph by Babcock and Babcock [7] in 1953 offered enormous insights. By monitoring the solar field on a daily basis, it has been established that the solar field is a dynamical structure and this realization lead to an avalanche of theoretical investigations regarding its origin. A key development in our understanding of cosmic magnetism took place with the discovery of the pulsars in 1968 [8]. Their discovery, besides establishing the existence of $\mathbf{B}$-fields in the range of 10^{12} G, had another important consequence. As has been pointed out by Lyne and Smith [9], analysis of the time arrivals of the radio pulses offered an independent method for determining the electron density along the line of sights and thus a method for estimating the dispersion measure. Consequently Faraday rotation has become a powerful tool in detecting and studying magnetic fields coherent over galactic, length scales and beyond. Currently cosmic magnetism is detected and studied via a number of complementary techniques. Spacecraft missions provide in situ observations of the planetary and the solar field. Measurements of the Zeeman splitting of spectral lines is employed for the study of the solar field and the field of nearby stars. Optical and infrared polarization data, observations of the polarized radio synchrotron emission from the Milky Way and nearby galaxies, overall X-ray observations of the sky, as well the detection and analysis of very high energy cosmic rays, are employed to detect and study magnetic structure galactic and extragalactic structures. All of these observational efforts leads to the conclusion that magnetic fields seem to be an attribute of our observable universe. Every cosmic object that has been placed under observational scrutiny exhibits magnetic activity. Cosmic structures on the planetary, stellar, galactic, cluster and supercluster scales all exhibit some form of magnetic activity. Moreover, the observed fields lack a uniformity in the sense that their properties vary substantially from one cosmic object to another, from a dipole-like structure of approximately 1 G for our own earth and planets, to a complex oscillatory field of our sun. Hot stars in the main sequence exhibit fields ranging from $(10^2 - 10^3)G$, while white dwarfs exhibit fields up to $10^8 G$. The gigantic field of the pulsars is inferred to be in the ranges $(10^8 - 10^{12})G$, while the recently discovered magnetars and anomalous $X - ray$ pulsars posses a surface B-field of order $(10^{14} - 10^{15})G$. Other galactic structures posess magnetic fields that exhibit markedly different properties than the observed stellar and planetary fields. Our Milky Way possess a weak field of a few μ-G coherent on length scales over Kpc. This component is referred to as the regular or mean field in contrast to a secondary component $\delta\mathbf{B}$ which fluctuates over length scales $\simeq 100pc$, the characteristic length scale of interstellar turbulence. The field exhibits a toroidal-like component directed along the spiral arms while the galactic center exhibits a poloidal-like structure. A detailed description of galactic field with references to original articles is discussed in Refs. 10 to 12. In addition to the field of the Milky Way, the fields of several near-by spiral galaxies have been estimated with high enough resolution. Their regular fields are in the range of a few μ-G and exhibit similar properties to the field of our Milky Way. Elliptical and irregular galaxies also exhibit magnetic activity, although their fields become a more difficult entity to detect. A review of the literature regarding properties of the $\mathbf{B}$-fields associated with elliptical galaxies can be found in Refs. 11 and 12. Thus it appears that magnetic activity is a common property of galactic structures. The next scale where magnetic activity has been detected involves the galaxy cluster scale. Galaxy clusters are the largest non-linear systems in the Universe. Detailed X-ray observations from Einstein, ROSAT, Chandra and XMM-Newton observatories show that the inter-cluster medium is filled with hot plasma emitting X-rays with energies $(1 - 10)Kev$. Magnetic fields in galaxy clusters have been detected via Faraday rotation and estimates of the regular $\mathbf{B}$-field found to be in the range $0.2 - 3\mu G$. Details of those observations and further references can be found in Refs. 11 and 12. Finally, we shall briefly mention observational efforts aimed at detecting magnetic activity on a supercluster scale. On this scale, the detection of magnetic fields becomes a very difficult issue. So far, radio emission from the region between the Coma cluster and the Abell cluster 1367 has been detected. These two clusters are $40Mpc$ apart and define the plane of the Coma supercluster. The observed emissivity has been interpreted as being due to the presence of a magnetic field of the Coma-Abell 1367 supercluster with strength $0.2 - 0.6\mu G$ (for details of those observation see Refs. 11 and 12).

This brief survey regarding the current status of cosmic magnetism leads to the conclusion that magnetism is a normal activity of the observable part of the universe. At the same time, this conclusion poses a number of interrelated questions being for an answer: What is the origin of those fields? Does there exist a single unified principle that underlies their existence? Why are they there? Are they telling us anything about the large-scale structure of the universe? Below we shall attempt to summarize the current ideas regarding the phenomenon of cosmic magnetism.

2. On the origin of cosmic magnetism

The observational evidence mentioned above shows that all fields observed so far share a common property: They are anchored either in a media in a state of plasma and high temperatures or in media that are excellent conductors. Maxwell‘s theory and the absence of any evidence supporting the existence of magnetic poles offers magnetization and electric currents as possible sources of cosmic magnetism. Due to the high temperatures involved, magnetization or its close relative, ferromagnetism, appear as to be unlike source of cosmic magnetism. According to all efforts focused on the electric currents as possible sources for cosmic fields. However electric currents in conducting media are subject to Ohmic dissipation and unless an electromotive force is operating, they are destined to decay. If an initial current $J = \sigma E$ of length scale L finds itself in a conducting medium, it decays exponentially with a characteristic time scale $T_{Ohm} = (4\pi\sigma L^2)/c^2$ [13]. The existence of this T_{Ohm} permit to draw some general conclusion regarding the origin and maintenance of cosmic fields. Let us first consider the magnetic field of our own earth. Seismological studies suggest that Earth's interior consists of a solid core made up of iron and nickel followed by a liquid core of iron and lighter elements, and finally a solid mantle. An estimate of Ohmic dissipation yields $T_{Ohm} = 10^4-$years, while Paleomagnetic studies suggest that the terrestrial field is at least 10^9-years old, and moreover it exhibits an aperiodic change in its polarity (for a detailed modeling of the earth see for instance

Ref. 14). These observations suggest that, if the earth's field originates in some initial current distribution confined in its interior, then the field could not have survived for such a long time. Moreover the a-periodic change in its polarity seems to incompatible with the properties of an exponentially decaying magnetic field due, to the action of the Ohmic dissipation. The general consensus of the community is that the observed properties of the earths field are the results of an electromotive force operating in its interior. Similar conclusions can be drawn for the solar field. Due to the close proximity of our earth to the sun, the solar field has been studied extensively and with high resolution. It shows a change in the polarity every 22 years as well as exhibiting other time-dependent phenomena such as sun spots, etc. (for a detailed description of this field, see Refs. 14 and 15). The highly conducting nature of the solar plasma and large linear dimensions involved yield a T_{Ohm} which exceeds the Hubble time. However, the observed properties of the solar field, as like in the case of the earth, require the action of an electromotive force. An electromotive force is also required to explain the properties of the galactic field. Here a naive estimate of the T_{Ohm} using for σ the value of the molecular conductivity yields $T_{Ohm} > 10^{16}$ years (see for instance [15]). However, this estimate overlooks the dissipative effects due to the turbulence motion of the interstellar medium. Using an estimate of turbulent conductivity σ_{tur} reduces T_{Ohm} to a value much shorter than the galactic age T_a, and thus the origin and maintenance of the observed $\mathbf{B}_G$ would again require the action of an electromotive force. As is well known, there exist many candidates for an electromotive force that would generate electric currents: Thermoeffects, inhomogeneities in the composition of a conducting medium or the difference in the mobility between electrons and ions due to acceleration can also generate electric currents. However none of those has been accepted as a mechanism for explaining cosmic magnetism. Motivated by Hale‘s discovery, Larmor proposed in 1918 the motion of conducting fluids in the presence of a magnetic field as the mechanism for the generation and maintenance of an electric current. However, in a landmark paper, Cowling in 1934 pointed out that matters are not that simple [16]. He presented arguments showing that an axisymmetric conducting fluid flow cannot maintain a steady axisymmetric magnetic field against the action of Ohmic dissipation. His conclusion was verified by a number of independent investigations and lead to the formulation of the so-called anti-dynamo theorems. In view of that development, the post-Cowling period was marked by intense efforts to bypass the conclusions of the antidynamo theorems. As an outcome of all those investigations, it has been gradually realized that even though axisymmetric flows cannot maintain an axisymmetric $\mathbf{B}$-field, nevertheless small deviations from axisymmetry may be able to maintain a suitably averaged magnetic field $\mathbf{B}$ (for a taste of the various approaches and references to original works consult [14]). This idea has lead to the development of mean field electrodynamics (for an introduction see Steenbeck, Krause and Radler (1966) [17,18]). In this approach, the magnetic field $\mathbf{B}$ and the velocity field $\mathbf{v}$ are considerd to be the sum of a mean field $\mathbf{B}, \mathbf{v}$ and fluctuating parts $\delta\mathbf{B}, \delta\mathbf{v}$. The component $\delta\mathbf{v}$ originates in the turbulent nature of the conducting fluid flow varying over some characteristic length scale l, while the mean part $\mathbf{v}$ of the velocity field varies over some length scale L, with $L \gg l$. Provided that Reynolds averaging hold true, the mean field $\mathbf{B}$ satisfies an effective induction equation that incorporates an electromotive force originating in the fluctuating parts $\delta\mathbf{B}$ and $\delta\mathbf{v}$ respectively. Mean field electrodynamics has been applied to explain the terrestrial planetary and solar magnetisms. For success and the status of those modelings, the reader is referred to the extensive literature on the subject (for an update, see Ref. 19. Although in general there are issues to be overcome, it appears that turbulent conducting fluid flows act as sources for planetary and solar magnetism. The turbulent nature of the conducting fluid combined with the fluctuating part of the magnetic field generate an additional component of an electromotive force that bypasses the antidynamo theorems.

However, turbulent dynamo theory has not been successfully implemente to provided an explanation of all so far observed fields. No definite theory exists to explain the gigantic field of the pulsars or the field of the Milky Way and other extragalactic structures mentioned earlier on. Let us first briefly consider the magnetic field $\mathbf{B}$ of the pulsars. So far around 1000 pulsars have been detected, and their magnetic fields, as inferred from the measurements of the period derivatives, vary considerably. Young neutron stars appear to be in the range of $(10^{11} - 10^{15})G$, and this class includes the classical radio pulsars and magnetars. Old neutron stars possess weaker fields $< 10^9$G, and this category includes the millisecond pulsars and low mass X-ray binaries. Despite the fact that direct observational evidence of their global magnetic fields is lacking, there exists strong evidence supporting the idea that their field is subject to decay. On theoretical grounds, this decay would depend on the location, strength and structure of the field. It would also depend upon the equation of state of the neutron star matter as well as upon the conducting properties of the neutron star matter and, moreover, thermal history of the star. Moreover, due to the, fact that gravity is no longer weak, general relativistic effects ought to be incorporated in to physical processes taking place in pulsars interiors. Even though their interiors are excellent conductors, an issue related to their field concerns the Ohmic decay timescale. What is the impact of relativistic gravity on this decay? These problems have been addressed in Ref. 20. The conclusion of the work is that relativistic gravity tends to decelerate the field decay, but not by much. Even though an understanding of the physics governing magnetic field decay in a pulsar still is not completely understood, another important open problem is related to the origin of their gigantic field. Several competing scenarios have been proposed to explain the inferred strengths of the pulsar fields, and the simplest scenario invokes the fossil field hypothesis. Despite the supernova explosion, the B-field of the progenitor stars survives; and since neutron stars interiors are excellent conductors, the effects of Ohmic dissipation are insignificant. Since, as we have mentioned above, direct observations of the pulsar field is lacking, one cannot really eliminate this hypothesis as in the case of the terrestrial or solar field. It may be noted that, even though flux freezing yields magnetic field strengths of the right order of magnitude, the hypothesis overlooks many important features of the precollapse and postcollapse state. It does not take into account either the convective core of the progenitor star or the convective nature of the newborn neutron star. One would expect in such environments that the dynamo action would be fully operational. Blandford *et al.* [21] have proposed that thermoelectric currents driven by temperature gradients during the cooling phase of a neutron star are responsible for generating the field of the pulsars. A different scenario has been proposed and elaborated by Thomson and Duncan [22]. It has been argued in that reference turbulent dynamo action is the most probable mechanism responsible for the magnetic field of the young pulsars. The convective nature of the progenitor as well as the newly-born neutron star provided ideal environments for a dynamo action. Even though dynamo action appears as the favorable model, it can also provide a natural setting to explain the magnetar fields; nevertheless, it has not been established that this is in fact the case.

As similar situation prevails for the origin of the very weak but highly coherent $\mathbf{B}$-field of our Milky Way and other galactic and extragalactic $\mathbf{B}$-fields. As in the case of a pulsar $\mathbf{B}$-field, they are the subject of controversy and scientific debate. Regarding the field of the Milky Way, the central issue is the following: Does this field originate in a process taking place within the galaxy itself or has it been generated via the compression or a dynamo process of a primordial $\mathbf{B}$-field? The first possibility is referred to as the astrophysical resolution, are the second one as the cosmological resolution. If the first alternative holds true, then the observed galactic fields originate in astrophysical processes taking place within the galaxy itself. Supernova explosions, stellar winds, and shock fronts are important ingredients for this scenario. These fields generate a seed field that is subsequently amplified by the galactic rotation. A concrete materalization of this idea is based on the so-called $a - \Omega$ turbulent dynamo theory. The differential rotation and turbulent flow of the interstellar medium in the spiral arms act as a dynamo that provides the observed $\mathbf{B}$-field. A detailed discussion of this model is presented in the Ref. 15, while a recent review of it, in the light of new observations, is presented in Ref. 23. An alternative scenario within the astrophysical resolution is based on the idea that the field of magnetized matter in supernova ejects combined with the stellar $\mathbf{B}$-fields via reconnection theory yields the largest scale coherent galactic field (for a review of this scenario see for instance [24]).

On the other hand, in the light of recent observations and theoretical advances in the field of cosmology, the cosmological resolution of the observed large scale fields is gaining momentum. The idea that primordial $\mathbf{B}$-fields may be responsible for the observed large scale was implicit in Fermi's postulate mentioned earlier on, although Fermi has not advanced any specific process or processes to explain such a field. One possibility is to endow the big-bang singularity with a non-vanishing $\mathbf{B}$-field. However this hypothesis is rather unsatisfactory. One would expect the primordial field $\mathbf{B}$ to have been generated via physical processes. Indeed, with the advent of the inflationary scenario and the infusion of ideas of particle physics into the physics of the early universe, scenarios have been proposed that would provide a mechanism for the generation of a primordial magnetic field. Broadly speaking, those scenarios are divided into two classes, scenarios operating after inflation or scenarios operating during inflation. It has been argued that inflation leads also to small amplitude-long wavelength electromagnetic fields that were subsequently amplified via a dynamo action, and such fields may be responsible for the observed magnetism in the universe (see for instance [25] and references therein). Scenarios operating after inflation are typically based on phase transitions. An electroweak or QCD phase transition, in the very early stages of the cosmological expansion may lead in the creation of a seed magnetic field $\mathbf{B}$ which is subsequently amplified via dynamo action, (see for instance [26] and references therein). Unfortunately, due to space time limitations, we shall not have the opportunity to discuss those approaches in detail.

Although all of the above proposals have their own merits and weak aspects, nevertheless neither the astrophysical proposal nor the cosmological one are free of shortcomings. So, for the moment, the origin of the galactic and extragalactic fields is not settled. On the other hand, the discussion in this and the previous section makes one point clear: conducting fluid and plasma constitute a part of the solution of the problem of cosmic magnetism. Whether, however, this component has to be implemented with a turbulent part, and whether or not dynamo action will be required, and whether or not the seed fields are primordial in nature or are generated by an astrophysical process, are issues to be resolved in the future. Here we shall offer only one comment: it is perhaps striking to note that both cases where controversy surrounding the origin of the observed fields, *i.e.* the case of pulsar fields and the case of galactic and extragalactic fields both settings besides the conducting matter also involve relativistic gravity. For the case of pulsars, the ratio $2MG/c^2$ is close to unity, while for the case of a large scale fields, if their origin is primordial, it envolves the background geometry of the early universe. For the rest of this paper we shall concentrate on the generation of electromagnetic fields by conducting fluids in an arbitrary spacetime (M, g).

3. Conducting Fluids on a Curved Spacetime

We start by considering an arbitrary smooth spacetime (M, g). By smooth we shall mean that M is a nice topological space, i.e. a Hausdorff space, connected and paracompact, while (M, g) as a Lorentzian manifold will be assumed to be chronological, and g a smooth enough tensor field so that all subsequent operations involving differentiation would be permissible (for a definition of the various terms see Ref. 27). On this (M, g), we consider a conducting medium described by an energy momentum tensor $T^{\mu\nu}$ interacting with an electromagnetic field $F^{\mu\nu}$. In general, the interaction of those systems is a complex problem even in the absence of gravitation. The polarizability and conducting properties of the medium ought to be accounted for, and thus the macroscopic form of the Maxwell equations ough to employed. Moreover, a set of constitutive relations ought to be specified and their form would reflect the nature of the medium as well as its thermodynamical state. Fortunately as we have mentioned earlier on, cosmic fields are defined on highly conducting fluids or media in a state of plasma. In the so-called magnetohydrodynamical regime (MHD hereafter), the collision time scale between the conduction electrons and ions is much shorter than the time scale characterizing the variations of the electromagnetic and gravitational field. Consequently, the medium moves as one component fluid and below we shall denote by $V = V^{\mu}\partial/\partial x^{\mu}$, $g(V, V) = -1$ its four velocities defined as the eigenvector of the baryon current. In the presence of an electromagnetic field $F_{\mu\nu}$, inductive effects generate a conduction current J_c, which as we have mentioned above, exhibits a complex dependence upon the microscopic and thermodynamical properties of the fluid as well as upon the strength of the field $F_{\mu\nu}$. Below we shall adopt the MHD regime and shall assume that the dielectric properties of the fluid are negligible. Moreover we shall assume that the conduction current J_c is described by the general relativistic version of Ohm's law: $J_c^{\mu} = \sigma g^{\mu\nu} F_{\nu\tau} V^{\tau}$, where σ stands for the electric conductivity of the medium. For this setting, Maxwell equations coupled with J_c imply that the components $F^{\mu\nu}$ within the fluid obey:

$$\nabla_{\mu} F^{\mu\nu} = -\frac{4\pi}{c} \sigma g^{\nu\mu} F_{\mu\tau} V^{\tau} \tag{1}$$

$$\nabla_{\mu} J_c^{\mu} = 0, \qquad \nabla_{[\mu} F_{\nu\sigma]} = 0 \tag{2}$$

while in the regions free of conducting matter, $F^{\mu\nu}$ obey the homogeneous version of the above equaions obtained by setting $J_c^{\mu} \equiv 0$. At the interface separating the conducting region from the vacuum, $F_{\mu\nu}$ would be required to fulfill suitable matching conditions but for the purpose of this paper we shall not require their specific forms. Note that in (1), (2) we have augmented the Maxwell equations with a conservation equation for the conduction current J_c^{μ}, since for an arbitrary velocity field V its covariant conservation ought to be checked explicitly. For a prescribed velocity field V and conductivity scalar σ, the above equations describe the electromagnetic field generated by the conducting fluid. We shall be interested in identifing particular solutions to those equations that would be of relevance to the problem related to the generation and maintainenace of cosmic magnetic fields. However it would be worth while at this point considering the same problem in a non-relativistic setting. For this latter setting, and relative to some global inertial frame, we assume that a conducting fluid flow is described by the smooth velocity field $\mathbf{v}(\mathbf{x}, t)$ that is confined in an open, connected, simply connected and bounded subset V of $I\!R^3$ with a smooth boundary ∂V. For simplicity, below we shall always assume that V is the interior of a spere. We may interpret this flow as describing the solar interior or the smooth component of the flow of any other cosmic object. Inertial observes at rest relative to the global frame measure a magnetic field $\mathbf{B}$ satisfying:

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{x}, t) &= \nabla \times (\mathbf{v}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t)) \\ &\quad - \nabla \times (\lambda \nabla \times \mathbf{B}(\mathbf{x}, t)), \\ \nabla \cdot \mathbf{B}(\mathbf{x}, t) &= 0, \quad \mathbf{x} \in V \end{aligned} \tag{3}$$

$$\begin{aligned} \nabla \times \mathbf{B}(\mathbf{x}, t) &= \nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0, \\ \mathbf{x} &\in I\!R^3 - V \cup \partial V, \end{aligned} \tag{4}$$

where $\lambda = c^2/4\pi\sigma$ stands for the magnetic diffusivity of the conducting fluid assumed hereafter to be constant. In view of the above discussion regarding Ohmic dissipation and electromotive forces, we assume that at an initial time $t = t_0$, a non-singular seed field $\mathbf{B}_0(\mathbf{x})$ is specified and we primarily seek non-singular solutions $\mathbf{B}(\mathbf{x}$ to $t)$ the above equations so that $\mathbf{B}(\mathbf{x}, t) = O(r^{-3})$ as $|\mathbf{x}| \to \infty$, and $\mathbf{B}(\mathbf{x}, t)$ is continuous across ∂V. If such solutions exist in general, there is no guarantee that their magnetic energy would remain non-zero in the infinite future. The challenge posed by the observed cosmic fields and their properties is captured in the so-called (laminar) dynamo problem. This problem calls for the formulation of necessary and sufficient conditions upon the smooth field $\mathbf{v}(\mathbf{x}, t)$ that would be capable of dynamo action. We are saying that the field $\mathbf{v}(\mathbf{x}, t)$ acts as a dynamo for a specified initial distribution $\mathbf{B}_0(\mathbf{x})$ and magnetic diffusivity λ, for this prescribed $\mathbf{v}$, and λ (3,4) permits at least one non-singular solution $\mathbf{B}(\mathbf{x}, t)$ so that $\lim_{t\to\infty} \int \mathbf{B}^2(\mathbf{x}, t) d^3x \neq 0$. As we have mentioned earlier, Cowlings theorem implies that not every $\mathbf{v}(\mathbf{x}, t)$ is capable of dynamo action. Below we shall outline a proof where for certain families of velocity fields and diffusivity λ, all axially symmetric non-singular solutions of (3,4) are decaying solutions. In order to see what is involved, we recall here that any smooth vector field $\mathbf{F}$ defined on Euclidean $(I\!R^3, \langle | \rangle)$ can be decomposed into its toroidal and poloidal components according to

$$\mathbf{F} = -\hat{\mathbf{r}} \times \nabla U + \hat{\mathbf{r}} V + \nabla W = \mathbf{F}_T + \mathbf{F}_P \tag{5}$$

where U, V and W are smooth functions, ∇ stands for the gradient operator, $\hat{\mathbf{r}} = r\mathbf{e}_r$, $r^2 = x^2 + y^2 + z^2$ and $\mathbf{e}_r$ is the unit vector along the position vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$, The so-called toroidal part is defined by $\mathbf{F}_T = -\hat{\mathbf{r}} \times \nabla U$, and the poloidal part is defined by $\mathbf{F}_P = \hat{\mathbf{r}} V + \nabla W$. Using decomposition (5), it follows that $\mathbf{F}_T$ and $\mathbf{F}_P$ satisfy $\nabla \times \mathbf{F}_T =$ poloidal, $\nabla \times \mathbf{F}_P =$ toroidal while $\mathbf{F}_T \times \mathbf{G}_T =$ poloidal. For the particular case where

$\nabla \cdot \mathbf{F} = 0$, then (5) implies that at least locally we may write

$$\mathbf{F} = \nabla \times \mathbf{A} = -\nabla \times (\mathbf{r} \times \nabla S) + \nabla \times (\hat{\mathbf{r}} T) = \mathbf{F}_T + \mathbf{F}_P \quad (6)$$

and thus any divergence-free vector field $\mathbf{F}$ is determined by the scalar functions $S = S(\mathbf{x})$ and $T = T(\mathbf{x})$,

Let $\mathbf{B}(\mathbf{x}, t)$ be a non-singular solution of (3,4) so that both $\mathbf{v}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are axisymmetric about the same axis and $\mathbf{v}$ is assumed to be incompressible. Decomposing $\mathbf{B}$ and $\mathbf{v}$ into their toroidal and poloidal parts according to

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_T(\mathbf{x}, t) + \mathbf{B}_P(\mathbf{x}, t),$$
$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_T(\mathbf{x}, t) + \mathbf{v}_P(\mathbf{x}, t),$$

the induction equation yields:

$$\frac{\partial \mathbf{B}_T}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})_P - \nabla \times (\lambda \nabla \times \mathbf{B}_T) \quad (7)$$

$$\frac{\partial \mathbf{B}_P}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})_T - \nabla \times (\lambda \nabla \times \mathbf{B}_P) \quad (8)$$

Since $\mathbf{B}_P$ is solenoidal, it may be represented in the form: $\mathbf{B}_P = \nabla \times \mathbf{A}_T$ for some toroidal field A_T. It follows from (8) that $\mathbf{A}_T$ obeys

$$\frac{\partial \mathbf{A}_T}{\partial t} = (\mathbf{v} \times \mathbf{B})_T + \lambda \nabla^2 \mathbf{A}_T \quad (9)$$

Introducing cylindrical coordinates (r, z, φ) so that the symmetry axis, is along the z-axis and setting $\mathbf{A}_T = A(r, x)\mathbf{e}_\varphi = (x(r, z)/r)\mathbf{e}_\varphi$, the "flux function" $x(r, z)$ obeys

$$\frac{\partial x}{\partial t} + \mathbf{u}_P \cdot \nabla x = \lambda D^2 x, \quad (10)$$

where the operator D^2 is defined by: $D^2 = \nabla^2 - (2/r)(\partial/\partial r)$, with ∇^2 the flat Laplacian operator acting on scalars. Upon multiplicaying (10) by $x(r, z)$ and integrating over $I\!R^3$, taking into account the asymptotic behavior of the field $\mathbf{B}$ it follows that a steady state with $x \neq 0$ is not possible, and thus ultimately $\mathbf{B}_P \to 0$. However, once $\mathbf{B}_P \to 0$, then by appealing to (7) it follows that the toroidal part $\mathbf{B}_T$ satisfies a sourceless equation and by an argument similar to the case of (8), ultimately $\mathbf{B}_T$ decays to zero as well. Thus, as long as $\mathbf{v}$ is incompressible and axisymmetric and the diffusivity λ is constant, any non-singular axisymmetric solution of (2,3) is destined to decay. In view however of the importance of Cowling's theorem to cosmic magnetism, and with reference to the set (1,2), naturally we are led to ask: does one expect axisymmetric solutions of (1,2) generated by special velocity field V to behave like those of (3,4)? What is the impact of the curvature and topology of (M, g) upon such solutions? Is there a relativistic version of Cowling's theorem?

Below we shall briefly discuss some of these issues; however, at the start we expect difficulties. At first the covariant form of Maxwell's equations involve, as primary variables, the components of the Maxwell tensor $F_{\mu\nu}$ and the four current J^μ. Moreover, and in contrast to Minkowski spacetime where the Poincaré group singles out the family of inertial observers, in an arbitrary (M, g) preferred families of observers may not exist and thus the concept of the electric or magnetic fields in general require the specification of a family of fiducial observers. Therefore they become observer-dependent concepts, and thus special care is required to draw physical conclusions. One option is to work with manifest covariant objects, and below we shall explore this option. Let us return to the system (1,2,) and suppose that a prescribed velocity field V and a conductivity scalar σ have been defined. Via manipulations of the Ricci identities, it can shown that any solution $F_{\mu\nu}$ also satisfies [28]:

$$\nabla^\alpha \nabla_\alpha F_{\gamma\delta} = \frac{4\pi\sigma}{c} \Big[V^\alpha \nabla_a F_{\gamma\delta} + F_\delta{}^\alpha \Big(\hat{w}_{\alpha\gamma} + \hat{\sigma}_{\alpha\gamma} + \frac{1}{3}\hat{\theta}\hat{h}_{\alpha\gamma} - \hat{a}_\alpha V_\gamma \Big) - F_\gamma{}^\alpha \Big(\hat{w}_{\alpha\delta} + \hat{\sigma}_{\alpha\delta} + \frac{1}{3}\hat{\theta}\hat{h}_{\alpha\delta} - \hat{a}_\alpha V_\delta \Big) \Big] + R_{\gamma\delta\alpha\mu} F^{\alpha\mu} + R_{\gamma\mu} F_\delta{}^\mu - R_{\delta\mu} F_\gamma{}^\mu, \quad (11)$$

while the part of (M, g) free of conducting fluid and currents, $F_{\mu\nu}$, obeys:

$$\nabla^\alpha \nabla_\alpha F_{\delta\gamma} = R_{\gamma\delta\alpha\mu} F^{\alpha\mu} + R_{\gamma\mu} F_\delta{}^\mu - R_{\delta\mu} F_\gamma{}^\mu, \quad (12)$$

where the tensors $\hat{w}_{\alpha\beta}$, $\hat{\theta}$, $\hat{\sigma}_{\alpha\beta}$ and $\hat{a}_\mu$ stand for the rotation, expansion, shear and four-acceleration of the flow lines, defined via the invariant decomposition:

$$\nabla_\mu V_\nu = \hat{\omega}_{\nu\mu} + \hat{\sigma}_{\nu\mu} + \frac{1}{3}\hat{\Theta}\hat{h}_{\nu\mu} - \hat{a}_\nu V_\mu, \quad (13)$$

while $\hat{h}_{\alpha\beta}(V) = g_{\alpha\beta} + V_\alpha V_\beta$ is the projection tensor associated with the four-velocity V.

Equations (11-12) show that, besides the scalar σ, the field $F^{\mu\nu}$ is influenced by two classes of factors: factors describing the state of the conducting fluid and secondly by the curvature of g manifesting itself via the coupling of $F^{\mu\nu}$ to the Ricci and Riemann tensor. Thus, and in contrast to non-relativistic settings, in an arbitrary spacetime the study of the electromagnetic field generated by a conducting fluid is a difficult problem to settle. Eqs. (11-12) ought to be analyzed in combination with Eqs. (1,2) and this requirement complicates the matter. Whether conducting fluids are capable of dynamo action and whether curvature or global topology of (M, g) can be influential factors for the maintenance of an initial electromagnetic field are open questions. An answer to these questions would require the understanding of the long term behavior of solutions of the system (1,2, 11- 12), a task that is by no means trivial.

On the other hand, the discussion of sections (1,2) indicates that, as far as issues of cosmic magnetism are concerned, the relevant background spacetimes are either stationary axisymmetric spacetimes (case of pulsar) or spatially homogeneous and isotropic cosmological spacetimes (case of galactic or extragalactic fields). Both families of those spacetimes permit a family of time-like world lines, and this property is very helpful. For such spacetimes we shall show below that Eqs. (1,2) can be cast into an equivalent form so that they are reminiscent of the Eqs. (3,4). For purposes of general we start with a background (M, g), permitting a smooth timelike congruence of world lines defining a non-singular, timelike unit vector field u possessing complete orbits. Making use of this field u, the Maxwell tensor $F_{\mu\nu}$ can be decomposed according to [27]:

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu + \epsilon_{\mu\nu\sigma\tau} u^\sigma B^\tau$$
$$\Leftrightarrow \quad E_\mu = F_{\mu\nu} u^\nu, \quad B_\mu = -\frac{1}{2}\epsilon_{\mu\nu}{}^{\sigma\tau} F_{\sigma\tau} u^\nu \tag{14}$$

Denoting by $h^\mu{}_\nu(u) = \delta^\mu{}_\nu + u^\mu u_\nu$ the projection tensor associated with the field u the fields E_μ and B_μ are spatials, *i.e.* $E^\mu = h^\mu{}_\nu E^\nu$, $B^\mu = h^\mu{}_\nu B^\nu$, and are interpreted as the electric and magnetic field measured by the u-observers. As before, the decomposition

$$\nabla_\mu u_\nu = \omega_{\nu\mu} + \sigma_{\nu\mu} + \frac{1}{3}\Theta h_{\nu\mu} - a_\nu u_\mu \tag{15}$$

defines the rotation $\omega_{\mu\nu}$, shear $\sigma_{\mu\nu}$, expansion Θ and acceleration a_μ of the world lines of the u observers. Forming $u_\nu \nabla_\mu F^{\mu\nu} = -(4\pi/c) J^\nu u_\nu$, making use of (14) and straightforward algebra in view of (15) yields:

$$\nabla_\mu E^\mu = -\epsilon_{\mu\nu\sigma\tau}\omega^{\mu\nu} u^\sigma B^\tau + a_\mu E^\mu + 4\pi\rho \tag{16}$$

where $c\rho = -J^\mu u_\mu$ is identified as the charge density measure by u^μ. Projecting the first pair of Maxwell's equations on the rest space of u, yields:

$$h^\sigma{}_\nu \nabla_\mu F^{\mu\nu} = h^\sigma{}_\nu \nabla_\mu (u^\mu E^\nu - u^\nu E^\mu + \epsilon^{\mu\nu\rho\tau} u_\rho B_\tau)$$
$$= -\frac{4\pi}{c} J^\nu h^\sigma{}_\nu,$$

which can eventually be rearranged so that

$$u^\mu \nabla_\mu E^\nu - (\omega^\nu{}_\mu + \sigma^\nu{}_\mu - \frac{2}{3} h^\nu{}_\mu \Theta + u^\nu a_\mu) E^\mu$$
$$+ \epsilon^{\nu\mu\sigma\tau}(u_\mu a_\sigma B_\tau + u_\mu \nabla_\sigma B_\tau + \omega_{\mu\nu} B_\tau)$$
$$+ u^\nu \epsilon^{\alpha\beta\mu\tau} u_\alpha \omega_{\beta\mu} B_\tau = -\frac{4\pi}{c} h_\mu{}^\nu J^\mu. \tag{17}$$

The decomposition of $\nabla_{[\mu} F_{\nu\sigma]} = 0 \iff \epsilon^{\mu\nu\sigma\tau}\nabla_\mu F_{\nu\sigma} = 0$ can be worked out analogously. Projecting $\epsilon^{\mu\nu\sigma\tau}\nabla_\mu F_{\nu\sigma} = 0$ along u, we obtain

$$\nabla_\mu B^\mu = \epsilon_{\mu\nu\sigma\tau}\omega^{\mu\nu} u^\sigma E^\tau + a_\mu B^\mu, \tag{18}$$

while the combination $h^\rho{}_\tau \epsilon^{\mu\nu\sigma\tau}\nabla_\mu F_{\nu\sigma} = 0$ yields

$$u^\mu \nabla_\mu B^\nu - (\omega^\nu{}_\mu + \sigma^\nu{}_\mu - \frac{2}{3} h^\nu{}_\mu \Theta + u^\nu a_\mu) B^\mu$$
$$-\epsilon^{\nu\mu\sigma\tau}(u_\mu a_\sigma E_\tau + u_\mu \nabla_\sigma E_\tau + \omega_{\mu\nu} E_\tau)$$
$$+ u^\nu \epsilon^{\alpha\beta\mu\tau} u_\alpha \omega_{\beta\mu} E_\tau = 0 \tag{19}$$

The current conservation equation $\nabla_\mu J^\mu = 0$ implies that the charge density $\rho = -J^\mu u_\mu / c$ and spatial current $J^\nu_{(u)} = h^\nu_\mu J^\mu$ seen by the u observers obey

$$u^\mu \nabla_\mu (\rho c) + \Theta\rho + \nabla_\mu J^\mu_{(u)} = 0. \tag{20}$$

The system (16-19), with (20), added to it, constitutes a first-order system of equations equivalent to the content of covariant Maxwell's eqs. For the particular case where J^μ is identified as the conduction current $J^\mu_c = \sigma g^{\mu\nu} F_{\nu\tau} V^\tau$, taking its components J^μ_c along and perpendicular to u^μ we obtain:

$$J^\mu_c = \sigma(E_\nu V^\nu) u^\mu + \sigma(\epsilon^{\mu\nu\sigma\tau} u_\nu B_\sigma V_\tau - u^\nu V_\nu E^\mu)$$
$$= -\rho c u^\mu + J^\mu_{(u)}. \tag{21}$$

This current and (16-19), adding suitable conditions across the hypersurface separating the fluid from the vacuum region, provide a framework for the description of the electromagnetic field generated by the flow of the conducting fluid. In fact bellow we shall show that the above framework leads to an effective set of equations that under suitable conditions on the flow are structurally similar to the set (3,4).

4. Conducting fluids on Cosmological spacetimes

In this section, we shall adapt the previous formalism to an (M, g) to permit a smooth timelike congruence of world lines so that

$$\omega_{\mu\nu} = \sigma_{\mu\nu} = a_\mu = 0 \qquad \text{and} \qquad \Theta \neq 0. \tag{22}$$

Spacetimes (M, g) consistent with (22) include the Friedman-Robertson-Walker (FRW) family and the family of the maximally symmetric spacetimes, i.e. de Sitter, anti-de Sitter and Minkowski spacetimes. Although we are not aware of any theorem guaranteeing that any (M, g) allow a smooth congruence that obeys (22) is necessary locally or globally isometric to an FRW spacetime or a spacetime of constant curvature, in this section we assume that the background (M, g) belongs either to an FRW family or is a spacetime of constant curvature. Making use of the symmetries of the background, we employ a coordinate gauge so that g is described by

$$g = -(dx^0)^2 + a^2(x^0)\gamma_{ab} dx^a dx^b = -(dx^0)^2 + a^2(x^0)$$
$$\times [d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2),] \tag{23}$$

where $a(x^0)$ is for the moment a smooth function, and $\Sigma(\chi)$ stands for $\sin\chi$, χ and $\sinh\chi$ depending on whether the surfaces of homogeneity and isotropy have a positive, zero respectively negative curvature. We shall assume those surfaces to be simply connected and geodesically complete (with respect to the induced Riemannian metric $a^2(x^0)\gamma_{ab}$), and thus are isometric to Euclidean $I\!R^3$, the sphere S^3, and hyperbole space H^3, respectively. Relative to (23), it is easily seen that the velocity field $u = (\partial/\partial x^0)$ satisfies (22), with $\Theta = 3(1/a)(da/dx^0) = 3H(x^0)$, where $H(x^0)$ stands for the Hubble constant. Specializing Eqs. (16-19) for the congruence satisfying condition (22), we obtain

$$\nabla_\mu E^\mu = 4\pi\rho,$$

$$u^\mu \nabla_\mu E^\nu + \frac{2}{3}\Theta E^\nu + \epsilon^{\nu\mu\sigma\tau} u_\mu \nabla_\sigma B_\tau = -\frac{4\pi}{c} h^\nu_\mu J^\mu, \quad (24)$$

$$\nabla_\mu B^\mu = 0,$$

$$u^\mu \nabla_\mu B^\nu + \frac{2}{3}\Theta B^\nu - \epsilon^{\nu\mu\sigma\tau} u_\mu \nabla_\sigma E_\tau = 0, \quad (25)$$

where E^μ and B^μ are the components of the electric and magnetic fields seen by the u-observers. Since the latter fields are spatial, the above equations can be rewritten in the form

$$D_a E^a = 4\pi\rho, \qquad D_a B^a = 0, \quad (26)$$

$$\frac{\partial E^a}{\partial x^0} + \frac{3}{a}\frac{da}{dx^0}E^a - \epsilon^{abc} D_b B_c = -\frac{4\pi}{c}J^a,$$

$$\frac{\partial B^a}{\partial x^0} + \frac{3}{a}\frac{da}{dx^0}B^a + \epsilon^{abc} D_b E_c = 0, \quad (27)$$

where currently (D, ϵ^{abc}) stands for the covariant derivative and coordinate components of the three dimensions Levi- Civita density associated with the spatial metric $a^2(x^0)\gamma_{ab}$. By introducing the scale factors, $h_\chi = a$, $h_\theta = a\Sigma$ and $h_\varphi = a\Sigma\sin\theta$, so that the intrinsic metric γ on each $x =$ const surface takes the form

$$\gamma = a^2(x^0)\left[d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$
$$= h^2_\chi d\chi^2 + h^2_\theta d\theta^2 + h^2_\varphi d\varphi^2. \quad (28)$$

and we also note that the fields

$$e_0 = \frac{\partial}{\partial x^0}, \qquad e_\chi = \frac{1}{a(x^0)}\frac{\partial}{\partial\chi},$$

$$e_\theta = \frac{1}{a(x^0)\Sigma}\frac{\partial}{\partial\theta}, \qquad e_\varphi = \frac{1}{a(x^0)\,\Sigma\sin\theta}\frac{\partial}{\partial\varphi}, \quad (29)$$

constitute an orthonormal, parallel basis propagated along the world lines of the u-observers. Relative to this anholonomic basis, the spacetime metric g and the three metric γ's can be written in the form

$$g = -(dx^0)^2 + d\widetilde{\chi}^2 + d\widetilde{\theta}^2 + d\widetilde{\varphi}^2,$$

$$c = d\widetilde{\chi}^2 + d\widetilde{\theta}^2 + d\widetilde{\varphi}^2,, \quad (30)$$

where the forms $\{dx^0, d\widetilde{\chi}, d\widetilde{\theta}, d\widetilde{\varphi}\}$ are dual to the basis vectors $\{e_0, e_\chi, e_\theta, e_\varphi\}$. The fields E and B can be expanded according to $\mathbf{E} = E^{\hat{a}}\mathbf{e}_{\hat{a}}$, $\mathbf{B} = B^{\hat{a}}\mathbf{e}_{\hat{a}}$, $\hat{a} = (\chi, \theta, \varphi)$ where $E^{\hat{a}}$, $B^{\hat{a}}$, stand for the physical components of the electric and magnetic field as measured by the u-observers. It follows then easily from (26-27) that the frame components $E^{\hat{a}}$, $B^{\hat{a}}$, $J^{\hat{a}}$ satisfy

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \qquad \nabla\cdot\mathbf{B} = 0, \quad (31)$$

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{a^2}\frac{\partial a^2\mathbf{E}}{\partial x^0}, \qquad \nabla\times\mathbf{E} = -\frac{1}{a^2}\frac{\partial a^2\mathbf{B}}{\partial x^0}, \quad (32)$$

$$\frac{\partial\rho}{\partial x^0} + 3\frac{\dot{a}}{a}\rho + \nabla\cdot\mathbf{J} = 0, \quad (33)$$

where $u^\mu\nabla_\mu \equiv (\partial/\partial x^0)$ and $(\nabla\cdot, \nabla\times)$ stand for the divergence and curl operators formed in terms of the scale factors (h_1, h_2, h_3) appearing in (28). Above, bold-type symbols stand for the wet of frame components taken relative to the basis $\{e_\chi, e_\theta, e_\varphi\}$, while $\nabla\cdot$ and $\nabla\times$ are defined by

$$(\nabla\cdot\mathbf{F}) = \frac{1}{h_1h_2h_3} \times\left[\frac{\partial}{\partial x^1}(h_2h_3F_1) + \frac{\partial}{\partial x^2}(h_1h_3F_2) + \frac{\partial}{\partial x^3}(h_1h_2F_3)\right] \quad (34)$$

$$\nabla\times\mathbf{F} = (\nabla\times\mathbf{F})^i e_i = \frac{\epsilon^{ijk}}{h_1h_2h_3}h_i\left[\frac{\partial}{\partial x^j}(h_kF_k)\right]e_i, \quad (35)$$

where $i, j, k = 1, 2, 3$, $(x^1, x^2, x^3) = (\chi, \theta, \varphi)$, while the totally antisymmetric Levi-Civita symbol ϵ^{ijk} has been normalized according to $\epsilon_{123} = 1$. Due to the dependence of the scale factors h_i, $i = 1, 2, 3$ upon x^0, the operators $\nabla\cdot$, $\nabla\times$ and $u^\mu\nabla_\mu \equiv (\partial/\partial x^0)$ no longer commute. A straightforward computation using (34, 35) shows that

$$\left[\frac{\partial}{\partial x^0},\ \nabla\times\right]\mathbf{F} = -\frac{\dot{a}}{a}\nabla\times\mathbf{F} \qquad \left[\frac{\partial}{\partial x^0},\ \nabla\cdot\right]\mathbf{F} = -\frac{\dot{a}}{a}\nabla\cdot\mathbf{F},$$

which establish the compatibility of (31-33).

We now consider a conducting fluid flow confined within an open, and for simplicity "spherical-like", region V of, say, the x^0= const hypersurface with a smooth boundary ∂V. We readjust the coordinate gauge (30) so that the region V is described by $0 \le \chi < \chi_0$, $0 \le \varphi \le 2\pi$, and $0 \le \theta \le \pi$ and denote by $\mathbf{n} = (\partial/\partial x)$ the outward point normal ∂V, defined by $\chi = \chi_0$. Due to the co-moving nature of the coordinates, the motion of the fluid remains within this "spherical region" for all subsequent times. Taking the frame components of the fluid four-velocity with respect to the basis (31) we obtain

$$V \equiv \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}e_0 + \frac{\mathrm{v}^i}{c\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}e_i, \quad (36)$$

where $\mathrm{v}^i(t, x)$, $i = 1, 2, 3$ are the components of the three-velocity as measured by the u-observers relative to their orthonormal frames, $\mathbf{v}^2 = (\mathrm{v}^1)^2 + (\mathrm{v}^2)^2 + (\mathrm{v}^3)^2$, and by assumption $\mathbf{v}\cdot\mathbf{n} = 0$, where $n = (\partial/\partial x)$ is the outward normal of ∂V. In view of (36), the conduction current J_c defined in (16) can be written in the form:

$$J_c = \frac{\sigma}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\left(\mathbf{E}\cdot\frac{\mathbf{v}}{c}\mathbf{e}_0 + \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right) = -\rho c e_0 + \mathbf{J}, \quad (37)$$

where we have expressed the spatial $\mathbf{J}$ part of J_c in terms of the frame components of the electric and magnetic field. In close analogy to the conditions underlying the derivation of (3,4) below, we shall restrict our attention to fluid flows which are "non-relativistic", *i.e.* flow so that the three-velocity $\mathbf{v}$ as seen by the u-observers satisfies $|\mathbf{v}|/c \ll 1$. Under such conditions, terms $|\mathbf{v}|^2/c^2$ or higher will be neglected. Besides this; we shall also neglect the time component of $\mathbf{J}_c$. The reasoning for this is identical to that involved in the non-relativistic MHD. The proper charge density in the fluid is leadingly described by $\nabla \cdot (\mathbf{v} \times \mathbf{B}/c)$, which is of the order $|\mathbf{v}|/c$. In contrast, the volumen charge density in (36) is a second-order effect, a point that will be elaborated below. Accordingly, the conduction current relative to the u-observers will be described by $\mathbf{J}_c = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c)$. In this approximation, Colomb's law will be of secondary importance. It will be imposed after the solution has been constructed, and the source term in $\nabla \cdot \mathbf{E}$ will specify the effective volume charge density of the fluid. Under these conditions, the fields $\mathbf{E}$ and $\mathbf{B}$ within the fluid satisfy

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{1}{a^2}\frac{\partial}{\partial x^0}(a^2\mathbf{B}) \tag{38}$$

$$\nabla \times \mathbf{B} = \frac{4\pi\sigma}{c}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) + \frac{1}{a^2}\frac{\partial}{\partial x^0}(a^2\mathbf{E}), \quad . \tag{39}$$

Based on those equations an order of magnitude estimate shows that the fields E and field B in Faraday's law are related via

$$|\mathbf{E}| \approx \frac{L}{c}(\frac{1}{T} + \frac{1}{T_{exp}})|\mathbf{B}|,$$

where (L, T) are the length (resp time scale) over which $\mathbf{E}$ (resp B) varies appreciably, while $T_{exp}^{-1} = c(1/a)(da/dx^0)$ corresponds to the expansion time scale of the background spacetime. For the particular case where $T_{exp} \gg T$, $\partial\mathbf{E}/\partial x^0$ in Ampere's law can be neglected in comparison to $\nabla \times \mathbf{B}$; moreover, the term $2\dot{a}/a\mathbf{E}$ has been incorporated into the current. Under these conditions, after taking another curl in Eq. (39) and rearaging, it follows that the field within the fluid satisfies

$$\frac{1}{a}\frac{\partial(a^2\mathbf{B})}{\partial x^0} = \nabla \times \left(\frac{\mathbf{B} \times \mathbf{v}}{c}\right) + \nabla \times \left(\frac{c}{4\pi\sigma}\nabla \times \mathbf{B}\right),$$
$$\nabla \cdot \mathbf{B} = 0 \quad 0 \le \chi < \chi_0 \tag{40}$$

while in the region exterior to V, field $\mathbf{B}$ would be assumed to obey

$$\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = 0 \qquad \chi > \chi_0. \tag{41}$$

As long as conductivity is finite, field $\mathbf{B}$ would be continuous across the $x = x_0$ timelike hypersurface and thus if by $[\mathbf{B}]$ we denote the jumb of $\mathbf{B}$ across ∂V we shall require that the any non singular solution of (40, 41) ought to satisfy $\left[\mathbf{B}\right]_{\partial V} = 0$. The set of Eqs. (40, 41) are structurally identical to the set (1,2). It is however important to realize that a solution to (40, 41) would require the specification of the behavior of the field $\mathbf{B}$ as $\chi \to \infty$, case of $I\!R^3$, H^3 or specific behavior of the $\mathbf{B}$-field at the antipodal point of S^3. Implementation of such conditions leads to the existence to global solutions of (40, 41) possessing properties markedly different from their non-relativistic counterpart. A detailed analysis of the behavior of solutions of above system will be discussed elsewhere.

5. Conducting fluids in a stationary-axisymmetric and circular spacetime

In this second application, we shall briefly discuss the form of (18-24) for a stationary-axisymmetric and circular background spacetime (M, g). For such spacetimes, there exists a local coordinate chart (t, φ, x^1, x^2) so that (t, φ) are adapted to the Killing fields, while g takes the form [27]

$$\begin{aligned} g &= g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{ij}dx^i dx^j \\ &= -V^2dt^2 + g_{\varphi\varphi}(d\varphi - \Omega_B dt)^2 + g_{ij}dx^i dx^j, \\ i, j &= 1, 2 \end{aligned} \tag{42}$$

where $V = V(x^1, x^2)$, $\Omega_B = \Omega_B(x^1, x^2)$, $g_{ij} = g_{ij}(x^1, x^2)$ and (x^1, x^2) are for the moment arbitrary coordinates of the family of 2-planes perpendicular to the planes spanned by the commuting Killing vector fields $\xi_t = (\partial/\partial t)$ and $\xi_\varphi = (\partial/\partial\varphi)$, and V and Ω_B are defined invariantly by [26]:

$$\begin{aligned} V^2 &= -g(\xi_t, \xi_t) + \frac{g^2(\xi_t, \xi_\varphi)}{g(\xi_\varphi, \xi_\varphi)}, \\ \Omega_B &= -\frac{g(\xi_t, \xi_\varphi)}{g(\xi_\varphi, \xi_\varphi)} \end{aligned} \tag{43}$$

The function Ω_B is related to the so-called dragging of inertial frames, but as we shall see shortly it also plays a very important role in the description of electromagnetic phenomena. Any stationary-axisymmetric circular spacetime accepts the so-called ZAMO family of observers, defined by the requirement that their four-velocity u be described by:

$$u = \frac{1}{V}(\xi_t + \Omega_B \xi_\varphi); \tag{44}$$

and note that

$$\nabla_\mu u_\nu = \sigma_{\nu\mu} - a_\nu u_\mu, \tag{45}$$

implying that the world lines of the ZAMO's are accelerated, shearing, non-expanding and possess zero rotation. On the other hand, (42) implies that the intrinsic geometry any $t =$ const family of hypersurfaces is described by

$$\begin{aligned} \gamma = g_{\varphi\varphi}d\varphi^2 + \gamma_{ij}dx^i dx^j &= e^{2\lambda}dr^2 + e^{2\mu}d\theta^2 + e^{2\psi}d\varphi^2 \\ &= h_r^2 dr^2 + h_\tau^2 d\theta^2 + h_\varphi^2 d\varphi^2, \end{aligned}$$

where we fixed the in the choice of the (x^1, x^2) coordinates freedom by introducing an arbitrary orthogonal set (r, θ), and scale factors h_r, h_τ and h_φ in (46) have been introduced for later use. We introduce a field of orthonormal tetrads carried by ZAMOS, defined by

$$e_{(t)} = u = e^{-\nu}\left[\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}\right], \quad e_{(\varphi)} = e^{-\psi}\frac{\partial}{\partial \varphi},$$

$$e_{(r)} = e^{-\lambda}\frac{\partial}{\partial r}, \quad e_{(\theta)} = e^{-\mu}\frac{\partial}{\partial \theta}, \quad (46)$$

and adopt Eqs. (16-19) to the congruence of ZAMOS [see (45)]. After some algebra, we find

$$\nabla_\mu E^\mu - a_\mu E^\mu = 4\pi\rho, \quad \nabla_\mu B^\mu - a_\mu B^\mu = 0, \quad (47)$$

$$u^\mu \nabla_\mu E^\nu - \sigma^\nu{}_\mu E^\mu - u^\nu a_\mu E^\mu$$
$$= -\epsilon^{\nu\mu\sigma\tau}(u_\mu a_\sigma B_\tau + u_\mu \nabla_\sigma B_\tau) - \frac{4\pi}{c} h_\mu{}^\nu J^\mu \quad (48)$$

$$u^\mu \nabla_\mu B^\nu - (\sigma^\nu{}_\mu + u^\nu a_\mu) B^\mu$$
$$= \epsilon^{\nu\mu\sigma\tau}(u_\mu a_\sigma E_\tau + u_\mu \nabla_\sigma E_\tau), \quad (49)$$

where (E^μ, B^μ) are the components of the electric and magnetic fields as seen by ZAMOS. Above, $u^\mu \nabla_\mu = d/d\tau$, where τ is the proper time measured along the world lines of ZAMOS. However those equations simplify considerably if we introduce the Killing time t. To transform the set (47-49) into this new time coordinate, we first consider an arbitrary, smooth, vector field X defined in the region of spacetime covered by the (t, φ, r, θ) chart. In view of the fact

$$Vu = \frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi},$$

it follows that the Lie derivative X along Vu satisfies

$$\mathcal{L}_{Vu} X = \mathcal{L}_{\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}} X, \quad (50)$$

from which we infer

$$[V u^\mu \nabla_\mu X^\nu - X^\mu \nabla_\mu (V u^\nu)] \frac{\partial}{\partial x^\nu} = \mathcal{L}_{\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}} X. \quad (51)$$

Developing the left hand side further, we arrive at

$$V(u^\mu \nabla_\mu X^\nu)\frac{\partial}{\partial x^\nu} = \mathcal{L}_{\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}} X$$
$$+ [(X^\mu \nabla_\mu V) u^\nu + V X^\mu \nabla_\mu u^\nu] \frac{\partial}{\partial x^\nu}. \quad (52)$$

Identifying X as the electric field E (respectively magnetic field B) we, obtain

$$(u^\mu \nabla_\mu E^\nu)\frac{\partial}{\partial x^\nu} = \frac{1}{V}\mathcal{L}_{\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}} E$$
$$+ [E^\mu a_\mu u^\nu + E^\mu(\sigma^\nu{}_\mu - a^\nu u_\mu)] \frac{\partial}{\partial x^\nu}, \quad (53)$$

and of course a similar expression holds true once E is replaced by B. Accordingly, the set of Eqs. (47-49) takes the form

$$\nabla_\mu E^\mu - a_\mu E^\mu = 4\pi\rho, \quad \nabla_\mu B^\mu - a_\mu B^\mu = 0, \quad (54)$$

$$\frac{1}{V}(\mathcal{L}_{\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}} E)^\nu = -\epsilon^{\nu\mu\sigma\tau}(u_\mu a_\sigma B_\tau$$
$$+ u_\mu \nabla_\sigma B_\tau) - \frac{4\pi}{c} h_\mu{}^\nu J^\mu \quad (55)$$

$$\frac{1}{V}(\mathcal{L}_{\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}} B)^\nu = \epsilon^{\nu\mu\sigma\tau}(u_\mu a_\sigma E_\tau + u_\mu \nabla_\sigma E_\tau), \quad (56)$$

where it should noted that, as a consequence of stationarity,

$$(\mathcal{L}_{\frac{\partial}{\partial t}} E) = \frac{\partial E^{\hat{i}}}{\partial x^0} e_{\hat{i}}.$$

As in the previous section, we rewrite those equations in terms of the quantities referring to the intrinsic geometry of the $t = $ const slice [see (46)]. Leaving intermediate computation aside, we find that they reduce to the form

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0 \quad (57)$$

$$\nabla \times (V\mathbf{B}) = \frac{4\pi}{c}\mathbf{J}V + \frac{1}{c}\mathcal{L}_{\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}} \mathbf{E} \quad (58)$$

$$\nabla \times (V\mathbf{E}) = -\frac{1}{c}\mathcal{L}_{\frac{\partial}{\partial t} + \Omega_B \frac{\partial}{\partial \varphi}} \mathbf{B}, \quad (59)$$

where the operators $\nabla\cdot$ and $\nabla\times$ are formed using the scale factor $(h_r,\ h_\tau\ h_\varphi)$ introduced in (46). Moreover, using the properties of the Lie derivatives, the set (30-32) can be cast in the form

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0 \quad (60)$$

$$\nabla \times (V\mathbf{B}) = \frac{4\pi}{c}\mathbf{J}V + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$
$$+ \Omega_B(\xi_\varphi \cdot \nabla)\mathbf{E} - h_\varphi(\mathbf{E} \cdot \nabla\Omega_B) e_{\hat{\varphi}} \quad (61)$$

$$\nabla \times (V\mathbf{E}) = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \Omega_B(\xi_\varphi \cdot \nabla)\mathbf{B}$$
$$+ h_\varphi(\mathbf{B} \cdot \nabla\Omega_B) e_{\hat{\varphi}}. \quad (62)$$

The above form of Maxwell's equations describes the orthogonal components of the electric and magnetic field generated by a charge density ρ and a spatial current $\mathbf{J}$. For axially symmetric sources and fields, *i.e.* $\mathcal{L}_{\xi_\varphi}\mathbf{E} = \mathcal{L}_{\xi_\varphi}\mathbf{B} = \mathcal{L}_{\xi_\varphi}\mathbf{J} = \mathcal{L}_{\xi_\varphi}\rho = 0$, after introducing the decomposition of all fields into their poloidal and toroidal parts, we find that Ampere's and Faraday's law can be writen in the form

$$\nabla \times (V\mathbf{B}^P) = \frac{4\pi}{c}\mathbf{J}^T + \frac{1}{c}\frac{\partial \mathbf{E}^T}{\partial t} + h_\varphi(\mathbf{E}^P \cdot \nabla\Omega_B) e_{\hat{\varphi}}. \quad (63)$$

$$\nabla \times (V\mathbf{B}^T) = \frac{4\pi}{c}\mathbf{J}^P + \frac{1}{c}\frac{\partial \mathbf{E}^P}{\partial t} \quad (64)$$

$$\nabla \times (V\mathbf{E}^P) = -\frac{1}{c}\frac{\partial \mathbf{B}^T}{\partial t} + h_\varphi(\mathbf{B}^P \cdot \nabla\Omega_B) e_{\hat{\varphi}}. \quad (65)$$

$$\nabla \times (V\mathbf{E}^T) = -\frac{1}{c}\frac{\partial \mathbf{B}^P}{\partial t}. \quad (66)$$

The analysis so far is general and poses no restriction upon the structure of the four-current. In principle, the above eqs can be adopted to a conduction current associated with a conducting fluid. Moreover, in order to be applied to a concrete case, a background geometry needs to be chosen. A physically important geometry would corespond to the geometry associated with a stationary rotating star. As is well known, the equations of a relativistic stellar structure for such a system is rather complex and, due to space limitations, we shall not discuss the form of the above equations in such a background. We shall however make comments regarding the structure of Faraday's and Ampere's law as described by (62,63). Their right-hand side shows a coupling of the term Ω_B, responsible for the dragging of inertial frames, to the electric and magnetic fields. This coupling acts as an effective current term and expresses an aspect of gravitomagnetism. Due to this coupling, the induction equation would contain additional "electromotive terms". Whether then a Cowling's theorem holds true for the resulting system is for the moment an open question.

6. Discussion

In this paper, we have concentrated on some problems that arise from the existence of the phenomenon of cosmic magnetism. Our treatment has been restricted to the dynamical equations describing the electromagnetic field generated by a conducting fluid described by a single component. An issue that needs to be addressed concerns the status of Cowling's theorem. Does some version of this theorem hold true in a general relativistic setting? The work of the last three sections offers a framework where this question can be properly addressed. On the other hand, and as the observational evidences make clear, large-scale magnetism appears to be an attribute of our observable universe. As we have seen, turbulent conducting fluid flows are of relevance. Understanding the properties of these flows in an arbitrary background spacetime and their impact on the generation of electromagnetic fields is also a relevant issue. Extension of the above work beyond the MHD regime, inclusion of additional terms in conduction current, and their relevance to the electromagnetic fields generated are problems worth pursuing and of relevance to the phenomenon of cosmic magnetism.

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