# An introduction to the brane world

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The study of the so called brane world models has introduced completely new ways of looking at standard problems in many areas of theoretical physics. Inspired in the recent developments of string theory, the Brane World picture involves the introduction of new extra dimensions beyond the four we see, which could either be compact or even open (infinite). The sole existence of those new dimensions may have non-trivial observable effects in short distance gravity experiments, as well as in our understanding of the cosmology of the early Universe, among many other issues. The goal of the present notes is to provide a short introduction to Brane World models, to their motivations and consequences. We cover some of their basic aspects. The discussion includes models with flat compact extra dimensions, as well as the so-called Randall-Sundrum models.

*Keywords:* Brane world; extra dimensions; Randall-Sundrum models.

En años recientes, el avance en el estudio de los llamados Mundos Brana ha provisto de nuevas formas de mirar viejos problemas de la física teórica. Los modelos de mundo brana se inspiran en desarrollos recientes de la teoría de cuerdas (la teoria de branas), y suponen la existencia de nuevas dimensiones espaciales más alla de las cuatro que vemos cotidianamente, las cuales pueden ser tanto compactas como infinitas. La existencia de dimensiones extras podría ser identificable en el comportamiento de la interacción gravitacional a pequeñas distancias. Su presencia puede, además, impactar de manera importante nuestra actual comprensión del Universo temprano y su cosmología, entre otras cosas. A lo largo de las presentes notas, cuyo interés es servir como una introducción breve al estudio de los mundos brana, revisamos algunos aspectos de estos modelos. Se discuten tanto los modelos con dimensiones extra planas y compactas, como los llamados modelos de Randall-Sundrum.

*Descriptores:* Mundos Brana; dimensiones extra; cosmología.

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## 1. Introduction

In the course of the last five years there has been considerable activity in the study of models that involve new, extra dimensions. The possible existence of such dimensions has got strong motivation from theories that try to incorporate gravity and (gauge) interactions in a unique scheme, in a reliable manner. The idea dates back to the 1920's to the works of Kaluza and Klein [1], who tried to unify electromagnetism with Einstein gravity by assuming that the photon originates from the fifth component of the metric. With the advent of string theory, the idea has gained support since all versions of string theory are naturally and consistently formulated only in a space-time of more than four dimensions (actually 10D, or 11D for M-theory). Until recently, however, it was conventional to assume that such extra dimensions were compactified to manifolds of small radii with sizes of the order of the inverse Planck scale, $\ell_P = M_P^{-1} = G_N^{1/2} \sim 10^{-33}$ cm. or so, such that they would remain hidden to the experiment. It was only during the last years of the 20th century when people started to ask the question of how large these extra dimensions could be without coming into conflict with observations, and even more interesting, where and how these extra dimensions could manifest themselves. The intriguing answer to the first question points towards the possibility that extra dimensions as large as millimeters [2] could exist and yet remain hidden to the experiments [3–7]. To allow that, however, matter should be localized on a hypersurface (the brane) embedded in a higher dimensional world (the bulk). Again, the main motivation for these models comes from string theories where the Horava-Witten solution [8] of the non perturbative regime of the $E_8 \times E_8$ string theory provided one of the first models of this kind. To answer the second question many phenomenological studies have been done in a truly bottom-up approach, often based on simplified field theoretical models, trying to provide new insights to the possible implications of the fundamental theory at the observable level, although it is unclear whether any of those models are realized in nature. Nevertheless, they may help to find the way to search for extra dimensions, if there are any.

It is fair to say that similar ideas were proposed in the 80's by several authors [9]; nevertheless, they were missed for some time, until recent developments on string theory, basically the rise of M-theory, provided an independent realization to such models [8, 10–12], given them certain credibility.

It is the goal of the present notes to provide a brief introduction for the beginner to the general aspects of theories with extra dimensions. Many variants of the very first model by Arkani-Hammed, Dimopoulos and D'vali [2] have been proposed over the years, and there is no way we could comment on all those results in a short review such as the present one. We shall rather concentrate on some of the most general characteristics shared by these models. With particular interest, we shall address dimensional reduction, which provides the effective four dimensional theory on which most calculations are actually based. The determination of the effective gravity coupling, and the also effective gravitational potential in four dimensions will be discussed in the notes. We will also cover some aspects of the cosmology of models in more than four dimensions. Of particular interest in our discussions are the models of Randall and Sundrum [13–16] for warped backgrounds, with compact or even infinite extra dimensions. We will show in detail how these solutions arise, as well as how gravity behaves in such theories. Some further ideas include: why and how the size of the compact extra dimensions remain stable; graviton localization at branes; and brane cosmology are also covered. The interested reader that would like to go beyond the present notes can consult any of the excellent reviews that are now in the literature, some of which are given in Ref. 17.

## 2. General Aspects: Flat Extra Dimensions

### 2.1. Planck versus the Fundamental Gravity Scale

The possible existence of more than four dimensions in nature, even if they were small, may not be completely harmless, and in principle, they could have some visible manifestations in our (now effective) four-dimensional world. To look for such signals, one has first to understand how the effective four dimensional theory arises from the higher dimensional one. Formally, this can be achieved by dimensionally reducing the complete theory, a concept that we shall discuss further in the following section. One of the first things we should notice is that since gravity is a geometric property of space, in a higher dimensional world, where Einstein gravity is assumed to hold, the gravitational coupling does not necessarily coincide with the well-known Newton constant $G_N$, which is, nevertheless, the gravity coupling we do observe. To explain this more clearly, let us assume as in Ref. 2 that there are $\delta$ extra space-like dimension which are compactified into circles of the same radius R (so the space is factorized as an $\mathcal{M}_4 \times T^\delta$ manifold). We will call the fundamental gravity coupling $G_*$, and then write down the higher dimensional gravity action:

$$S_{grav} = -\frac{1}{16\pi G_*}\int d^{4+\delta}x\,\sqrt{|g_{(4+\delta)}|}\,R_{(4+\delta)}\ , \tag{1}$$

where $g_{(4+\delta)}$ stands for the metric in the whole $(4+\delta)$D space,

$$ds^2 = g_{MN}dx^M dx^N\ , \tag{2}$$

for which we will always use the $(+,-,-,-,\dots)$ sign convention, and $M,N = 0,1,\dots,\delta+3$. The above action must have the proper dimensions, meaning that the extra length dimensions that come from the extra volume integration must be equilibrated by the dimensions on the gravity coupling. Notice that in natural units, $c=\hbar=1$, $S$ has no dimensions. We are also assuming for simplicity that $g_{(4+\delta)}$ is taken to be dimensionless, so $[R_{(4+\delta)}] = [\text{length}]^{-2}$ or $[\text{Energy}]^2$ in natural units.

Now, in order to extract the four dimensional gravity action, let us assume that the extra dimensions are flat; thus, the metric has the form

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu - \delta_{ab}dy^a dy^b, \tag{3}$$

where $g_{\mu\nu}$ gives the four-dimensional part of the metric which depends only on the four-dimensional coordinates $x^\mu$, for $\mu = 0,1,2,3$; and $\delta_{ab}dy^a dy^b$ gives the line element on the torus, whose coordinates are parameterized by $y^a$, $a=1,\dots,\delta$. It is now easy to see that $\sqrt{|g_{(4+\delta)}|} = \sqrt{|g_{(4)}|}$ and $R_{(4+\delta)} = R_{(4)}$, so that one can integrate over the extra dimensions in Eq. (1) to obtain the effective action

$$S_{grav} = -\frac{V_\delta}{16\pi G_*}\int d^4x\,\sqrt{|g_{(4)}|}\,R_{(4)}\ , \tag{4}$$

where $V_\delta$ stands for the volume of the extra space, for the torus $V = (2\pi R)^\delta$. This last equation is precisely the standard gravity action in 4D if one makes the identification

$$G_N = G_*/V_\delta\ . \tag{5}$$

The newton constant is therefore given by a volumetric scaling of the truly fundamental gravity scale. Thus, $G_N$ is in fact an effective quantity. Notice that, even if $G_*$ were a large coupling, one can still understand the smallness of $G_N$ via the volumetric suppression.

To obtain a more physical meaning of these observations, let us consider a simple experiment. Let us assume a couple of particles of masses $m_1$ and $m_2$, respectively, located on the hypersurface $y^a = 0$, and separated from each other by a distance $r$. The gravitational flux between these two particles would spread over the entire $(4+\delta)$ D space; however, since the extra dimensions are compact, the effective strength of the gravity interaction would have two clear limits:

(i) If the two test particles are separated by a distance $r \gg R$, the torus would effectively disappear for the four-dimensional observer; the gravitational flux is then diluted by the extra volume and the observer would see the usual (weak) 4D gravitational potential

$$U_N(r) = -G_N\frac{m_1 m_2}{r}\ . \tag{6}$$

(ii) However, if $r \ll R$, the 4D observer would be able to feel the presence of the bulk through the missing flux that goes into the extra space, and thus, the potential between each particle would appear to be stronger:

$$U_*(r) = -G_*\frac{m_1 m_2}{r^{\delta+1}}. \tag{7}$$

It is precisely the volumetric factor which matches both regimes of the theory. The change in the short distance behavior of Newton's gravity law should be observable in the experiment when measuring $U(r)$ for distances less than $R$. The current search for such deviations has reached as low as 200 microns, with no signs of extra dimensions so far [3].

We should now recall that the Planck scale, $M_P$, usually assumed to be the fundamental energy scale typically associated with the scale at which quantum gravity (or string theory) should make itself manifest, is defined in terms of the Newton constant, via

$$M_P c^2 = \left[\frac{\hbar c^5}{8\pi G_N}\right]^{1/2} \sim 2.4 \times 10^{18}\ \mathrm{GeV}\ . \quad (8)$$

In the present picture, it is clear then that $M_P$ is not fundamental anymore. The true scale for quantum gravity should be given in terms of $G_*$ instead. We then define the string scale as

$$M_* c^2 = \left[\frac{\hbar^{1+\delta} c^{5+\delta}}{8\pi G_*}\right]^{1/(2+\delta)} . \quad (9)$$

Switching to natural units ($c = \hbar = 1$) from here on, both scales are then related to each other by [2]

$$M_P^2 = M^{\delta+2} V_\delta\ . \quad (10)$$

From the particle physics world we already know that there is no evidence of quantum gravity (either supersymmetry, or string effects) well up to energies around one hundred GeV, which means that $M_* \geq 1$ TeV. If the volume were large enough, then the fundamental scale could be as low as the electroweak scale, and there would be no hierarchy in the fundamental scales of physics, which so far has been considered a puzzle. Of course, the price of solving the hierarchy problem this way would be now to explain why the extra dimensions are so large. Using $V \sim R^\delta$, one can reverse the above relation and get a feeling of the possible values of $R$ for a given $M_*$. This is done precisely for our above-mentioned wish for having the quantum gravity scale as low as possible, although the actual value is unknown. As an example, if one takes $M_*$ to be 1 TeV then, for $\delta = 1$, $R$ turns out to be about the size of the solar system ($R \sim 10^{11}$ m)!, whereas for $\delta = 2$ one gets $R \sim 0.2$ mm, that is, just at the current limit of the experiments. More than two extra dimensions are in fact expected (strings predict six more), but in the final theory these dimensions may turn out to have different sizes, or even geometries. More complex scenarios with a hierarchical distribution of the sizes could be natural. To have an insight into the theory, however, one usually relies on toy models with a single extra dimension, compactified into circles or orbifolds.

## 2.2. Brane World theory prescriptions

While submillimeter dimensions remain untested for gravity, particle physics forces have certainly been accurately measured up to weak scale distances (about $10^{-18}$ cm). Therefore, the matter particles cannot freely propagate in those large extra dimensions, but must be constrained to live in a four-dimensional submanifold. Then the scenario we have in mind is one where we live in a four-dimensional surface embedded in a higher dimensional space. Such a surface shall be called a "brane" (a short name for membrane). This picture is similar to the D-brane models [12], as in the Horava-Witten theory [8]. We may also imagine our world as a domain wall of size $M_*^{-1}$, where the particle fields are trapped by some dynamical mechanism [2]. A hypersurface or brane would then be located at a specific point on the extra space, usually, at the fixed points of the compact manifold. Clearly, this picture violates translational invariance, which may be reflected in two ways in the physics of the model, affecting the flatness of the extra space (which compensates for the required flatness of the brane), and introducing a source of violation of the extra linear momentum. The first would drive us to the Randall-Sundrum Models, that we shall discuss latter on. The second will be a constant issue throughowt our discussions.

What we have called a brane in our previous paragraph is actually an effective theory description. We have chosen to think of them as topological defects (domain walls) of almost zero width, which could have fields localized on its surface. String theory D-branes (Dirichlet branes) are, however, surfaces where an open string can end on. Open strings give rise to all kinds of fields localized on the brane, including gauge fields. In the supergravity approximation, these D-branes will also appear as solitons of the supergravity equations of motion. In our approach, we shall care little about where these branes come from, and rather simply assume there is some consistent high-energy theory that would give rise to these objects, and which should appear at the fundamental scale $M_*$. Thus the natural cutoff of our models would always be given by the quantum gravity scale.

D-branes are usually characterized by the number of spatial dimensions on the surface. Hence, a p-brane is described by a flat space time with p space-like and one time-like coordinates. Unless otherwise stated, we shall always work with models of 3-branes. We need to be able to describe theories that live both in the brane (as the Standard Model) and in the bulk (like gravity), as well as the possible interactions among these two theories. To do so we use the following prescriptions:

(i) Bulk theories are, as usual, described by the higher dimensional action, defined in terms of a Lagrangian density of $\phi(x, y)$ fields valued on all space-time coordinates of the bulk

$$S_{\mathrm{bulk}}[\phi] = \int d^4x\, d^\delta y\, \sqrt{|g_{(4+\delta)}|}\mathcal{L}(\phi(x,y))\ , \quad (11)$$

where, as before, $x$ stands for the (3+1) coordinates of the brane and $y$ for the $\delta$ extra dimensions.

(ii) Brane theories are described by the (3+1)D action of the brane fields, $\varphi$, which is naturally promoted to a higher dimensional expression by the use of a delta density:

$$S_{\mathrm{brane}}[\varphi] = \int d^4x d^\delta y \sqrt{|g_{(4)}|}\mathcal{L}(\varphi(x))\delta^\delta(\vec{y}-\vec{y}_0), \quad (12)$$

where we have taken the brane to be located at the position $\vec{y} = \vec{y}_0$ along the extra dimensions, and the metric $g_{(4)}$ stands for the 4D induced metric on the brane.

(iii) Finally, the action may contain terms that couple bulk to brane fields. The latter are localized on the space, thus, it is natural that a delta density would be involved in such terms, say for instance

$$\propto \int d^4x\, d^\delta y\, \sqrt{|g_{(4+\delta)}|}\, \phi^2(x,y)\, \varphi(x)\, \delta^\delta(\vec{y}-\vec{y}_0)$$
$$= \int d^4x\, \sqrt{|g_{(4)}|}\phi^2(x,0), \varphi(x)\ . \tag{13}$$

## 2.3. Dimensional reduction: Kaluza-Klein Decomposition

The presence of delta functions in the previous actions does not allow for a transparent interpretation, nor for an easy reading of the theory dynamics. When they are present it is more useful to work in the effective four-dimensional theory which is obtained after integrating over the extra dimensions. This procedure is generically called dimensional reduction. It also helps to identify the low energy limit of the theory (where the extra dimensions are not visible).

To get some insight into what the effective 4D theory looks like, let us consider a simplified five-dimensional toy model where the fifth dimension has been compactified on a circle of radius $R$. The generalization of these results to more dimensions would be straightforward. Let $\phi$ be a bulk scalar field for which the action on flat space time has the form

$$S[\phi] = \frac{1}{2}\int d^4x\ dy\ \left(\partial^A\phi\partial_A\phi - m^2\phi^2\right)\ ; \tag{14}$$

where now $A = 1,\ldots,5$, and $y$ denotes the fifth dimension. The compactness of the internal manifold is reflected in the periodicity of the field, $\phi(y) = \phi(y+2\pi R)$, which allows for a Fourier expansion of the field as

$$\phi(x,y) = \frac{1}{\sqrt{2\pi R}}\phi_0(x) + \sum_{n=1}^{\infty}\frac{1}{\sqrt{\pi R}}\left[\phi_n(x)\cos\left(\frac{ny}{R}\right)\right.$$
$$\left. + \hat{\phi}_n(x)\sin\left(\frac{ny}{R}\right)\right]. \tag{15}$$

The very first term, $\phi_0$, with no dependence on the fifth dimension, is usually referred to as the zero mode. Other Fourier modes, $\phi_n$ and $\hat{\phi}_n$; are called the excited or Kaluza-Klein (KK) modes of the field. Notice the different normalization on all the excited modes, $\phi_n$ and $\hat{\phi}_n$, with respect to the zero mode.

By introducing the last expansion into the action and integrating over the extra dimension, one obtains

$$S[\phi] = \sum_{n=0}^{\infty}\frac{1}{2}\int d^4x\left(\partial^\mu\phi_n\partial_\mu\phi_n - m_n^2\phi_n^2\right)$$
$$+\sum_{n=1}^{\infty}\frac{1}{2}\int d^4x\left(\partial^\mu\hat{\phi}_n\partial_\mu\hat{\phi}_n - m_n^2\hat{\phi}_n^2\right), \tag{16}$$

where the KK mass is given as $m_n^2 = m^2 + n^2/R^2$. Therefore, in the effective theory, the higher dimensional field appears as an infinite tower of fields with masses $m_n$.

Notice that all excited modes are fields with the same spin, and quantum numbers as $\phi$. But they differ in the KK number $n$, which is also associated with the fifth component of the momentum. From a formal point of view, KK modes are only a manifestation of the discretization of the (otherwise continuum) extra momentum of the particle. We would see particles with a different higher dimensional momentum as having different masses. This can also be understood from the higher dimensional invariant $p^A p_A = m^2$, which can be rewritten as the effective four-dimensional squared momentum invariant $p^\mu p_\mu = m^2 + \vec{p}_\perp{}^2$, where $\vec{p}_\perp$ stands for the extra momentum components.

Dimensionally reducing any higher dimensional field theory would indeed give a similar spectrum for each particle. For $m = 0$, it is clear that, for energies below $1/R$, only the massless zero mode will be kinematically accessible, making the theory looking four-dimensional. The appreciation of the impact of KK excitations thus depends on the relevant energy of the experiment, and on the compactification scale $1/R$:

(i) For energies $E \ll 1/R$, physics would behave purely four dimensionally.

(ii) At larger energies, $1/R < E < M_*$, or equivalently as we do measurements at shorter distances, a large number of KK excitations, $\sim (ER)^\delta$, become kinematically accessible, and their contributions relevant for physics. Therefore, right above the threshold of the first excited level, the manifestation of the KK modes will start showing the higher dimensional nature of the theory.

(iii) At energies above $M_*$, however, our effective approach must be replaced by the use of the fundamental theory that describes quantum gravity phenomena.

Furthermore, notice that the five dimensional field $\phi$ we considered before has mass dimension $3/2$, in natural units. This can be easily see from the kinetic part of the Lagrangian, which involves two partial derivatives, each with mass dimension one, and the fact that the action should be dimensionless. In contrast, by similar arguments, all excited modes have mass dimension one, which is consistent with the KK expansion (15). In general, for $\delta$ extra dimensions we get the mass dimension for an arbitrary field to be $[\phi] = d_4 + \delta/2$, where $d_4$ is the natural mass dimension of $\phi$ in four dimensions. Because this change in the dimensionality of $\phi$, most interaction terms on the Lagrangian (apart from the mass term) would all have dimensionful couplings. To keep them dimensionless, a mass parameter should be introduced to correct the dimensions. It is common to use as the natural choice for this parameter the cut-off of the theory, $M_*$. For instance, let us consider the quartic couplings of $\phi$ in 5D. Since all potential terms should be of dimension five, we should write down $(\lambda/M_*)\phi^4$. After integrating the fifth dimension, this operator will generate quartic couplings among all KK modes. Four normalization factors containing $1/\sqrt{R}$ appear in the expansion of $\phi^4$. Two of them will be removed by the integration; thus, we are left with the effective coupling $\lambda/M_*R$. By introducing Eq. (10), we observe that the effective couplings have the form

$$\lambda\left(\frac{M_*}{M_P}\right)^2\phi_k\phi_l\phi_m\phi_{k+l+m}, \tag{17}$$

where the indices are arranged to respect the conservation of the fifth momentum. From the last expression, we conclude that, in the low energy theory ($E < M_*$), even at the zero mode level, the effective coupling appears suppressed with respect to the bulk theory. Therefore, the effective four-dimensional theory would be weaker interacting compared to the bulk theory. Let us recall that same happened to gravity on the bulk, where the coupling constant is stronger than the effective 4D coupling, due to the volume suppression given in Eq. (5), or equivalently in Eq. (10). Similar arguments apply in general for the brane-bulk couplings. We shall use these facts when considering gravity interactions more carefully for test particles on the brane, which we shall do throughout the next section.

Different compactifications would lead to different mode expansions. Eq.(15), would had to be chosen according to the geometry of the extra space, by typically using wave functions for free particles on this space as the basis for the expansion. Extra boundary conditions associated with specific topological properties of the compact space may also help for a proper selection of the basis. A useful example is the one dimensional orbifold, $U(1)/Z_2$, which is built out of the circle, by identifying the opposite points around zero. The operations can be seen as reducing the interval of the original circle to $[0,\pi]$ only. Operatively, this is done by requiring the theory to be invariant under the extra parity symmetry $Z_2{:}y \to -y$. Under these symmetries, all fields should pick up a specific parity. Even (odd) fields would then be expanded into only cosine (sine) modes. Thus, odd fields do not appear at the zero mode level of the theory, which also means that the orbifolding projects half of the modes out of the KK expansion.

### 2.4. Graviton couplings and the effective gravity interaction law

One of the first physical examples of a brane-bulk interaction one may be interested in analyzing with some care is the effective gravitational coupling of particles located on the brane, which needs to understand the way gravitons couple to brane fields. The problem has been extensively discussed by Giudice, Ratazzi and Wells [18] and independently by Han, Lykken and Zhang [19], assuming a flat bulk. Here we summarize some of the main points. Starting from the action that describes a particle on the brane

$$S = \int d^4x\sqrt{|g(y^a=0)|}\,\mathcal{L}\,, \tag{18}$$

where the induced metric $g(y^a = 0)$ now includes the small metric fluctuations $h_{M,N}$ over the flat space, which are also called the graviton, such that

$$g_{M,N} = \eta_{M,N} + \frac{1}{2M_*^{\delta/2+1}}h_{M,N}. \tag{19}$$

The source of those fluctuations are of course the energy on the brane, *i.e.* the matter energy momentum tensor $\sqrt{g}T^{\mu\nu} = \delta S/\delta g_{\mu\nu}$, that apperars on the RHS of the Einstein equations:

$$R_{M,N} - \frac{1}{2}R_{(4+\delta)}g_{M,N} = -\frac{1}{M_*^{2+\delta}}T_{\mu\nu}\eta^\mu_M\eta^\nu_N\delta^{(\delta)}(y)\ .$$

The effective linearized coupling of matter to graviton fields is then described by the action

$$S_{int} = \int d^4x\frac{h_{\mu\nu}}{M_*^n/2+1}T^{\mu\nu}. \tag{20}$$

It is clear that, from the effective four-dimensional point of view, the fluctuations $h_{M,N}$ would have different 4D Lorentz components.

(i) $h_{\mu\nu}$ clearly contain 4D Lorentz tensors, the true, actual four-dimensional gravitons.

(ii) $h_{a\mu}$ behaves as a vector, the graviphotons.

(iii) Finally, $h_{ab}$ behaves as a group of scalars (graviscalar fields), one of which corresponds to the partial trace of $h$ ($h^a_a$) that we shall later call the radion field.

To count the number of degrees of freedom in $h_{M,N}$ we should first note that $h$ is an $n \times n$ a symmetric tensor, for $n = 4+\delta$. Next, general coordinate invariance of general relativity can be translated into $2n$ independent gauge fixing conditions, half usually chosen as the harmonic gauge $\partial_M h^M_N = (1/2)\partial_N h^M_M$. In all, there are $n(n-3)/2$ independent degrees of freedom. Clearly, for $n = 4$, one has the usual two helicity states of a massless spin-two particle.

All those effective fields would of course have a KK decomposition,

$$h_{MN}(x,y) = \sum_{\vec{n}}\frac{h^{(\vec{n})}_{MN}(x)}{\sqrt{V_\delta}}\ e^{i\vec{n}\cdot\vec{y}/R}\ . \tag{21}$$

Here $\vec{n} = (n_1,\ldots,n_\delta)$, with all $n_a$ integer numbers. Once we insert the above expansion back into $S_{int}$, it is not hard to see that the volume suppression will exchange the $M_*$ by an $M_P$ suppression in the effective interaction Lagrangian of a single KK mode. Therefore, all modes couple with the strength of standard gravity. Briefly, only the 4D gravitons, $G_{\mu\nu}$ and the radion field $\sigma$, are coupled at the first order level to the brane energy momentum tensor [18, 19]

$$\mathcal{L} = -\frac{1}{M_P}\sum_{\vec{n}}\left[G^{(\vec{n})\mu\nu} - \frac{\kappa}{3}\sigma^{(\vec{n})}\eta^{\mu\nu}\right]T_{\mu\nu}. \tag{22}$$

Here, $\kappa$ is a parameter of order one. Notice that $G^{(0)\mu\nu}$ is massless since the higher dimensional graviton $h_{MN}$ has no mass itself. That is the source of long range, four-dimensional gravity interactions. It is worth remarking that on the contrary, $\sigma^{(0)}$ would not be massless, otherwise it should violate the equivalence principle, since it would mean a long-range scalar (gravitational) interaction also. $\sigma^{(0)}$ should get a mass from the stabilization mechanism that keeps the extra volume finite. We shall come back to this problem later on. From supernova constraints, such a mass should be larger than $10^{-3}$ eV [6].

Above Lagrangian runs over all KK levels, meaning that brane particles can release any kind of KK gravitons into the bulk. KK index $\vec{n}$ is also the extra component of the momentum, so they leave the brane, taking its energy away, in a clear violation of the 4D conservation of energy. This could appear in the future high-energy collider experiments, for instance, as missing energy [5, 7].

We started the section asking for the actual form of the effective gravitation interaction among particles on the brane. Now that we know how gravitons couple to brane matter, we can use this effective field theory point of view to calculate what the effective gravitational interaction law should be. KK gravitons are indeed massive, thus, the interaction mediated by them is short-range. More precisely, each KK mode contribute to the gravitational potential between two test particles of masses $m_1$ and $m_2$ located on the brane, separated by a distance $r$, with a Yukawa potential

$$\Delta_{\vec{n}} U(r) \simeq -G_N \frac{m_1 m_2}{r} e^{-m_{\vec{n}} r} = U_N(r) e^{-m_{\vec{n}} r} . \qquad (23)$$

The total contribution of all KK modes, the sum over all KK masses $m_{\vec{n}}^2 = \vec{n}^2/R$, can be estimated by the continuum KK modes limit, to get

$$U_T(r) \simeq -G_N V_\delta (\delta - 1)! \frac{m_1 m_2}{r^\delta + 1} \simeq U_*(r) . \qquad (24)$$

Experimentally, however, for $r$ just below $R$, only the very first excited modes would be relevant, and so, the potential one would see in short distance tests of Newton's law [3] should rather be of the form

$$U(r) \simeq U_N(r) \left(1 + \alpha e^{-r/R}\right) , \qquad (25)$$

where $\alpha$ is to account for the multiplicity of the very first excited level.

## 3. Cosmology in models with flat extra dimensions

### 3.1. Limits on Reheating Temperature due to Graviton emission

Graviton production by brane processes may not be such a harmless phenomenon. It may rather possess strong constraints on the theory when considering that the early Universe was an important resource of energy, which in the present picture lies completely on the brane. How much of this energy could have gone into the bulk without affecting cosmological evolution? For large extra dimensions, the splitting between two excited modes is pretty small, $1/R$. For $\delta = 2$ and $M_*$ at the TeV scale this means a splitting of just about $10^{-3}$ eV! For a process where the center mass energy is E, up to $N = (ER)^\delta$ KK modes would be kinematically accessible. During Big Bang Nucleosynthesis (BBN), for instance, where $E$ was about a few MeV, this already means more than $10^{18}$ modes. So many modes may be troublesome, and one has to ask the question how hot the Universe could go without losing too much energy. By looking at the effective Lagrangian in Eq. (22), one can immediately notice that the graviton creation rate, per unit time and volume, from brane thermal processes at temperature $T$ is

$$\sigma_{total} = \frac{(TR)^\delta}{M_P^2} = \frac{T^\delta}{M_*^{\delta+2}} .$$

The standard Universe evolution would be conserved, as far as the total number density of KK gravitons produced remains small when compared to photon number density. This is a sufficient condition that however can be translated into a bound for the reheating energy, since the hotter the medium the more gravitons can be excited. It is not hard to see that this condition implies [2]

$$\frac{n_g}{n_\gamma} \approx \frac{T^{\delta+1} M_P}{M_*^{\delta+2}} < 1 . \qquad (26)$$

Equivalently, the maximun temperature our Universe could reach without producing to many gravitons must satisfy

$$T_r^{\delta+1} < \frac{M_*^{\delta+2}}{M_P} . \qquad (27)$$

To give numbers, consider for instance $M_* = 10$ TeV and $\delta = 2$, which means $T_r < 100$ MeV, just about to what is needed to have BBN working. The brane Universe with large extra dimensions is then rather cold. This would be reflected in some difficulties for those models trying to implement baryogenesis or leptogenesis based on electroweak energy physics or higher.

As a complementary note, we should mention that, since thermal graviton emission is not restricted to the early Universe, one can expect this to be happening in any other environment. We have already mentioned colliders as an example. But even the hot astrophysical objects can be sources of gravitons. Gravitons emitted by stellar objects take away energy, which contribute to cooling the star. The stringent bounds on $M_*$ actually come from the study of this process [6].

### 3.2. Dimensional reduction and the radion field

We are now ready to postulate the model that should describe the brane Universe evolution where the 4D Friedmann-Robertson-Walker model must now be obtained as the effective zero limit, after dimensional reduction. To simplify the discussion, we will assume the extra space compactified into an orbifolded torus, $T^\delta/Z_2$, such that the extra coordinates $y^a$ take on values in the interval $[0,1]$, and the theory is invariant under the mapping $Z_2{:}\vec{y} \to -\vec{y}$. As before, the brane should be located at $y^a = 0$. Consider the metric in $4+\delta$ dimensions parameterized by

$$ds^2 = g_{AB}dx^A dx^B = g_{\mu\nu}dx^\mu dx^\nu - h_{ab}dy^a dy^b. \quad (28)$$

Note that since we have taken $y^a$ to be dimensionless, $h_{ab}$ has length dimension two. Also note that we are not considering in our parameterization the presence of vector-like connection $A^a_\mu$ pieces, which are common in Kaluza Klein theories. This is because we would only be interested in the zero mode part of the metric, and $A^a_\mu$ is odd under $Z_2$ parity transformation, and therefore it vanishes at the zero mode level.

Next, let us reduce the Einstein-Hilbert action

$$S = \frac{1}{2k_*^2}\int d^4x\, d^\delta y\, \sqrt{|g_{(4+\delta)}|}\, R_{(4+\delta)} \quad (29)$$

to four dimensions, considering only the zero mode level. Here, $1/k_*^2 = M_*^{2+d}$. One then obtains

$$S = \frac{1}{2k^2}\int d^4x\, \sqrt{-g_{(4)}}\, \frac{\sqrt{h}}{V_\delta}\left\{R_{(4)} - \frac{1}{4}\partial_\mu h^{ab}\,\partial^\mu h_{ab} - \frac{1}{4}h^{ab}\partial_\mu h_{ab}\cdot h^{cd}\partial^\mu h_{cd}\right\}, \quad (30)$$

where $1/k^2 = M_p^2$. Here $V_\delta$ stands for the stable volume of the extra space that corresponds to relationship (10).

In order to obtain the $4$ dimensional scalar curvature term in canonical form, we need to perform a conformal transformation on the metric,

$$g_{\mu\nu} \to e^{2\varphi}g_{\mu\nu}\,, \quad (31)$$

designed to cancel the extra $\sqrt{h}/V_\delta$ coefficient of $R_{(4)}$ in Eq. (30). We take $\varphi$ such that

$$e^{2\varphi}\sqrt{h}/V_\delta = 1\,. \quad (32)$$

The action in Eq. (30) is then transformed into

$$S = \frac{1}{2k^2}\int d^4x\, \sqrt{-g_{(4)}}\left\{R_{(4)} - \frac{1}{4}\partial_\mu h^{ab}\,\partial^\mu h_{ab} + \frac{1}{8}h^{ab}\partial_\mu h_{ab}\cdot h^{cd}\partial^\mu h_{cd}\right\}. \quad (33)$$

Next, for the four-dimensional part of the metric, $g_{\mu\nu}$, we can now assume the standard Friedmann-Robertson-Walker (FRW) metric with a flat geometry, *i.e.*

$$g_{\mu\nu} = diag(1, -a(t), -a(t), -a(t))\,, \quad (34)$$

for an isotropic and homogeneous (brane) Universe, whereas we consider a diagonal form for the $h$ part of the metric:

$$h_{ab} = b(t)^2\delta_{ab} \quad (35)$$

Obviously, the physical volume of the extra space is dynamical, and given as

$$vol_{\rm phys} = \sqrt{h} = b^\delta(t)\,.$$

If the bulk is stable, meaning that $b \neq b(t)$, the physical size of the extra dimension is given by the identification $b = R$. This turns out to be the stabilized condition, when one assumes the volume to have some dynamics, which should be reached at some given finite time $t$.

The action can be simplified by defining the radion field by

$$\sigma(t) = M_P\sqrt{\frac{\delta(\delta+2)}{2}}\ln\left(\frac{b(t)}{R}\right)\,. \quad (36)$$

This has a straightforward physical interpretation: it is related to the variation of the physical size of the volume. Notice that set to zero when the stabilized volume is reached. In these terms one gets the effective action

$$S{=}\int d^4x\frac{\sqrt{-g_{(4)}}}{2k^2}R_{(4)}{+}\int d^4x\frac{\sqrt{-g_{(4)}}}{2}\left(\partial^\mu\sigma\right)\left(\partial_\mu\sigma\right). \quad (37)$$

The very first term corresponds precisely to the 4D gravity action of the FRW model. On the other hand, last term can be identified as the action of a running mode. It is unstable under perturbations, which means that any small perturbation on the radion field can make the volume of the extra space expand or contract without control. This is what is called the radion stabilization problem, and it is a particular case of the more general moduli problem inherited from string theory. Understanding the stability of the volume of the compact space can be seen as finding the mechanism that provides the force that keeps the radion fixed at its zero value. Thus, one must find the potential $\sigma$ which provides such a force. The origin of this potential is largely unknown so far, although some ideas can be found in the literature, see for instance [20, 21]. We shall not comment on this here, but rather assume that such a potential should exist. As we shall discuss later on, the detailed form of $U(\sigma_v)$ may also be important to understanding the dynamics of the radions during and after inflation. As a parenthetical note, the physical mass of the radion is actually related to the second derivative of the potential at its minimum, and certainly, to avoid violation of the equivalence principle, it should be larger than about $10^{-3}$ eV.

As already mentioned, radion couples to matter fields. This is regardless of whether they are brane or bulk fields. After dimensionally reducing the action in Eq. (11), and including the conformal transformation that we performed on the metric, we get for the scalar field the effective action at the zero mode level

$$S[\phi]{=}\int d^4x\sqrt{-g_{(4)}}\left[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi{-}e^{-\alpha\sigma/M_p}V(\phi)\right], \quad (38)$$

where the coupling constant is given by $\alpha = \sqrt{2/\delta(\delta+2)}$. The case of a brane scalar field turns out to have the same functional form for the effective action as above.

### 3.3. Inflation

It is still possible that, due to some dynamical mechanism, the extra dimension gets stabilized long before the Universe exited from inflation, as in some scenarios in Kaluza-Klein (KK) theories, where the stabilization potential is generated by the Casimir force [20, 21]. Other possible sources for this stabilizing potential could be present in brane-bulk theories; for instance, the formation of the brane at very early times may give rise to vacuum energy that plays a role in eventually stabilizing the extra dimension. Let us for the moment consider stable bulk ($b = R$), and then address the problem of brane cosmological evolution. It is clear from the results of the previous section, by taking $\sigma = 0$, that the brane cosmological theory behaves as four-dimensional. The usual FRW model is therefore a good set up to analyze cosmology on the brane. During the inflation period, Hubble expansion is given as usual by

$$H \sim \sqrt{\frac{V(\phi)}{3M_P^2}} \,, \tag{39}$$

with $V(\phi)$ the potential of the slow rolling inflaton. A brane inflaton, however, is troublesome [22]. Consider for instance a typical chaotic inflation scenario [23], where the potential is simply given by $V(\phi) = (1/2)m^2\phi^2$. If the highest scale in the theory is $M_*$, during inflation, a brane inflaton potential can-not have values larger than $M_*^4$, regardless of the number of extra dimensions, just as in the usual 4D theories where the scale of the potential is not supposed to be larger than Planck scale. Next, since successful inflation (the slow roll condition) requires that the inflaton mass be less than the Hubble parameter, we have the inequality

$$m \leq H \leq M_*^2/M_P. \tag{40}$$

For $M_* \sim 1$ TeV, one then gets the bound $m \leq 10^{-3}$ eV, which is a severe fine tuning constraint on the parameters of the theory. Furthermore, such a light inflaton would certainly face troubles for reheating. Such a light inflaton would only decay into photons. The inflaton is believed to be chargeless, so that, such a decay can only occur via suppressed loop processes. The above constraint further implies that inflation occurs on a time scale $H^{-1}$ much greater than $M_*^{-1}$. As emphasized by Kaloper and Linde [22], this is conceptually very problematic since it requires that the Universe should be large and homogeneous enough from the very beginning so as to survive the large period of time from $t = M_*^{-1}$ to $t = H^{-1}$.

Moreover, for chaotic inflation one gets a tiny contribution to density perturbations

$$\frac{\delta\rho}{\rho} \sim 50\frac{m}{M_P} \leq 10^{-31}. \tag{41}$$

The situation does not improve for more elaborate models. For the case where the $\lambda\phi^4$ term dominates the density, for instance, one gets the same fine tuning condition as in four dimensions, $(\delta\rho/\rho) \sim \lambda^{1/2}$. Assuming Hybrid inflation [24], with the potential

$$V(\phi,\sigma) = \frac{1}{4\lambda}\left(M^2 - \lambda\sigma^2\right)^2 + \frac{1}{2}m^2\phi^2 + g^2\phi^2\sigma^2,$$

does not help either [22], since it needs either a value of $m$ six orders of magnitude smaller or a strong fine tuning on the parameters, to match the COBE result $(\delta\rho/\rho) \sim 10^{-5}$. Certainly, the problem would be relaxed if the fundamental scale $M_*$ were much larger than a few TeV; nevertheless, this means a shorter radius and most of the phenomenological interest in the model would also be gone.

A simple way to solve this problem could be by assuming that the inflaton is the zero mode of a bulk scalar field [25]. As such, its effective potential energy is enhanced by the volume of the extra space, which allows it to have larger densities contributing to the Hubble expansion. Indeed, one now has

$$V_{eff}(\phi_0) = V_\delta\, V_{bulk}(\phi_0) \leq M_*^2 M_P;$$

where the RHS comes from the natural upper bound $V_{bulk}(\phi_0) < M_*^{4+\delta}$. This immediately means that a non-stringent bound exists for Hubble, $H \leq M_*$, and non super-light inflaton is required. This also keeps the explanation of the flatness and horizon problems as usual, since now the time for inflation could be as short as in the standard theory. A hybrid inflation model now accommodates a nice prediction for density perturbations [25],

$$\frac{\delta\rho}{\rho} \sim \left(\frac{g}{2\lambda^{3/2}}\right)\,\frac{M_*^3}{m_0^2 M_P} \,, \tag{42}$$

which can easily give COBE normalization. Reheating would now be produced by the decay of the inflaton into brane standard fields. The effective brane-bulk coupling has a Planck suppression on it, which amounts to a low reheating temperature, that nevertheless comes out to be just right to allow for a successful BBN process. To give numbers, let us consider $T_R \sim 0.1\sqrt{\Gamma_\phi M_{P\ell}}$ and a typical rate for the decay into higgses $\Gamma_\phi \sim M_*^4/(32\pi M_P^2 m_\phi)$, and use $m_\phi$ around $0.1M_*$, with $M_* \sim 100TeV$, one then gets $T_R \sim 100$ MeV. It is worth mentioning that this number is well within the constraints due to graviton thermal production (27), discussed previously.

It is worth noticing that the number of e-foldings is usually much less than $60$ if the scale of inflation is low, especially if the scale of inflation is as low as $H_{inf} \approx M_* \sim$ TeV. It has already been shown in extra dimensional models [26, 27] that the number of e-foldings required for structure formation would be $43$, provided the universe reheats by $T_{rh} \sim 10 - 100$ MeV. Therefore, it is only the last $43$ e-foldings of inflation which are important for the purpose of density perturbations.

As an alternative way to solve inflaton problems, Arkani-Hamed *et al.* [28] proposed a scenario where it was assumed that inflation occurs before the stabilization of the internal dimensions. With the dilaton field playing the role of the inflaton field, they argued that early inflation, when the internal dimensions are small, can overcome the complications we mention at the beginning of the present section. However, one cannot allow the extra dimension to grow too much during inflation, since large changes in the internal size will significantly affect the scale invariance of the density perturbations, The radius of the extra dimension must remain essentially static while the Universe expands (slow rolling); this needs quite a flat potential which may cause trouble for later stabilization. The scenario may also pose some complications for the understanding of reheating since the radion is long-lived, and its mass could be very small (about $1/R$).

Another possible way out was proposed in Ref. 30, where it was suggested that the brane could be out of its stable point at early times, and inflation is induced on the brane by its movement through the extra space. The common point of last two scenarios is that they assume an unstable extra dimension throughout the inflation process. This may also be troublesome, as we shall discuss in the next section

### 3.4. Cosmological Radion problem

It is conceptually hard to accept the fact that the extra dimensions were at all times as large as suggested by the ADD model. Just as it is natural to think that all our observable Universe started as a small patch of size $\sim M_*^{-1}$, it seems so for the extra dimensions. From its definition, an initial value $b_{in} = M_*^{-1}$ means that the radion started with a large negative value,

$$\sigma_v(0) \; = \; -M_p\sqrt{\frac{2(\delta+2)}{\delta}}\ln\left(\frac{M_p}{M_*}\right) \; . \qquad (43)$$

Since the stable point has been defined such that $\sigma|_{b=R} = 0$, one has to conclude that the radion should roll down its potential from negative values as the extra dimensions expand. If the radion potential were flat enough in this range of values, it would be natural to think that the radion could play the role of the inflaton. The idea is enforced by the large absolute initial value of $\sigma$, which may already satisfy chaotic initial conditions for driving inflation. To complete the picture, one has at some point to be able to calculate the radion potential from some fundamental physics and prove that it is indeed flat for negative $\sigma$ values, whereas it grows fast enough for positive $\sigma$'s to avoid dynamically driving the volume much beyond the expected stable value $b = R$. Without the actual potential, it is hard to make any serious calculation for density perturbations or even reheating temperatures. Certainly a simple mass term like potential $U(\sigma) \sim m^2\sigma^2$ would not do it. The reason is two-fold. First, we are far away from the stable point where the radion mass is defined, so the potential would hardly be well described by just the second term of its Taylor expansion. Second, and more important, the simple chaotic potential predicts insufficient density fluctuations for small $M_*$. Indeed, the calculation gives [25] $\delta\rho/\rho \simeq m/M_P \ll M_*/M_P$.

Also interesting is the possibility of producing inflation with the help of a bulk inflaton in the presence of the radion field. This analysis can help us to understand whether our previous discussion makes sense at all . We now turn our attention to the action in Eq. (38). Notice that the inflaton to radion coupling is given only through the inflaton potential. The coupling induces an effective mass term for the radion which is proportional to the Hubble scale, $m_{eff}^2 \sim \alpha^2 V_{eff}(\phi)/M_P = \alpha^2 H^2$. Consequently, the radion gets a very steep effective potential term, which easily drives the radion towards the minimum within a Hubble time [30]. Indeed, once the Hubble induced mass term is switched on the radion will follow the evolution

$$\sigma(t) \sim \sigma_0\, e^{-(m_{eff}^2/3H)t} \, , \qquad (44)$$

where $\sigma_0$ is the initial amplitude. Naively, one could think that this should solve the problem of stabilization, provided inflation lasts a bit longer than this dynamical stabilization process. Nevertheless, it has been realized that, although this stabilization does take place, it happens that the effective minimum of the potential does not coincide with $\sigma = 0$ [31,32]. Actually, it generally happens that the global minimum for $\sigma$ is displaced during inflation when the radion potential is quite flat [31]. To see this, let us notice that the total potential has the form

$$U_{total}(\sigma,\phi) = U(\sigma) + e^{-\alpha\sigma/M_P}\, V_{eff}(\phi) \, , \qquad (45)$$

where by definition, $U(\sigma)$ has a minimum at $\sigma = 0$. During inflation, $V_{eff}$ defines the Hubble scale, as already mentioned, and it is taken as a constant. The global minimum for $\sigma$ of this potential is the solution to the equation

$$U'(\sigma) - \frac{\alpha}{M_P}\, e^{-\alpha\sigma/M_P} V_{eff} = 0 \, . \qquad (46)$$

Clearly, $\sigma = 0$ is not a global minimum. In fact, in order to match both terms of the equation, the minimum should lie within the positive range of $\sigma$ values. This implies that, during inflation, the extra space grows beyond its stable size ($b = R$), and gradually comes back to the final stable volume $V_\delta$ as the inflaton energy diminishes [31,32]. How far we are from the expected value $V_\delta$ depends on the actual profile of the radion potential. For steeper potentials, it is easy to see that the displacement could be negligible for practical proposes; however, that is not so for flatter radion potentials [32], since a larger value on $\sigma$ would be needed in the exponential of the last equation to match a small value of $U'(\sigma)$. One interesting conclusion arises: inflation could in principle be consistently analyzed in the setup of a stable bulk, as we did in the previous section; nevertheless, post-inflationary effects of the radion dynamics have yet to be studied carrefully.

## 4. Non Factorizable Geometries: Randall-Sundrum Models.

### 4.1. Warped Extra Dimensions

So far we have been working on the simplest picture, where the energy density on the brane does not affect the space-time curvature, but rather it has been taken as a perturbation on the flat extra space. For large brane densities, this may not be the case. The first approximation to the problem can be done by considering a five dimensional model where branes are located at the two ends of a closed fifth-dimension. Clearly, with a single extra dimension, the gravity flux produced by a single brane at $y = 0$ cannot softly close into itself at the other end of the space, making the model unstable, just as a charged particle living in a one-dimensional world does not define a stable configuration. Stability can only be insured by the introduction of a second charge (brane). Furthermore, to balance brane energy and still get flat (stable) brane metrics, one has to compensate for the effect on the space by the introduction of a negative cosmological constant on the bulk. Thus, the fifth dimension would be a slice of an Anti de-Siter space with a flat brane at its edges. Thus, one can keep the branes flat by paying the price of curving the extra dimension. Such curved extra dimensions are usually referred to as warped extra dimensions. Historically, the possibility was first mentioned by Rubakov and Shaposhnikov in Ref. [9], suggesting that the cosmological constant problem could be understood in this light: the matter field vacuum energy on the brane could be canceled by the bulk vacuum, leaving a zero (or almost zero) cosmological constant for the brane observer. No specific model was given there, though. It was actually Gogberashvili [13] who provided the first exact solution for a warped metric; nevertheless, the model is best now after Randall and Sundrum (RS) presented the solution in the context of the hierarchy problem [14]. Later developments suggested that the warped metrics could even provide an alternative to compactification for the extra dimensions [15, 16]. In what follows we shall discuss the concrete example as presented by Randall and Sundrum.

### 4.2. Randall-Sundrum background

Let us consider the following setup. A five dimensional space with an orbifolded fifth dimension of radius $r$ and coordinate $y$ which takes values in the interval $[0, \pi r]$. Consider two branes at the fixed (end) points $y = 0, \pi r$. with tensions $\tau$ and $-\tau$ respectively. For reasons that should become clear later on, the brane at $y = 0$ ($y = \pi r$) is usually called the hidden (visible) or Planck (SM) brane. We shall also assign to the bulk a negative cosmological constant $-\Lambda$. Contrary to our previous philosophy in the ADD model, we shall here assume that all parameters are of the order of the Planck scale. Next, we ask for the solution that gives a flat induced metric on the branes such that the 4D Lorentz invariance is respected. To get a consistent answer, one has to require that, at every point along the fifth dimension, the induced metric should be the ordinary flat 4D Minkowski metric. Therefore, the components of the 5D metric depend only on the fifth coordinate. Hence, one gets the ansatz

$$ds^2 = g_{AB} dx^A dx^B = \omega^2(y) \eta_{\mu\nu} dx^\mu dx^\nu - dy^2 \ , \qquad (47)$$

where we parameterize $\omega(y) = e^{-\beta(y)}$. The metric, of course, can always be written in different coordinate systems. Particularly, notice that one can easily go to the conformally flat metric, where there is an overall factor in front of all coordinates, $ds^2 = \omega^2(z)[\eta_{\mu\nu} dx^\mu dx^\nu - dz^2]$, where the new coordinate $z$ is a function of the old coordinate $y$ only.

Classical action contains $S = S_{grav} + S_h + S_v$, where

$$S_{grav} = \int d^4x\, dy \sqrt{g_{(5)}} \left( \frac{1}{2k_*^2} R_5 + \Lambda \right) \qquad (48)$$

gives the bulk contribution, whereas the visible and hidden brane actions are given by

$$S_{v,h} = \pm\, \tau \int d^4x \sqrt{-g_{v,h}} \ , \qquad (49)$$

where $g_{v,h}$ stands for the induced metric at the visible and hidden branes, respectively.

Five-dimensional Einstein equations for the given action,

$$\begin{aligned} G_{MN} &= R_{MN} - \frac{1}{2} g_{MN} R_{(5)} \\ &= -k_*^2 \Lambda\, g_{MN} + k_*^2 \tau \sqrt{\frac{-g_h}{g_{(5)}}} \delta_M^\mu \delta_N^\nu g_{\mu\nu} \delta(y) \\ &- k_*^2 \tau \sqrt{\frac{-g_v}{g_{(5)}}} \delta_M^\mu \delta_N^\nu g_{\mu\nu} \delta(y - \pi r) \end{aligned} \qquad (50)$$

are easily reduced into two simple, independent equations. First, we can expand the $G_{MN}$ tensor components on the LHS of the last equation, using the metric ansatz (47), to show

$$G_{\mu\nu} = -3\, g_{\mu\nu} \left( -\beta'' + 2(\beta')^2 \right) \ ; \qquad (51)$$

$$G_{\mu 5} = 0 \ ; \quad \text{and} \quad G_{55} = -6\, g_{55} (\beta')^2 \ . \qquad (52)$$

Next, using the RHS of Eq. (50), one gets

$$6(\beta')^2 = k_*^2 \Lambda \ ; \qquad (53)$$

and

$$3\beta'' = k_*^2 \tau \left[ \delta(y) - \delta(y - \pi r) \right] . \qquad (54)$$

This last equation clearly defines the boundary conditions for the function $\beta'(y)$ at the two branes (Israel conditions). Clearly, the solution is $\beta(y) = \mu |y|$, where

$$\mu^2 = \frac{k_*^2 \Lambda}{6} = \frac{\Lambda}{6 M_*^3} \ , \qquad (55)$$

with the subsidiary fine tuning condition

$$\Lambda = \frac{\tau}{6 M_*^3} \ , \qquad (56)$$

obtained from the boundary conditions, which is equivalent to the exact cancellation of the effective four-dimensional cosmological constant. The background metric is therefore

$$ds^2 = e^{-2\mu|y|}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2 \ . \tag{57}$$

The effective Planck scale in the theory is then given by

$$M_P^2 = \frac{M_*^3}{\mu}\left(1 - e^{-2\mu r\pi}\right) . \tag{58}$$

Notice that for a large $r$, the exponential piece becomes negligible, and the above expression has the familiar form given in Eq. (10) for one extra dimension, of (effective) size $1/\mu$.

### 4.3. Visible versus Hidden Scale Hierarchy

The RS metric has a peculiar feature. Consider a given distance, $ds_0^2$, defined by fixed intervals $dx_\mu dx^\mu$ from brane coordinates. If one maps the interval from hidden to visible brane, it would appear here exponentially smaller than what is measured at the hidden brane, i.e. $ds_0^2|_v = \omega^2(y)ds_0^2|_h$. This scaling property would have interesting consequences when introducing fields to live on any of the branes. Particularly, let us discuss what happens for a theory defined on the visible brane.

The effect of the RS background on visible brane field parameters is non-trivial. Consider for instance the scalar field action for the visible brane at the end of the space given by

$$S_H = \int d^4x\omega^4(\pi r)\left[\omega^{-2}(\pi r)\partial^\mu H\partial_\mu H - \lambda\left(H^2 - \hat{v}_0^2\right)^2\right].$$

As a rule, we choose all dimensionful parameters on the theory to be naturally given in terms of $M_*$, and this to be close to $M_P$. So we take $v_0 \sim M_*$. After introducing the normalization $H \to \omega^{-1}(\pi r)H = e^{\mu r\pi}H$ to recover the canonical kinetic term, the above action becomes

$$S_H = \int d^4x\sqrt{-g}\left[\eta^{\mu\nu}\partial_\mu H\partial_\nu H - \lambda\left(H^2 - v^2\right)^2\right], \tag{59}$$

where the vacuum $v = e^{-\mu r\pi}\hat{v}_0$. Therefore, by choosing $\mu r \sim 12$, the physical mass of the scalar field, and its vacuum, would naturally appear at the TeV scale rather than at the Planck scale, without the need for any large hierarchy on the radius [14]. Notice that, on the contrary, any field located on the other brane will have a mass of the order of $M_*$. Moreover, it also implies that no particles exist in the visible brane with masses larger than TeV. This observation has been considered a nice possible way of solving the scale hierarchy problem. For this reason, the original model proposed that our observable Universe resided on the brane located at the end of the space, the visible brane. So the other brane really becomes hidden. This two brane model is sometimes called RSI model.

### 4.4. Kaluza Klein decomposition

As a further note, notice that since there is 4D Poincaré invariance everywhere, every bulk field on the RS background can be expanded into four-dimensional plane waves $\phi(x,y) \propto e^{ip_\mu x^\mu}\phi_p(y)$. This would be the basis of the Kaluza Klein decomposition, that we shall now discuss. Note also that the physical four momentum of the particle at any position of the brane goes as $p^\mu_{phys}(y) = \omega^{-1}(y)p^\mu$. Therefore, modes which are soft on the hidden brane, become harder at any other point of the bulk.

Let us consider again a bulk scalar field, now on the RS background metric. The action is then

$$S[\phi] = \frac{1}{2}\int d^4x\, dy\sqrt{g_{(5)}}\left(g^{MN}\partial_M\phi\partial_N\phi - m^2\varphi^2\right) \ . \tag{60}$$

By introducing the factorization $\phi(x,y) = e^{ip_\mu x^\mu}\varphi(y)$ into the equation of motion, one gets that the KK modes satisfy

$$\left[-\partial_y^2 + 4\mu\,\mathrm{sgn}(y)\partial_y + m^2 + \omega^{-2}(y)\,p^2\right]\varphi(y) = 0 \ , \tag{61}$$

where $p^2 = p^\mu p_\mu$ can also be interpreted as the effective four-dimensional invariant mass, $m_n^2$. It is possible, through a functional re-parameterization and a change of variable, to show that the solution for $\varphi$ can be written in terms of Bessel functions of index $\nu = \sqrt{4 + m^2/\mu^2}$ [33,34], as follows

$$\varphi_n(z) = \frac{1}{N_n\omega^2(y)}\left[J_\nu\left(\frac{m_n}{\mu\omega(y)}\right) + b_{n\nu}Y_\nu\left(\frac{m_n}{\mu\omega(y)}\right)\right], \tag{62}$$

where $N_n$ is a normalization factor, $n$ labels the KK index, and the constant coefficient $b_{n\nu}$ has to be fixed by the continuity conditions at one of the boundaries. The other boundary condition would serve to quantize the spectrum. For more details the interested reader can see Ref. 34. Here we will just make some few comments about. First, for $\omega(\pi r) \ll 1$, the discretization condition that one gets for $x_{n\mu} = m_n/\mu\omega(y)$ looks as

$$2J_\nu(x_{n\mu}) + x_{n\nu}J'_\nu(x_{n\mu}) = 0 \ . \tag{63}$$

Therefore, the lowest mode satisfies $x_{1\mu} \sim \mathcal{O}(1)$, which means that $m_1 \simeq \mu e^{-\mu r\pi}$. For the same range of parameters we considered before to solve the hierarchy problem, one gets that lightest KK mode would have a mass of order TeV or so. Next, for the special case of an originally massless field ($m = 0$), one has $\nu = 2$, and thus the first solution to Eq. (63) is just $x_{12} = 0$, which indicates the existence of a massless mode in the spectrum. The next zero of the equation would be of order one again, and thus the KK tower would start at $\mu e^{-\mu r\pi}$. The spacing between two consecutive KK levels would again be of about the same order. There is no need to stress that this would actually be the case of the graviton spectrum. This makes the whole spectrum completely distinct from the former ADD model. With such heavy graviton modes, one would not expect to have visible deviations on the short distance gravity experiments, nor constraints from BBN or star cooling.

### 4.5. Radion Stabilization

The way the RSI model solves the hierarchy problem between $m_{EW}$ and $M_P$ depends on the interbrane spacing $\pi r$. Stabilizing the bulk becomes in this case an important issue if one is willing to keep this solution. The dynamics of the extra dimension would give rise to a runaway radion field, as it does for the ADD case. A simple exploration of the metric (57), by setting a time dependent bulk radius $r(t)$, shows that

$$ds^2 \to e^{-2\mu r(t)|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu - r^2(t)d\phi^2 \ , \qquad (64)$$

with $\phi$ the angular coordinate on the half circle $[0,\pi]$. This suggests that, if the interbrane distance changes, the visible brane expands (or contracts) exponentially. The radion field associated whit the fluctuations of the radius, $b(t) = r(t)-r$, is again massless and thus it violates the equivalence principle. Moreover, without a stabilization mechanism for the radius, our brane could expand forever. Some early discussions on this and other issues can be found in Refs. 34, 36, and 37.

The simplest and most elegant solution for stabilization in RSI was proposed by Goldberger and Wise [33]. The central idea is really simple: if there is a vacuum energy on the bulk, whose configuration breaks translational invariance along a fifth dimension, say $\langle E\rangle(y)$. Then, the effective four-dimensional theory would contain a radius-dependent potential energy

$$V(r) = \int dy\, \omega^4(y)\, \langle E\rangle(y) \ .$$

Clearly, if such a potential has a non-trivial minimum, stabilization would be insured. The radion would feel a force that would tend to keep it at the minimum. The vacuum energy $\langle E\rangle(y)$ may come from many sources. The simplest possibility one could think of is a vacuum induced by a bulk scalar field, with non-trivial boundary conditions,

$$\langle\phi\rangle(0) = v_h \qquad \text{and} \qquad \langle\phi\rangle(\pi r) = v_v \ . \qquad (65)$$

The boundary conditions would amount for a non-trivial profile of $\langle\phi\rangle(y)$ along the bulk. Such boundary conditions may arise, for instance, if $\phi$ has localized interaction terms on the branes, as $\lambda_{h,v}(\phi^2 - v_{h,v}^2)^2$, which by themselves develop non zero vacuum expectation values for $\phi$ located on the branes. The vacuum is then the $x$-independent solution to the equation of motion (61), which can be written as

$$\langle\phi\rangle(y) = \omega^{-1}(y)\left[A\omega^{-\nu}(y) + B\omega^{\nu}(y)\right] \ , \qquad (66)$$

where $A$ and $B$ are constants to be fixed by the boundary conditions. One then obtains the effective 4D vacuum energy

$$V_\phi(r) = \mu(\nu+2)A^2\left(\omega^{-2\nu}(\pi r) - 1\right) + \mu(\nu-2)B^2\left(1-\omega^{2\nu}(\pi r)\right) \qquad (67)$$

After a lengthly calculation, and in the limit where $m \ll \mu$, one finds that the above potential has a non trivial minimum for

$$\mu r = \left(\frac{4}{\pi}\right)\frac{\mu^2}{m^2}\ln\left[\frac{v_h}{v_v}\right] \ . \qquad (68)$$

Hence, for $\ln(v_h/v_v)$ of order one, the stable value for the radius is proportional to the curvature $\mu$ parameter, and inversely to the squared mass of the scalar field. Thus, one only needs that $m^2/\mu^2 \sim 10$ to get $\mu r \sim 10$, as needed for the RSI model.

One might get a bit suspicious about whether the vacuum energy $\langle\phi\rangle(y)$ may disturb the background metric. It actually does, although the correction is negligible as the calculations for the Einstein-scalar field coupled equations may show [33, 36].

## 5. Infinite Extra Dimensions

The background metric solution (57) does not actually need the presence of the negative tension brane to hold as an exact solution to Einstein equations. Indeed the warp factor $\omega(y){=}e^{-\mu|y|}$ has been determined only by the Israel conditions at the $y = 0$ boundary, that is, by using $\omega''{=}\mu^2\omega{-}\mu\omega\delta(y)$ in Einstein equations, which implies equations (55) and (56). It is then tempting to 'move' the negative tension brane to infinity, which gives a non-compact fifth dimension. The picture becomes esthetically more appealing; it has no need for compactification. Nevertheless, one must now ask the question of whether such a possibility is at all consistent with observations. It is clear that the Newton's constant is now simply

$$G_N = \mu G_* \qquad (69)$$

–just take the limit $r \to \infty$ in Eq. (58)–, which reflects the fact that although the extra dimension is infinite, gravity remains four dimensional at large distances (for $\mu r \gg 1$). This is, in other words, only a consequence of the flatness of the brane. We shall expand our discussion on this point in the following sections. Obviously, with this setup, usually called the RSII model, we are giving up the possibility of explaining the hierarchy between Planck and electroweak scales. The interest on this model remains, however, due to potentially interesting physics at low energy, and also due to its connection to the AdS/CFT correspondence [37].

Although the fifth dimension is infinite, the point $y = \infty$ is in fact a particle horizon. Indeed, the first indication comes from the metric, since $\omega(y \to \infty){=}0$. The confirmation would come from considering a particle moving away from the brane on the geodesics $y_g(t){=}(1/2\mu)\ln(1+\mu^2t^2)$ [38]. The particle accelerates towards infinity, and its velocity tends to the speed of light. The proper time interval is then

$$d\tau^2 = \omega^2(y_g(t))dt^2 - \left(\frac{dy_g}{dt}\right)^2 dt^2 \ . \qquad (70)$$

Thus, the particle reaches infinity at an infinite time $t$, but in a finite proper time $\tau = \pi/2\mu$.

### 5.1. Graviton Localization

In order to understand why gravity on the brane remains four-dimensional at large distances, even though the fifth dimension is non-compact, one has to consider again the KK decomposition for the graviton modes, with particular interest in the shape for the zero mode wave function. Consider first the generic form of the perturbed background metric

$$ds^2 = \omega^2(y) g_{\mu\nu} dx^\mu dx^\nu + A_\mu dx^\mu dy - b^2 dy^2 \, .$$

Due to the orbifold projection, $y \to -y$, the vector component $A_\mu$ must be odd, and thus it does not contain a zero mode. Therefore, at the zero mode level only the true four-dimensional graviton and the scalar (radion) should survive. Let us concentrate on the 4D graviton perturbations only. Introducing the small field expansion as $g_{\mu\nu} = \eta_{\mu\nu} + \omega^{-2} h_{\mu\nu}$, and using the gauge fixing conditions $\partial_\mu h^\mu_\nu = 0 = h^\mu_\mu$, one obtains the wave equation

$$\left[\partial_y^2 - 4\mu^2 - \frac{m^2}{\omega^2(y)} - 4\mu\delta(y)\right] h = 0 \, , \qquad (71)$$

where the Lorentz indices should be understood. In the above equation the mass $m^2$ stands for the effective four-dimensional mass $p^\mu p_\mu = m^2$. It should be noticed that the mass spectrum would now be continuous, and starts at $m = 0$. In this situation the KK are normalized to the delta function, $\int dy \omega^{-2}(y)\, h_m(y)\, h_{m'} = \delta(m - m')$.

Introducing the functional re-parameterization

$$z = \frac{1}{\mu} sgn(y) \left(\omega^{-1}(y) - 1\right)$$

and

$$\Psi(z) = \omega^{-1/2}(y) h(y) \, ,$$

one can write the equation of motion for the KK modes as the Schödinger equation

$$\left[-\frac{1}{2}\partial_z^2 + V(z)\right] \Psi(z) = m^2 \Psi(z) \qquad (72)$$

with a 'volcano potential'

$$V(z) = \frac{15\mu^2}{8(\mu|z| + 1)^2} - \frac{3\mu}{2}\delta(z) \, , \qquad (73)$$

which peaks as $|z| \to 0$ but has a negative singularity right at the origin. It is well known from the quantum mechanics analog that such delta potential has a bound state, whose wave function peaks at $z = 0$, which also means at $y = 0$. In other words, it appears to be localized at the brane. Such a state is identified as our four-dimensional graviton. Its localization is the physical reason why gravity still behaves as four dimensional at the brane.

Indeed, the wave function for the localized state is

$$\Psi_o(z) = \frac{1}{\mu(|z| + 1/\mu)^{3/2}}, \qquad (74)$$

whereas the KK mode wave functions in the continuum are written in terms of Bessel functions, in close analogy to Eq. (62), as

$$\Psi_m \sim s(z) \left[Y_2\left(m|z| + \frac{1}{\mu}\right) + \frac{4\mu^2}{\pi m^2} J_2\left(m|z| + \frac{1}{\mu}\right)\right],$$

where $s(z) = (|z| + 1/\mu)^{1/2}$. By properly normalizing these wave functions using the asymptotics of the Bessel functions, it is possible to show that for $m < \mu$ the wave function at brane has the value

$$h_m(0) \approx \sqrt{\frac{m}{\mu}}. \qquad (75)$$

The coupling of gravitons to the brane is therefore weak for the lightest KK graviton states. The volcano potential acts as a barrier for those modes. The production of gravitons at low energies would then be negligible.

### 5.2. Gravity on the RSII brane

The immediate application of our last calculations is on the estimation of the effective gravitational interaction law at the brane. The reader should remember that the effective interaction of brane matter to gravitons is $h_{\mu\nu}(0) T^{\mu\nu}$. So it involves the evaluation of the graviton wave function at the brane position, as expected. Therefore the graviton exchange between two test particles on the brane separated by a distance $r$ gives the effective potential

$$U_{RSII}(r) \approx U_N(r) \left[1 + \int_0^\infty \frac{dm}{\mu} \frac{m}{\mu} e^{-mr}\right] \qquad (76)$$

$$= U_N(r) \left[1 + \frac{1}{\mu^2 r^2}\right] \, . \qquad (77)$$

Notice that the correction looks exactly like in the two extra dimensional ADD case, with $1/\mu$ as the effective size of the extra dimensions. Thus, from the brane point of view, the bulk should appear as compact, at least from the gravitational point of view. The conclusion is striking. There could be non-compact extra dimensions and yet scope to our observations!.

### 5.3. Higher dimensional generalization

The RSII model, which provides a serious alternative to compactification, can immediately be extended to a larger number of dimensions. First, notice that the metric (57) has come from the peculiar properties of co-dimension one objects in gravity. Thus, it is obvious that the straightforward generalization should also contain some co-dimension one branes in the configuration. Our brane, however should have a larger co-dimension. Let us consider a system of $\delta$ mutually intersecting $(2+\delta)$ branes in a $(4+\delta)$ dimensional AdS space, of cosmological constant $-\Lambda$. All branes should have a positive tension $\tau$. Clearly, the brane intersection is a 4 dimensional brane, where we assume our Universe lives. Intuitively, each of the $(2+\delta)$ branes would try to localize the graviton to itself, just as the RSII brane does. Consequently, the zero mode graviton would be localized at the intersection of all branes. This naive observation can indeed be confirmed by solving the Einstein equations for the action [16]

$$S = \int d^4x\, d^\delta y\, \sqrt{g_{(4+\delta)}} \left( \frac{1}{2k_*} R_{(4+\delta)} + \Lambda \right) - \sum_{\text{all branes}} \tau \int d^4x\, d^{\delta-1} y\, \sqrt{g_{(3+\delta)}}. \quad (78)$$

If the branes are all orthogonal to each other, it is straightforward to see that the space consists of $2^\delta$ equivalent slices of AdS space, glued together along the flat branes. The metric, therefore, would be conformally flat. Thus, one can write it, using appropriate bulk coordinates, as

$$ds^2_{(4+\delta)} = \Omega(z) \left( \eta_{\mu\nu} dx^\mu dx^\nu - \delta_{kl}\, dz^k dz^l \right) \quad (79)$$

with the warp factor

$$\Omega(z) = \frac{1}{\mu \sum_j |z^i| + 1}, \quad (80)$$

where the $\mu$ curvature parameter is now

$$\mu^2 = \frac{2k_*^2 \Lambda}{\delta(\delta+2)(\delta+3)}, \quad (81)$$

which is a generalization of the relation given in Eq. (55). Similarly, the fine tuning condition (56) now looks like

$$\Lambda = \frac{\delta(\delta+3)}{8(\delta+2)} \tau^2 k_*^2 \,. \quad (82)$$

The effective Planck scale is now calculated to be

$$M_P^2 = M_*^{(\delta+2)} \int d^\delta z \Omega^{(2+\delta)} = \frac{2^\delta \delta^{\delta/2}}{(\delta+1)!} M_*^{(\delta+2)} L^\delta \,, \quad (83)$$

for $L = 1/\sqrt{\delta}\mu$. Notice that this expression resembles the ADD relationship given in Eq. (10), with $L$ as the effective size of the extra dimensions.

Graviton localization can now be seen by perturbing the metric with $\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$ in Eq. (79), and writing the equation of motion for $h_{\mu\nu}$ in the gauge $h^\mu{}_\mu = 0 = \partial_\mu h^{\mu\nu}$, and in conformal coordinates, to obtain for $\Psi = \Omega^{(\delta+2)/2} h$ the linearized equation

$$\left[ -\frac{1}{2} m^2 + \left( -\frac{1}{2} \nabla_z^2 + V(z) \right) \right] \hat{\Psi} = 0 \,, \quad (84)$$

which is again nothing other than a Schödinger equation with the effective potential

$$V(z) = \frac{\delta(\delta+2)(\delta+4)\mu^2}{8} \Omega - \frac{(\delta+2)\mu}{2} \Omega \sum_j \delta(z^j). \quad (85)$$

Indeed, the spectrum has a massless bound state localized around the intersection of all delta function potentials $(z=0)$, which is $\Psi_{bound} \sim \Omega^{(\delta+2)/2}(z)$. Since the potential falls off to zero for a large $z$, there would also be a continuum of modes. Since the height of the potential near the origin is $\mu^2$, all modes with small masses, $m < \mu$ will have suppressed wave functions, while those with large masses will be unsuppressed at the origin. Therefore, the contribution of the lightest modes to the gravitational potential for two test particles at the brane would again be suppressed as in the RSII case. The correction to Newton's law is [16]

$$\Delta U(r) \sim U_N(r) \left( \frac{L}{r} \right)^\delta \,, \quad (86)$$

which again behaves as in the ADD case, mimicking the case of compact dimensions, though this is not the case.

## 6. Brane Cosmology

Let us now discuss what modifications are introduced to cosmology if one considers the RSII setup. One of the first things that one needs to know is the time dependence of the metric. As the bulk curvature arose to compensate for the brane tension in order to keep the brane flat, once a time dependent energy density $T_{\mu\nu}$ is introduced on the brane, as needed for our Universe, the warping will also become time dependent, and so one must reconsider the RS solution to five-dimensional Einstein equations. The problem was first addressed by Binetruy *et al.* in Refs. 40 and 41. It has also been noted that the Friedmann equation could be recovered on the brane in the low energy limit ($\rho <\ll \tau$), on the basis of the fine tuning (56), even thought the metric is not static [41–43]. Here we shall discuss some of these results.

We start by considering what the metric ansatz should be for brane cosmology, assuming that the only energy sources are the brane energy momentum tensor, $T_{\mu\nu}$, and the negative cosmological constant of the brane. We adopt the cosmological principle of isotropy and homogeneity in the three space dimensions of the brane; thus, the most general $T_{\mu\nu}$ has a diagonal form, parameterized by energy density, $\rho$, and pressure, $P$,

$$T^\mu{}_\nu = diag(\rho, -P, -P, -P) \,.$$

Also, this implies that $g_{MN} = g_{MN}(y,t)$, only, where the $y$ dependence reflects the breaking of translational invariance along the fifth dimension due to the presence of the brane. Finally, and for simplicity, we shall consider that distance intervals along the fifth dimensions are fixed along time. We then choose the following ansatz for the metric

$$ds^2 = w^2(y,t)\, dt^2 - a^2(y,t)\, \gamma_{ij}\, dx^i dx^j - dy^2 \,, \quad (87)$$

where $\gamma_{ij} = f(r)\delta_{ij}$, with $f^{-1}(r) = 1 - kr^2$ being the usual Robertson-Walker curvature term, where $k = -1, 0, 1$. For simplicity, we shall restrict our discussion to the flat Universe case. It is worth noticing that such a metric does reduce to the standard Friedmann-Robertson-Walker metric (34) when evaluating distance intervals at the brane, provided $w(0,t) = 1$. As in RS models, we shall here assume that the orbifold symmetry $P : y \to -y$ is present, thus, $a$ and $n$ would depend only on $|y|$.

The five-dimensional Einstein equations take the form

$$\begin{aligned} G_{AB} &= R_{AB} - \frac{1}{2} g_{AB} R \\ &= k_*^2 \left[ -\Lambda g_{AB} + T_{\mu\nu}\, \delta^\mu_A\, \delta^\nu_B\, \delta(y) \right], \end{aligned} \tag{88}$$

The delta function on the RHS of Eqs. (88) can be understood as a boundary condition. We then proceed by solving first the equations away from the brane. The global metric solution clearly shall be continuous everywhere. Is naturally solved by the orbifold condition. Metric derivatives on $y$, however, are not continuous; they should have a gap at $y = 0$, which should match the brane energy momentum tensor,

$$\int_{0-}^{0+} dy\, G_{\mu\nu} = k_*^2 T_{\mu\nu}. \tag{89}$$

Let us now proceed to the details. We use the metric ansatz (87) to explicitely expand the RHS of Eqs. (88) to get [39]

$$\begin{aligned} G_{00} &= 3 \left\{ \left(\frac{\dot a}{a}\right)^2 - w^2 \left[ \left(\frac{a'}{a}\right)^2 + \frac{a''}{a} \right] \right\} \\ G_{ij} &= \left[ a^2 \left\{ \frac{a'}{a} \left( 2\frac{w'}{w} + \frac{a'}{a} \right) + 2\frac{a''}{a} + \frac{w''}{w} \right\} \right. \\ &\qquad \left. + \frac{a^2}{w^2} \left\{ \frac{\dot a}{a} \left( 2\frac{\dot w}{w} - \frac{\dot a}{a} \right) - 2\frac{\ddot a}{a} \right\} \right] \delta_{ij}, \\ G_{04} &= 3 \left( \frac{w'}{w}\frac{\dot a}{a} - \frac{\dot a'}{a} \right), \\ G_{44} &= 3 \left\{ \frac{a'}{a} \left( \frac{a'}{a} + \frac{w'}{w} \right) - \frac{1}{w^2} \left[ \frac{\dot a}{a} \left( \frac{\dot a}{a} - \frac{\dot w}{w} \right) + \frac{\ddot a}{a} \right] \right\} \end{aligned} \tag{90}$$

where primes denote derivatives with respect to $y$ and dots derivatives with respect to $t$. Our boundary (Israel) conditions are then

$$\left.\frac{\Delta a'}{a}\right|_{y=0} = -\frac{k_*^2}{3}\rho, \tag{91}$$

$$\left.\frac{\Delta w'}{w}\right|_{y=0} = \frac{k_*^2}{3}(3P + 2\rho); \tag{92}$$

here the gap function $\Delta a'(0) = a'(0^+) - a'(0^-) = 2a'(0^+)$ give the size of the jump for the $y$ derivative of $a$ at zero. The same applies to $\Delta w'$. We can straightforwardly use these boundary conditions to evaluate the Einstein tensor components at the brane. In particular, the equation for $G_{04}$ gives the energy conservation formula

$$\dot\rho + 3H_0\,(\rho + P) = 0,$$

where we have introduced the brane Hubble function, $H_0(t)$, as the $y = 0$ value of the more general bulk Hubble function [43]

$$H(y,t) \equiv \left(\frac{\dot a}{a}\right). \tag{93}$$

It is also not difficult to show that the equation $G_{04}$=0 implies the more general condition

$$w(y,t) = \lambda(t)\dot a(y,t). \tag{94}$$

It is very illustrative to rewrite the equation for $G_{00}$ as the bulk Friedmann equation [43]

$$H^2(y,t) = w^2 \left[ -\frac{k_*^2}{3}\Lambda + \left(\frac{a'}{a}\right)^2 + \frac{a''_R}{a} \right], \tag{95}$$

where $a''_R$ stands for the regular part of the function. It is then clear that, upon evaluation at the brane, the Friedmann equation presents the 'wrong' dependence $H_0^2 \propto \rho^2$ [39] coming from the second term on the RHS of Eq. (95). Also, we may identify the last term of the same equation as the contribution of the Weyl tensor of the bulk [44]. We shall come back to this point a bit later. Now we turn to the other equations. First, $G_{ij}$ gives a non-independent equation. Indeed, it can be derived from Eq. (95) by taking a time derivative, and combining the result with $G_{04}$. This is the same as in the usual 4D case, where $G_{ij}$ gives the acceleration equation. Finally, $G_{44}$ represents the only truly new equation in the system. It is also the window to solving the $y$ dependence of the metric since, combined with the bulk Friedmann equation (95) and the acceleration equation, it simplifies to

$$3\frac{a''}{a} + \frac{w''}{w} = \frac{2}{3}k_*^2\Lambda. \tag{96}$$

Solving this equation, together with (94), we get the result

$$\begin{aligned} a(y,t) &= a_0(t) \left( \cosh(\mu|y|) - \frac{k_*^2}{6\mu}\,\rho\,\sinh(\mu|y|) \right), \\ w(y,t) &= \cosh(\mu|y|) + \frac{k_*^2}{6\mu}\,(3P + 2\rho)\,\sinh(\mu|y|), \end{aligned} \tag{97}$$

where the bulk curvature parameter $\mu^2 = k_*^2\Lambda/6$. Notice that at the static limit ($a_0 = 1$), where $\rho = -P = \tau$, with the brane tension obeying the fine tuning condition (56), we recover the RS metric solution $a(y) = w(y) = e^{-\mu|y|}$. Notice also that in the time-dependent case, we get the FRW metric on the brane.

We can now completely evaluate the Friedmann equation (95) on the brane by using the fact that the total energy density $\rho = \rho_m + \tau$, with $\rho_m$ the actual brane matter density, to get [42]

$$H_0(t) = \frac{\rho_m}{3M_P} \left( 1 + \frac{\rho_m}{2\tau} \right), \tag{98}$$

which has a quadratic term on the matter density. A Universe described by such a modified Friedmann equation evolves faster than the standard one. This may not be a problem during the very early stages of the Universe, whereas just before Nucleosynthesis the standard cosmological evolution $H_0^2 \sim \rho_m$ must had be restored in order not to disturb the success of the theory. Clearly, for small matter densities $\rho_m \ll \tau$, one recovers the standard Hubble expansion.

It is interesting to note, on the other hand, that inflation when driven by a scalar field whose energy density exceeds the brane tension, is more efficient in the brane world. This can be seen from the equation of motion $\ddot{\varphi}+3H_0\varphi+V'(\varphi)=0$, where the friction term becomes larger for larger energy densities. Thus the slow roll is enhanced by the modification to the Friedmann equation (98). Inflation would then last longer than in the standard 4D models, and even some steep potentials that were unable to drive inflation in the 4D case could now be successful [45]. Expansion at high energies drives the tilt of the spectrum of adiabatic density perturbations to zero and it seems not to alter their expected amplitude [45]. The physics of reheating [46,47], preheating, and other pre-BBN phenomena may also be affected, depending on the energy scale at which they take place.

### 6.1. Geometric approach

A more formal and general treatment of the brane model for the derivation of the effective Einstein equations on the brane was presented in Ref. 45. It uses a covariant geometric approach that does not rely on the metric ansatz, and I believe, it is worth underlining in here. Let us denote the unit vector normal to the brane by $n^A$, and the induced brane metric as $\gamma_{AB} = g_{AB} - n_A n_B$. Next we consider the extrinsic curvature of the brane $K_{AB} = \gamma_A^C\,\gamma_B^D \nabla_C n_D$ writing down the Gauss–Codacci equations

$$R_{(4)}{}^{A}_{BCD} = R_{(5)}{}^{M}_{NPQ}\, \gamma_M^A\, \gamma_B^N\, \gamma_C^P\, \gamma_D^Q + 2K^A_{[C}K_{B]D}\ , \quad (99)$$

$$D_N K^N_M - D_M K = R_{(5)}{}_{PQ} n^Q \gamma_M^P\ , \qquad (100)$$

where $D_M$ is the covariant differentiation with respect to $\gamma_{MN}$. The brane Ricci tensor is then obtained by contracting the first of above equations on $A$ and $C$, and one gets $R_{(4)}{}_{MN} = R_{(4)}{}_{CD}\, \gamma_M^C\, \gamma_N^D - R_{(4)}{}^{A}_{BCD} n_A\, \gamma_M^B\, n^C \gamma_N^D + KK_{MN} - K^A_M K_{NA}$. A further contraction on $M$ and $N$ will provide us with the scalar curvature $R_{(4)}$. All together this shall define the five-dimensional Einstein equation with a source given by

$$T_{MN} = \Lambda g_{MN} + S_{MN}\delta(\chi)\ , \qquad (101)$$

where we have explicitely chosen $\chi$ as the locally orthogonal coordinate to the brane, without los of generality. Here, $S_{MN}$ represents the brane energy density, which is given as

$$S_{MN} = -\tau\, \gamma_{MN} + t_{MN}, \qquad (102)$$

with $t_{MN}$ the brane energy momentum tensor, which clearly satisfies $t_{MN} n^M = 0$. It should be noted that, properly speaking, $S_{MN}$ should be evaluated by the variational principle of the 4D Lagrangian for matter fields. The decomposition (102) can be ambiguous. Again, the delta function would lead us to the Israel junction conditions

$$[\gamma_{MN}] = 0 \quad \text{and}$$
$$[K_{MN}] = -k_*^2 \left( S_{MN} - \frac{1}{3}\gamma_{MN} S \right)\ , \qquad (103)$$

where $[X] \equiv \Delta X(0) = 2X(0^+)$, due to the $Z_2$ symmetry. The first of these expressions only states the continuity of the metric at the brane, whereas the other allows us to completely determine the extrinsic curvature of the brane in terms of the energy momentum tensor. Putting it all these equations together, one gets the effective brane Einstein equations

$$G_{(4)}{}_{MN} = \Lambda_4 \gamma_{MN} + 8\pi G_N t_{MN} + k_*^2 \pi_{MN} - E_{MN}\ , \quad (104)$$

where the effective brane cosmological constant

$$\Lambda_4 = \frac{1}{2}k_*^2 \left( \frac{1}{6}k_*^2 \tau - \Lambda \right) \qquad (105)$$

is null only if the fine condition (56) holds. The Newton constant is defined in terms of the brane tension by

$$8\pi G_N = \frac{k_*^2 \tau}{6}\ , \qquad (106)$$

an expression that is equivalent to Eq. (69), with the proper insertion of the parameter $\mu$ as defined in Eq. (55). $E_{MN}$ stands for the limit value at $\chi = 0^+$ of the 5D Weyl tensor $C^{(5)}{}_{MANB} n^A n^B$, with $C^{(5)}$ the 5D Weyl curvature tensor. This gives the non-local effects from the free gravitational field in the bulk, and it cancels when the bulk is purely AdS. And last but not least, the tensor $\pi_{MN}$ gives the local quadratic contribution of the brane energy momentum tensor that arises from the extrinsic curvature terms. They are given as

$$\pi_{MN} = \frac{1}{12} t\, t_{MN} - \frac{1}{4} t_{MA} t^A_M + \frac{1}{24}\gamma_{MN}\left[3 t_{AB} t^{AB} - t^2\right]. \qquad (107)$$

## 7. Concluding remarks

Throughout the present notes, I have introduced the reader to some aspects of models with extra dimensions, where our Universe is constrained to live on a four-dimensional hypersurface. The study of the brane world has become a fruitful industry that has involved several areas of theoretical physics in the matter of a few years. It is fair to say, however, that many of the current leading directions of research obey speculative ideas more than well-established facts. Nevertheless, as happens with any other physics speculations, the studies on the brane world are guided by the principle of physical and mathematical consistency, and the further possibility of connecting the models with the more fundamental theories, i.e. string theory, from which the idea of extra dimensions and branes has been borrowed; and so with experiments in the near future.

It is difficult to address the very many interesting topics of the area, in the detail I intended here, without facing difficulties with the space of these short notes. In exchange, I have concentrated the mainly on the construction of the main frameworks (ADD, and RS models), the calculation of the effective gravity interactions on the brane, and brane cosmology. I hope these will serve the propose of introducing the reader to this area of research. The list of what is left out is extensive, it includes the recent discussions on the cosmological constant problem [48], higher dimensional warped spaces [49] dark matter from KK modes [50], Black Holes in both ADD and RS models [51, 52], deconstruction of extra dimensions [53], and the list goes on. I urge the interested reader to turn to the more extensive reviews [17] and to hunt for further references.

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