On quantum scattering on magnetic monopole

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For infinite dimensional representations of $SU(2)$ an addition theorem is found and used to obtain an analytic expression for the quantum mechanical scattering amplitude for an electrically charged particle moving in the field of a pointlike magnetic monopole.

Keywords: Representations of $SU(2)$; addition theorem; quantum scattering; magnetic monopole.

Se construyen las representaciones acotadas, de dimensión infinita, de $SU(2)$ y se obtiene una expresión analítica para la amplitud de dispersión cuántica correspondiente a una partícula eléctricamente cargada moviéndose en el campo de un monopolo magnético puntual.

Descriptores: Representaciones de $SU(2)$; teorema de adición; dispersión cuántica; monopolo magnético.

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The representations of the rotation group are widely used in quantum mechanics. However, as in common rotationally invariant quantum mechanical applications the Hilbert space is finite dimensional, most popular representations are also finite dimensional. This has not always been the case: in the late 60's, complex angular momenta where considered for some applications in particle physics [1, 2] and infinite dimensional representations where used, among others. Such approaches were left and darkness fell on infinite dimensional representations of $SU(2)$. However, recently we have found that working out a magnetic monopole problem with the generalized quantization condition forces us to use precisely such representations [3,4]. In the present paper, using our previous results and bounded infinite dimensional representations of $SU(2)$, we look for an analytical expression for the scattering amplitude in the monopole problem with arbitrary magnetic charge.

As is well known, the generators of the rotation group satisfy the $su(2)$ algebra

$$[J_i, J_j] = i\varepsilon_{ijk} J_k, \quad i, j, k = 1, 2, 3. \tag{1}$$

The casimir operator is

$$\mathbf{J}^2 = (J_+J_- + J_-J_+)/2 + J_3^2, \quad J_\pm = J_1 \pm iJ_2,$$

and one can choose, as a basis for the Hilbert space, simultaneous eigenvectors of $\mathbf{J}^2$ and J_3, say

$$\mathbf{J}^2|\jmath, m\rangle = \jmath(\jmath+1)|\jmath, m\rangle, \quad J_3|\jmath, m\rangle = m|\jmath, m\rangle.$$

Infinite dimensional representations of $SU(2)$ are classified as follows [1, 2, 5]:

- $\jmath + m$ integer. This case is known as *bounded below*. It means that there exists a state $|\jmath, -\jmath\rangle$ such that $J_-|\jmath, -\jmath\rangle = 0$.
- $\jmath - m$ integer, in which case the representation is known as *bounded above* because there exists a state $|\jmath, \jmath\rangle$ such that $J_+|\jmath, \jmath\rangle = 0$.
- Neither $\jmath - m$ nor $\jmath + m$ are integers. This type of representation is called *unbounded*.

Even if only irreducible representations are considered, for fixed $\jmath$ all of them are infinite dimensional.

For the **representation bounded above**, let us consider the action of the group $SU(2)$ on any function of the complex variable z [5, 6]:

$$T_g f(z) = (\bar\alpha - \beta z)^{2\jmath} f\left(\frac{\bar\beta - \alpha z}{\bar\alpha - \beta z}\right),$$

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1$$

So, in particular for the basis state $|\jmath, n\rangle = N_n z^n$ such that $\mathbf{J}^2|\jmath, n\rangle = \jmath(\jmath+1)|\jmath, n\rangle$ and $J_3|\jmath, n\rangle = (\jmath - n)|\jmath, n\rangle$ one has

$$T_g|\jmath, n\rangle = N_n(\bar\alpha - \beta z)^{2\jmath - n}(\bar\beta + \alpha z)^n = \sum_{k=0}^{\infty} A_{kn}^{(\jmath)}(g)|\jmath, k\rangle,$$

with the matrix elements $A_{kn}^{(\jmath)}$ given by

$$A_{kn}^{(\jmath)}(g) = \frac{N_n}{N_k}\frac{1}{k!}\frac{d^k}{dz^k}[(\bar\alpha - \beta z)^{2\jmath - n}(\bar\beta + \alpha z)^n]\bigg|_{z=0},$$

where the normalization factor is

$$N_n = (n!|\Gamma(2\jmath - n + 1)|)^{-1/2} \qquad [4].$$

Introducing the Euler angles φ, θ, ψ as

$$\alpha = e^{i(\varphi+\psi)/2}\cos\frac{\theta}{2}, \quad \beta = ie^{i(\varphi-\psi)/2}\sin\frac{\theta}{2},$$

we find that

$$A^{(j)}_{kn}(g) = \frac{N_k}{N_n} i^{n-k} 2^{k-j} \mathrm{e}^{i\varphi(k-j)} \mathrm{e}^{i\psi(n-j)} (1-x)^{(n-k)/2} \times (1+x)^{(2j-n-k)/2} P_k^{(n-k,2j-n-k)}(x), \quad n > k, \qquad (2)$$

where $x = \cos\theta$ and $P_n^{(\alpha,\beta)}(x)$ are *Jacobi polynomials* [5,6].

The matrix elements obtained have the following symmetry properties:

$$A^{(j)}_{nk}(g) = (-1)^{\sigma(n,k)} \mathrm{e}^{i(n-k)(\varphi-\psi)} A^{(j)}_{kn}(g),$$
$$A^{(j)}_{nk}(g^{-1}) = (-1)^{\sigma(n,k)} \overline{A^{(j)}_{kn}(g)},$$

where $\sigma(n,k) = \mathrm{sgn}(\sin\pi(2j-n))/\mathrm{sgn}(\sin\pi(2j-k))$.

For the **representation bounded below**, T_g acts on complex functions as [5, 6]

$$T_g f(z) = (\bar\beta + \alpha z)^{2j} f\left(\frac{\bar\alpha - \beta z}{\bar\beta + \alpha z}\right), \quad |z| \in \mathbb{C}, \qquad (3)$$

and the matrix elements of the representation are given by

$$B^{(j)}_{-k,-n}(g) = \frac{N_k}{N_n} i^{n-k} 2^{k-j} \mathrm{e}^{-i\varphi(k-j)} \mathrm{e}^{-i\psi(n-j)} \times (1-x)^{(n-k)/2} (1+x)^{(2j-n-k)/2} P_k^{(n-k,2j-n-k)}, \qquad (4)$$

where $k > n$ and again $N_n = (n!|\Gamma(2j-n+1)|)^{-1/2}$ [4].

The matrix elements obtained have the following symmetry properties:

$$B^{(j)}_{-n,-k} = (-1)^{\sigma(n,k)} \mathrm{e}^{-i(n-k)(\varphi-\psi)} B^{(j)}_{-k,-n}, \qquad (5)$$

$$B^{(j)}_{-n,-k}(g^{-1}) = (-1)^{\sigma(n,k)} \overline{B^{(j)}_{-k,-n}(g)}, \qquad (6)$$

where $\sigma(n,k) = \mathrm{sgn}(\sin\pi(2j-n))/\mathrm{sgn}(\sin\pi(2j-k))$.

Note by comparing (4) with (2), that one can be changed into the other by the transformations $\varphi \to -\psi$, $\psi \to -\varphi$. Thus the properties of this representation are similar to those of the one bounded above; one just need to make the changes mentioned in (5) and the other relations.

By expressing Jacobi polynomials in terms of hypergeometric functions [6], properties of the latter can be used to obtain expressions for matrix elements with different index values. According to this procedure, the representation bounded above has the block structure

$$A = \begin{pmatrix} A_{--} & 0 \\ A_{+-} & A_{++} \end{pmatrix},$$

while the representation bounded below,

$$B = \begin{pmatrix} B_{--} & B_{-+} \\ 0 & B_{++} \end{pmatrix},$$

In any of both expressions $A_{+-} = (A_{n,-k})$ and so on.

Now let us apply these results to scattering on a magnetic monopole. The classical Hamiltonian for an electrically charged particle of charge e and mass m moving in the field of a pointlike magnetic monopole of strength q is [9]

$$H = \frac{1}{2mr^2}\left((\mathbf{p}\cdot\mathbf{r})^2 + \mathbf{J}^2 - \mu^2\right), \quad \mu \equiv eq,$$

where

$$\mathbf{J} = \mathbf{r} \times (\mathbf{p} - e\mathbf{A}) - \mu\frac{\mathbf{r}}{r} \qquad (7)$$

is the total angular momentum, $\mathbf{A}$ being the vector potential. Following Dirac we choose $\mathbf{A}$ as [7]

$$\mathbf{A}(\mathbf{r}) = q\frac{1-\cos\theta}{r\sin\theta}\mathbf{e}_\varphi \qquad (8)$$

where θ and φ are, respectively, the usual polar and azimuthal angles taken about the z-axis. The components of the operator version of (7) become

$$J_\pm = \mathrm{e}^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi} - \frac{\mu\sin\theta}{1+\cos\theta}\right),$$
$$J_3 = -i\frac{\partial}{\partial\varphi} - \mu, \qquad (9)$$

Solutions to the Schrödinger equation are of the form

$$\Psi(\mathbf{r},t) = \mathrm{e}^{-iEt}R(r)Y(\theta,\varphi),$$

where

$$R(r) = (kr)^{-1/2}J_\nu(kr),$$
$$k \equiv \sqrt{2mE},$$
$$\nu = \sqrt{(j+1/2)^2 - \mu^2}$$

and the functions depending on angular variables are written as [3, 4]

$$Y \propto \mathrm{e}^{i(m+\mu)\varphi} z^p (1-z)^q F(a,b;c;z), \qquad (10)$$
$$(p+q)(p-q) = m\mu,$$

where $z \equiv \sin^2(\theta/2)$, and $F(a,b;c;z)$ is a hypergeometric function with parameters

$$a = p+q-j, \quad b = p+q+j+1, \quad c = 2p+1.$$

If a or b are negative integers then the hypergeometric series reduces to a polynomial and (10) are called *generalized monopole harmonics* [3]. The equality in (10) is recovered when the normalization factor is included [4].

Any wave function can be expressed as a linear combination of monopole harmonics (10). Expansion must be made over all available values of j, m. For our case, in having single-valued eigenfunctions from (10), one must impose $m+\mu$ integers. In standard quantum mechanical calculations, only positive half integer or integer values of j are considered and $-j \le m \le j$. However, in our case, such selection implies $\mu = n/2$, which is Dirac's quantization rule, precisely the case we wish to avoid. So, from nowon we shell use j and m, not integers or half integers. This forces us to use infinite dimensional representations of the rotation group. Solutions

(10), with parameters corresponding to each bounded representation, are given in [4]. Note that for the representation bounded above, $\jmath + \mu \in \mathbb{Z}_+$ while for that bounded below, $\jmath - \mu \in \mathbb{Z}_+$. Then any arbitrary solution must have the form

$$\Psi(\mathbf{r}) = \sum_{\jmath+\mu,m} C^{(+)}_{\jmath m\mu} \frac{J_\nu(kr)}{\sqrt{kr}} \overset{+}{Y}{}^{(\mu,m)}_{\jmath}(\theta,\varphi) + \sum_{\jmath-\mu,m} C^{(-)}_{\jmath m\mu} \frac{J_\nu(kr)}{\sqrt{kr}} \overset{-}{Y}{}^{(\mu,m)}_{\jmath}(\theta,\varphi), \quad (11)$$

where $\pm$ corresponds to *representations bounded above* and *below*, respectively [4].

Under the scattering condition, a wave function must have the asymptotic behavior [9]

$$\Psi_{\text{scatt}} \sim \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta,\varphi)\frac{\mathrm{e}^{ikr}}{kr},$$

where $f(\theta,\varphi)$ is known as scattering amplitude; by using a similar procedure to that in [9], one finds that asymptotically coefficients in (11) as

$$C^{(\pm)}_{\jmath m\mu} = \mathrm{e}^{-i(\nu+1/2)\pi/2}\overline{\overset{\pm}{Y}_{\jmath m\mu}(\pi-\theta',\pi+\varphi')}, \quad (12)$$

where one appeals to (11) as a valid expansion for any solution.

Substitution of (12) in (11) leads us to

$$f(\bar\theta,\bar\varphi) = \sum_{\jmath+\mu,m} \mathrm{e}^{-i(\nu+1/2)\pi/2}\overline{\overset{+}{Y}{}^{(\mu,m)}_{\jmath}(\tilde\theta,\tilde\varphi)}\,\overset{+}{Y}{}^{(\mu,m)}_{\jmath}(\theta,\varphi) + \sum_{\jmath-\mu,m} \mathrm{e}^{-i(\nu+1/2)\pi/2}\overline{\overset{-}{Y}{}^{(\mu,m)}_{\jmath}(\tilde\theta,\tilde\varphi)}\,\overset{-}{Y}{}^{(\mu,m)}_{\jmath}(\theta,\varphi), \quad (13)$$

where $\tilde\theta = \pi - \theta$ and $\tilde\varphi = \pi + \varphi$; $\bar\theta$ and $\bar\varphi$ are obtained from θ, φ and $\tilde\theta$, $\tilde\varphi$ with the standard rules for the addition of angles [9]. Thus in (13), the first problem is making sums over m, so one needs to find an *addition theorem for monopole harmonics*. In the case of working only with unitary representation, such an addition theorem is know [9], so here we consider specifically infinite dimensional bounded representations.

Analyzing these cases, one concludes that correspondence between solutions (10) and matrix representations (2, 4) is established as

Representation bounded above

If $m = \jmath - n$ and $0 \le n \le \jmath + \mu$, then

$$\overset{+}{Y}{}^{(\mu,n)}_{\jmath} = \sqrt{\frac{2\jmath+1}{4\pi}}\, i^{\mu+\jmath-n}\mathrm{e}^{-i\mu\psi}\mathrm{e}^{i\varphi(\mu+2\jmath-2n)}A^{(\jmath)}_{n,\jmath+\mu},$$

while for $n > \jmath + \mu$, then

$$\overset{+}{Y}{}^{(\mu,n)}_{\jmath} = \sqrt{\frac{2\jmath+1}{4\pi}}\, i^{n-\jmath-\mu}\mathrm{e}^{i(\jmath-n)(\varphi+\psi)}A^{(\jmath)}_{\jmath+\mu,n}.$$

In the case $m{=}\jmath{+}n$, one has $\overset{-}{Y}{}^{(\mu,n)}_{\jmath} \sim B^{(-\jmath-1)}_{-(\jmath+\mu),-n}$ where as harmonic has norm zero so the matrix element.

Representation bounded below

For $m = -\jmath + n$ and $0 \le n \le \jmath - \mu$, one finds

$$\overset{-}{Y}{}^{(\mu,n)}_{\jmath} = \sqrt{\frac{2\jmath+1}{4\pi}}\, i^{\jmath-\mu-n}\mathrm{e}^{i(n-\jmath)(\psi+\varphi)}B^{(\jmath)}_{-n,-(\jmath-\mu)};$$

and if $n > \jmath - \mu$, then

$$\overset{-}{Y}{}^{(\mu,n)}_{\jmath} = \sqrt{\frac{2\jmath+1}{4\pi}}\, i^{n-\jmath+\mu}\mathrm{e}^{-i\mu\psi}\mathrm{e}^{i\varphi(2n+\mu-2\jmath)}B^{(\jmath)}_{-(\jmath-\mu),-n}.$$

Finally, for $m = -\jmath - n - 1$, $\overset{+}{Y}{}^{(\mu,n)}_{\jmath} \sim A^{(-\jmath-1)}_{\jmath-\mu,n}$.

Note that for all our cases, correspondence between solutions to th Schrödinger equation and matrix representations of $SU(2)$ are made not over the whole matrices if not only over such blocks which under consecutive group transformations go into a block with the same structure as the original one (specifically A_{++} and B_{--}). This fact will be important in calculating scattering amplitude.

Substituting these relations in (13), and using properties such as (5, 6) we obtain the formula

$$f(\theta) = -\left(\frac{1-\cos\theta}{2}\right)^{\mu}\sum_{\jmath-\mu=0}^{\infty}(-1)^{\jmath-\mu}(2\jmath+1)\mathrm{e}^{-i(\nu+1/2)\pi/2}P^{(2\mu,0)}_{\jmath-\mu}(\cos\theta) - \left(\frac{1-\cos\theta}{2}\right)^{-\mu}\sum_{\jmath+\mu=0}^{\infty}(2\jmath+1)\mathrm{e}^{-i(\nu+1/2)\pi/2}P^{(0,-2\mu)}_{\jmath+\mu}(\cos(\pi-\theta)) \quad (14)$$

which in contrast with the usual results [9, 10] includes two contributions which may give different results from those known at the present time. The results obtained by working out specific calculations using (14) will be presented elsewhere.

1. S.S. Sannikov, *Sov. J. Nucl. Phys.* **6** (1968) 788.
2. S.S. Sannikov, *Sov. J. Nucl. Phys.* **6** (1968) 939.
3. A.I. Nesterov and F. Aceves de la Cruz, *Phys. Lett. A* **302** (2002) 253 [hep-th/0208210].
4. A.I. Nesterov and F. Aceves de la Cruz, *On infinite-dimensional representations of the rotation group and Dirac monopole*, hep-th/0503040.
5. B.G. Wybourne, *Classical groups for Physicists* (Wiley, New York, 1974).
6. G.E. Andrews, R. Askey, and R. Roy, *Special functions* (Cambridge, Cambridge University Press, 2000).
7. P.A.M. Dirac, *Proc. Roy. Soc. Lond. A* **133** (1931) 60.
8. R. Jackiw, *Phys. Rev. Lett.* **54** (1985) 159.
9. J. Schwinger, K.A. Milton, W.-Y. Tsai, L.L. DeRaad Jr, and D.C. Clark, *Ann. Phys.* **101** (1976) 451.
10. K.W. Ford and J.A. Wheeler, *Ann. Phys.* **7** (1959) 287.