

A method for generating Gowdy cosmological models

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Numerical methods have been extensively used to investigate Gowdy models, but only recently it has been argued that solutions-generating techniques can be applied in this case to generate new solutions. In this work, we concentrate on T^3 Gowdy cosmological models and shall see that a complex coordinate transformation, together with a complex change of metric functions, allows us to apply, in a straightforward manner, the well-known solution-generating techniques that have been intensively used for stationary axisymmetric solutions.

Keywords: Exact solutions; classical differential geometry; classical relativity; cosmology.

Métodos numéricos han sido extensamente usados para investigar modelos de Gowdy, pero sólo recientemente se ha propuesto que las técnicas de generación de soluciones pueden ser aplicadas en éste caso para generar nuevas soluciones. En este trabajo nos concentraremos en los modelos cosmológicos de Gowdy T^3 y mostraremos como una transformación coordenada compleja junto con un cambio complejo de las funciones métricas nos permite aplicar de una manera sencilla las técnicas de generación de soluciones que han sido extensamente usadas para soluciones estacionarias axisimétricas.

Descriptores: Soluciones exactas; geometría diferencial clásica; relatividad clásica; cosmología.

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1. Stationary axisymmetric solutions and Gowdy T^3 models

Consider the line element for stationary axisymmetric space-times in the Lewis-Papapetrou form [13]

$$ds^2 = -e^{2\psi}(dT + \omega d\phi)^2 + e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (1)$$

where ψ , ω , and γ are functions of the nonignorable coordinates ρ and z . The ignorable coordinates T and ϕ are associated with the two Killing vector fields $\eta_I = \partial/\partial T$ and $\eta_{II} = \partial/\partial\phi$. The field equations take the form

$$\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz} + \frac{e^{4\psi}}{2\rho^2}(\omega_\rho^2 + \omega_z^2) = 0, \quad (2)$$

$$\omega_{\rho\rho} - \frac{1}{\rho}\omega_\rho + \omega_{zz} + 4(\omega_\rho\psi_\rho + \omega_z\psi_z) = 0, \quad (3)$$

$$\gamma_\rho = \rho(\psi_\rho^2 - \psi_z^2) - \frac{e^{4\psi}}{4\rho^2}(\omega_\rho^2 - \omega_z^2), \quad (4)$$

$$\gamma_z = 2\rho\psi_\rho\psi_z - \frac{1}{2\rho}e^{4\psi}\omega_\rho\omega_z, \quad (5)$$

where the lower indices represent the partial derivative with respect to the corresponding coordinate.

Consider now the following coordinate transformation $(\rho, t) \rightarrow (\tau, \sigma)$ and the complex change of coordinates

$(\phi, z) \rightarrow (\delta, \chi)$ defined by: $\rho = e^{-\tau}$, $T = \sigma$, $z = i\chi$, $\phi = i\delta$, and introduce the functions P , Q and λ by means of the relationships:

$$\psi = \frac{1}{2}(P - \tau), Q = i\omega, \gamma = \frac{1}{2}\left(P - \frac{\lambda}{2} - \frac{\tau}{2}\right).$$

Introducing the last two equations into the line element (1), we obtain

$$-ds^2 = e^{-\lambda/2}e^{\tau/2}(-e^{-2\tau}d\tau^2 + d\chi^2) + e^{-\tau}[e^P(d\sigma + Qd\delta)^2 + e^{-P}d\delta^2]. \quad (6)$$

Let us take $\tau \geq 0$ and “compactify” the new coordinates as $0 \leq \chi$, σ , $\delta \leq 2\pi$, with the coordinates τ , χ , σ and δ in the range given above known as the line element for Gowdy T^3 cosmological models [3–6]. Applying the transformations to the field equations (2)–(5), we obtain the field equations for the Gowdy cosmological models which, after some algebraic manipulations, can be written as a set of two second-order differential equations for P and Q

$$P_{\tau\tau} - e^{-2\tau}P_{\chi\chi} - e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\chi^2) = 0, \quad (7)$$

$$Q_{\tau\tau} - e^{-2\tau}Q_{\chi\chi} + 2(P_\tau Q_\tau - e^{-2\tau}P_\chi Q_\chi) = 0, \quad (8)$$

and two first order differential equations for λ ,

$$\lambda_\tau = P_\tau^2 + e^{-2\tau}P_\chi^2 + e^{2P}(Q_\tau^2 + e^{-2\tau}Q_\chi^2), \quad (9)$$

$$\lambda_\chi = 2(P_\chi P_\tau + e^{2P}Q_\chi Q_\tau). \quad (10)$$

It should be emphasized that this method for “deriving” the Gowdy line element from the stationary axisymmetric one involves real as well as complex transformations at the level of coordinates and metric functions. It is, therefore, necessary that the resulting metric functions P , Q , and λ be real. That means that in general it is not possible to take an axisymmetric stationary solution and apply the transformations to obtain a Gowdy cosmological model [11]. If the resulting functions are not real, they cannot be physically reasonable solutions to the real equations (7)–(10). These transformations can be used only as a guide, to get some insight into the form of the new solutions. In any case, the corresponding field equations must be invoked in order to confirm the correctness of the solution.

2. The Ernst representation

An alternative approach for exploring the symmetries inherent in the Ernst equation [8–11, 13–16] was explicitly developed by Sibgatullin [17] and consists in constructing exact solutions to the Ernst equation from initial data specified only on a certain hypersurface (submanifold) of the spacetime. For instance, in the case of stationary axisymmetric spacetimes, Sibgatullin’s method allows one to construct exact solutions from their data on the axis of symmetry. Next we shall show that Sibgatullin’s method can be applied in the case of Gowdy cosmological models and shall present an example of its application.

For the Ernst representation of the main field Eqs. (7) and (8), we need to introduce a new coordinate t and a new function $R = R(t, \chi)$ by means of the equations [8, 12]

$$t = e^{-\tau}, \quad R_t = te^{2P}Q_\chi, \quad R_\chi = te^{2P}Q_t. \quad (11)$$

Then, the field equation (7) can be expressed as

$$t^2 \left(P_{tt} + \frac{1}{t} P_t - P_{\chi\chi} \right) + e^{-2P} (R_t^2 - R_\chi^2) = 0, \quad (12)$$

whereas Eq. (8) for the function Q turns out to be equivalent to the the integrability condition $R_{t\chi} = R_{\chi t}$. However, an alternative and convenient equation is obtained by introducing Eq. (11) directly into Eq. (8). Thus we obtain

$$te^P \left(R_{tt} + \frac{1}{t} R_t - R_{\chi\chi} \right) - 2[(te^P)_t R_t - (te^P)_\chi R_\chi] = 0, \quad (13)$$

an equation which of course becomes an identity if the integrability condition $R_{t\chi} = R_{\chi t}$ is satisfied. We can now introduce the complex Ernst potential E and the complex gradient operator D as

$$E = te^P + iR, \quad \text{and} \quad D = \left(\frac{\partial}{\partial t}, i \frac{\partial}{\partial \chi} \right), \quad (14)$$

which allow us to write the main field equations in the *Ernst-like representation*

$$Re(E) \left(D^2 E + \frac{1}{t} D t D E \right) - (DE)^2 = 0. \quad (15)$$

It is easy to verify that the field equations (12) and (13) can be obtained as the real and imaginary part of the Ernst equation (15), respectively. In terms of the Ernst potential, the system of first order, partial, differential equations (9) and (10) take the form:

$$\lambda_t = -\frac{t}{2} (C_+ C_+^* + C_- C_-^*), \quad (16)$$

$$\lambda_\chi = -\frac{t}{2} (C_+ C_+^* - C_- C_-^*), \quad (17)$$

where

$$C_\pm = \frac{1}{\text{Re}(E)} (E_t \pm E_\chi) - \frac{1}{t},$$

and the asterisk denotes complex conjugation.

If the Ernst potential E is known, then it is easy to recover the metric functions P , Q and λ which enter the line element (6) of Gowdy T^3 cosmological models. In fact, from Eq. (14) one can algebraically construct the functions P and R . Then the function Q can be obtained by solving the system of two first order partial differential equations given in (11). Notice that the integrability condition of this last system is satisfied by virtue of Eq. (15). Finally, the system (16) and (17) for the function λ can be solved by quadratures since its integrability condition coincides with the Ernst equation (15). Consequently, all the information about any Gowdy T^3 cosmological model is contained in the corresponding Ernst potential.

3. Asymptotic Behavior (AVTD)

The AVTD behavior [7] implies that, at the singularity all spatial derivatives of the field equations can be neglected and only the temporal behavior is relevant. In the case of T^3 models, the transformation seen in Sec. 1 indicates that the limit $\tau \rightarrow \infty$ is equivalent to the limit $\rho \rightarrow 0$; however, this is true only at the level of coordinates and a more detailed analysis is necessary to make sure that this analogy is also valid at the level of explicit solutions. If we neglect the spatial dependence on z in the system of partial differential equations for ψ and ω given in Eqs. (2) and (3), which according to the transformation used is equivalent to the spatial dependence on χ in Gowdy models, then we obtain a system of differential equations which can be solved by quadratures, and yields

$$\begin{aligned} \psi &= \frac{1}{2} \ln[a(\rho^{1+c} + b^2 \rho^{1-c})] \\ \omega &= \frac{ib}{a(\rho^{1+c} + b^2 \rho^{1-c})} + id, \end{aligned} \quad (18)$$

where a , b , c and d are arbitrary real functions of z . If we now follow the prescription given in the complex change of coordinates and the relationships between P , Q and λ with

the metric functions for obtaining Gowdy models, we find that solution (18) “corresponds” to the Gowdy model

$$P = \ln[a(e^{-c\tau} + b^2 e^{c\tau})], \quad Q = \frac{b}{a(e^{-2c\tau} + b^2)} + d, \quad (19)$$

where now a , b , c and d are to be considered as arbitrary real functions of the coordinate χ . The solution (19) is known in the literature as the AVTD solution for Gowdy T^3 models [6] and dictates the behavior of these models near the singularity $\tau \rightarrow \infty$. Thus, we have “derived” the AVTD solution starting from its stationary axisymmetric counterpart. This is a further indication that the behavior of Gowdy models at the initial singularity is mathematically equivalent to the behavior of stationary axisymmetric solutions at the axis. The value of the function λ corresponding to the AVTD solution (19) can be obtained by integrating Eq. (9): $\lambda = \lambda_0 - c^2 \ln t$, where λ_0 is an additive constant. Furthermore, the corresponding AVTD Ernst potential can be obtained by introducing Eq. (19) into Eqs. (14) and (15). Then

$$E = a[e^{-(1+c)\tau} + b^2 e^{-(1-c)\tau}] + iR^{avtd}. \quad (20)$$

with $R_\chi^{avtd} = -2abc$. If we define

$$E(\tau \rightarrow \infty, \chi) = e(\chi)$$

as the Ernst potential at the singularity, we see from Eq. (20) that for $c \in (-1, 1)$ only the imaginary part remains, $e(\chi) = iR^{avtd}$. This means that the real part of $e(\chi)$ is arbitrary and, since R^{avtd} is given in terms of the real part, it is also arbitrary. If $c \notin (-1, 1)$, the Ernst potential diverges at the singularity for arbitrary values of the functions a and b . In the limiting case $c = \pm 1$, the Ernst potential at the singularity is regular, but again no conditions appear for the behavior of the functions a and b . Consequently, the AVTD behavior does not impose any conditions on the function $e(\chi)$. We shall now see that it is possible to use this function to construct the corresponding Ernst potential $E(\tau, \chi)$.

4. Sibgatullin’s method

Sibgatullin’s method [17] was developed to construct exact stationary axisymmetric solutions starting from their data on the axis of symmetry. It is based upon the fact that the Ernst equation possesses symmetry properties associated with an infinite-dimensional Lie group which transforms one solution of this equation into another solution of the same equation. This implies remarkable analyticity properties that make it possible to reduce the Ernst equation to a system of linear integral equations which can be integrated explicitly if the initial data are known, for instance, on the axis of symmetry. It is clear that the Ernst-like representation (15) possesses similar symmetry properties. On the other hand, we have shown that the behavior of stationary axisymmetric solutions near the axis is mathematically equivalent to the behavior of Gowdy T^3 cosmological models near the singularity. Thus, it

should be possible to construct Gowdy cosmological models starting from the value of the corresponding Ernst potential at the singularity. It turns out that Sibgatullin’s method can be generalized in a straightforward manner to include the case of Gowdy models.

Assuming that the value of the Ernst potential is known at the initial singularity, i.e. $e(\chi)$ is given, then the Ernst potential can be generated by means of the integral equation

$$E(t, \chi) = \frac{1}{\pi} \int_{-1}^1 \frac{e(\xi)\mu(\xi)}{\sqrt{1-s^2}} ds, \quad (21)$$

where the unknown function $\mu(\xi)$ be found from the singular integral equation

$$\int_{-1}^1 \frac{\mu(\xi)[e^*(\eta) + e(\xi)]}{(s - \kappa)\sqrt{1-s^2}} ds = 0, \quad (22)$$

with the normalization condition:

$$\int_{-1}^1 \frac{\mu(\xi)}{\sqrt{1-s^2}} ds = \pi,$$

where $\xi = \chi + ts$, $\eta = \chi + t\kappa$, with $s, \kappa \in [-1, 1]$.

Notice that, for this method, no condition is imposed on the behavior of $e(\chi)$. This is in accordance with the result obtained concerning about the AVTD behavior of the Ernst potential near the singularity. Once $e(\chi)$ is given in any desired form, one need only calculate the integral (21) to find the Ernst potential. However, to calculate this integral, one must first find the function $\mu(\xi)$ by means of the singular equation (22) and the normalization condition. In practice, for a given $e(\xi)$, one must make a reasonable ansatz for $\mu(\xi)$ that permits solutions to the integral singular equation (22).

5. Example of Gowdy T^3 model

The cases where the Ernst potential at the initial singularity behaves as a rational function are relatively easy to analyze [11, 18]. In this section we shall present an example. Let us consider the following simple example of an Ernst potential at the singularity

$$e(\chi) = \frac{\chi_0 - \chi}{\chi_0 + \chi},$$

where χ_0 is a real constant. The first step in the construction is to find the unknown function μ according to Eq. (22) and the normalization condition. A reasonable ansatz is again a rational function [17]

$$\mu = A_0 + \frac{A_1}{\xi - \xi_1}, \quad (23)$$

where ξ_1 is the root of the equation $e(\xi) + \tilde{e}(\xi) = 0$ (in this case $\xi_1 = \chi_0$) and A_0 , A_1 are functions of t and χ .

Introducing μ into the normalization condition and into the integral singular equation (22) we obtain

$$A_0 + \frac{A_1}{r_-} = 1; \quad A_0 - \frac{A_1(r_+ + r_-)}{2\chi_0 r_-} = 0, \quad (24)$$

where $r_- = \sqrt{(\chi - \chi_0)^2 - t^2}$. The last two equations can be used to find the explicit values of A_0 and A_1 which can then be replaced in the result of the integration of Eq. (21) and yield

$$E(t, \chi) = -A_0 - \frac{A_1 - 2\chi_0 A_0}{r_+} = \frac{2\chi_0 - r_+ - r_-}{2\chi_0 + r_+ + r_-}, \quad (25)$$

where $r_+ = \sqrt{(\chi + \chi_0)^2 - t^2}$. It is easy to check that this is indeed a solution to the Ernst equation (15). Since the resulting Ernst potential is real, the solution corresponds to a polarized ($Q = 0$) Gowdy model [1, 2]. The expression for metric function P can be easily obtained from the definition (14) and Eq. (25), and the remaining function λ can be calculated (up to an additive constant) by quadratures from Eq. (16) and (17):

$$\lambda = \ln \left[\frac{1}{t} \frac{(r_+ r_-)^2}{(r_+ + r_- + 2\chi_0)^4} \right].$$

The physical significance of this solution becomes plausible in a different system of coordinates x and y which we introduce in two steps. Let us first introduce in the (τ, χ) -sector of the line element (6) by means of the relationships: $e^{-2\tau} = t^2 = \chi_0^2(1 - x^2)(1 - y^2)$, $\chi = \chi_0 xy$, or the inverse transformation law

$$x = \frac{r_+ + r_-}{2\chi_0}, \quad y = \frac{r_+ - r_-}{2\chi_0},$$

so that the metric functions become

$$P = \ln \left[\frac{1 - x}{\chi_0 \sqrt{(1 - x^2)(1 - y^2)(1 + x)}} \right]$$

$$\lambda = \ln \left[\frac{(x^2 - y^2)^2}{\chi_0 \sqrt{(1 - x^2)(1 - y^2)(1 + x)^4}} \right]. \quad (26)$$

The second transformation now affects all the sectors of line element (6) and is defined by

$$x = \frac{T}{\chi_0} - 1, \quad y = \cos \theta, \quad \sigma = r \delta = \phi.$$

Then, after some algebraic manipulations, the metric can be written as

$$-ds^2 = - \left(\frac{2\chi_0}{T} - 1 \right)^{-1} dT^2 + \left(\frac{2\chi_0}{T} - 1 \right) dr^2 + T^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (27)$$

an expression that can immediately be recognized as the Kantowski-Sachs cosmological model [19, 20]. Thus, we have shown that the Kantowski-Sachs metric can be constructed from the value of its Ernst potential at the singularity.

6. Conclusions

We have shown that it is possible to generate a Gowdy T^3 cosmological model starting from the data near the initial singularity. To this end, we first show that the Gowdy T^3 line element can be obtained from the line element of stationary axisymmetric solutions by means of a complex transformation that involves the metric functions and the coordinates. The behavior of stationary axisymmetric solutions at the axis of symmetry is shown to be mathematically equivalent to the behavior of Gowdy T^3 models near the singularity. In particular, we have derived the AVTD solution from its stationary axisymmetric counterpart. We then use the Ernst representation of the field equations and apply Sibgatullin's method to the Ernst potential, which can be given at the singularity as any arbitrary function of the angle coordinate χ . In particular, we have shown that the Kantowski-Sachs cosmological model can be derived in this manner by starting from a specific form of the Ernst potential in terms of a rational function.

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