Noncommutativity in the Theory of Gravity

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Noncommutative gravity is a very interesting subject that has not yet been successfully related to string theory. However, it can be motivated by itself by the consideration of a description of the microscopic structure of spacetime, leaving for the future its precise connection to string theory or M-theory. In this paper we review some of the recent attempts to make sense of the noncommutative description of some classical theories of gravity by using the Seiberg-Witten map. In particular we describe noncommutative topological gravity and a gauge invariant proposal generalizing Plebański-Ashtekar Self-dual gravity.

Keywords: Noncommutative field theory; Seiberg-Witten map; topological gravity; self-dual gravity.

La gravedad no conmutativa es un tema muy interesante que, hasta ahora, no ha sido incluido en la teoría de cuerdas. Sin embargo, este tema puede ser motivado mediante la consideración de una descripción microscópica de la estructura del espacio-tiempo, dejando para el futuro su relación precisa con la teoría de cuerdas o la teoría M. En este artículo, revisamos algunos de los intentos recientes para dar sentido a la descripción noconmutativa de algunas teorías clásicas de la gravedad, mediante el uso del mapeo de Seiberg-Witten. En particular, describimos la gravedad topológica no conmutativa y una propuesta no conmutativa e invariante de norma que generaliza la gravedad auto-dual de Plebański-Ashtekar.

Descriptores: Teoría de campos no conmutativa; mapeo de Seiberg-Witten; gravedad topológica; gravedad auto-dual.

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1. Introduction

The noncommutative nature of spacetime is an old idea which seems to have originated in the 1930's, soon after the formulation of quantum mechanics, by Heisenberg. It was later worked out systematically by Snyder [1]. Recently it has been studied by many authors from the mathematical perspective [2], as well as from the field theoretical point of view (for recent review papers, see for instance, [3, 4]).

In the last years, in connection with Matrix and string theory, noncommutative gauge theory has once again attracted attention [5, 6]. In particular, Seiberg and Witten [6] have found noncommutativity in the description of the low energy excitations of open strings (possibly attached to D-branes) in the presence of a constant NS bulk $B-$field. They have pointed out that, depending on the regularization scheme of correlation functions of the worldsheet theory, regularization by point splitting or the Pauli-Villars method lead to noncommutative gauge theories or the usual commutative Yang-Mills theory, respectively. Thus, the independence of the regularization scheme tells us that the resulting theory of noncommutative gauge fields (deformed by the Moyal star-product) should be equivalent to a gauge theory in terms of the usual commutative fields. These two theories turn out to be related through the so-called Seiberg-Witten map.

It is well known that gravity and gauge theories are realized in very different ways in string theory. Moreover, as mentioned, string theory predicts a noncommutative effective Yang-Mills theory as a theory defined in the worldvolume of a D-brane. Furthermore, in a number of works [7], it is shown how, starting from the Seiberg-Witten map, gauge theories based on *any* nonabelian gauge groups can be constructed on a noncommutative spacetime. In these developments, the key argument is that no additional degrees of freedom have to be introduced in order to formulate noncommutative gauge theories. Thus, we can ask ourselves whether a noncommutative description of gravity would arise from these considerations. In this context, a number of noncommutative approaches to Einstein's theory of gravity in four dimensions have recently been proposed [8–15]. Noncommutative gravity has been previously considered in the context of the Connes triplet in Refs. 16 and 17. Other noncommutative theories in other dimensions have been considered in Refs. 18 to 21 as well in Refs. 16 and 17.

All these formulations have a constant deformation parameter θ, and in Ref. 9 it is argued that Kontsevich's formality theorem [22] would allow us to relate these theories to diffeomorphism invariant theories. In Refs. 9 and 11, the gauge approach to gravitation has been used to obtain the noncommutative fields through the Seiberg-Witten map, although the actions are not invariant under the corresponding noncommutative gauge symmetries. In Ref. 14, a formulation for topological gravity is given in terms of a noncommutative gauge invariant action, and in Ref. 15, a proposal for a noncommutative formulation of the Plebański-Ashtekar self-dual gravity [23–25]), based on the Seiberg-Witten construction is given. The action is fully invariant under the noncommutative symmetry, and the noncommutative torsion-free condition is straightforwardly solved. These proposals will be outlined in this paper.

2. Weyl-Wigner-Moyal Correspondence

If we wish to have a noncommutative field theory, we can start from the noncommutative spacetime defined by the commutation relations: $[\widehat{x}^\mu, \widehat{x}^\nu] = i\theta^{\mu\nu}$, with $\widehat{x}^\mu$ being linear operators acting in some suitable Hilbert space. The algebra of these operators is associative and noncommutative: $\widehat{x}^\mu(\widehat{x}^\nu\widehat{x}^\rho) = (\widehat{x}^\mu\widehat{x}^\nu)\widehat{x}^\rho$. In this case we need to know how to define field operators, because now the ordering of these operators will make a difference. One way to do this is by means of the Weyl ordering prescription. This ordering is natural in the context of the *Weyl-Wigner-Moyal correspondence* (see [26] for a review), which assigns to each usual function $f(x)$ one operator function $W(f)(\widehat{x})$ with the most symmetric ordering. We start from the Fourier transform of the function $f(x)$,

$$\widetilde{f}(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^dx e^{-ik_\mu x^\mu} f(x), \tag{1}$$

to which the following operator is assigned:

$$\begin{aligned} W(f)(\widehat{x}) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^dk e^{ik_\mu \widehat{x}^\mu} \widetilde{f}(k) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^dk \sum_n k_{\mu_1} \dots k_{\mu_n} \widehat{x}^{\mu_1} \dots \widehat{x}^{\mu_n} \widetilde{f}(k), \end{aligned} \tag{2}$$

where obviously each term is completely symmetric under the permutations of $\widehat{x}$. If we consider the derivative of the function, then we have that the corresponding operator is

$$\begin{aligned} W(\partial_\mu f)(\widehat{x}) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^dk e^{ik_\nu \widehat{x}^\nu} k_\mu \widetilde{f}(k) \\ &= \left[-i\theta^{-1}_{\mu\nu}\widehat{x}^\nu, W(f)(\widehat{x})\right]. \end{aligned} \tag{3}$$

One important property is that there is a function in this way associated whith the operator product $W(f)W(g)$. Indeed, we have

$$\begin{aligned} W(f)(\widehat{x})W(g)(\widehat{x}) &= \frac{1}{(2\pi)^d} \int d^dk d^dp e^{ik_\mu \widehat{x}^\mu} e^{ip_\nu \widehat{x}^\nu} \widetilde{f}(k)\widetilde{g}(p) = \frac{1}{(2\pi)^d} \int d^dk d^dp e^{i(k_\mu + p_\mu)\widehat{x}^\mu} e^{-\frac{i}{2}k_\mu \theta^{\mu\nu} p_\nu} \widetilde{f}(k)\widetilde{g}(p) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^dk e^{ik_\mu \widehat{x}^\mu} \int d^dp e^{-\frac{i}{2}k_\mu \theta^{\mu\nu} p_\nu} \widetilde{f}(k-p)\widetilde{g}(p). \end{aligned} \tag{4}$$

The last integral is the Fourier transform of the function

$$\begin{aligned} (f * g)(x) &= \int d^dy e^{\frac{i}{2}\frac{\partial}{\partial x_\mu}\theta^{\mu\nu}\frac{\partial}{\partial x_\nu}} f(x)g(y)\delta(x-y) \\ &= f(x) e^{\frac{\overleftarrow{\partial}}{\partial x^\mu}\theta^{\mu\nu}\frac{\overrightarrow{\partial}}{\partial x^\nu}} g(x), \end{aligned} \tag{5}$$

which is the Moyal product of the functions f and g. Thus we have $W(f)W(g)(\widehat{x}) = W(f * g)(\widehat{x})$, and the spacetime noncommutative algebra obtained by the Weyl prescription is homomorphic to the function algebra with the Moyal product.

This product is associative, *i.e.* $f * (g * h) = (f * g) * h$, and satisfies the cyclicity property under integrals on closed manifolds,

$$\begin{aligned} \int d^dx f(x) * g(x) * h(x) &= \int d^dx g(x) * h(x) * f(x) \\ &= \int d^dx h(x) * f(x) * g(x). \end{aligned} \tag{6}$$

As an example, we can take the function $f(x) = x^\mu$; then,

$$[x^\mu \overset{*}{,} x^\nu] \equiv x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}. \tag{7}$$

The cyclicity property has the important consequence that if we have an integral of the form

$$\int d^dx L(x),$$

whose integrand transforms as

$$L(x) \to L'(x) = g(x) * L(x) * g^{-1}(x),$$

it is invariant under this transformation:

$$\int d^dx L'(x) = \int d^dx L(x).$$

In particular, if these are matrix-valued functions, we have

$$\mathrm{Tr} \int d^dx L'(x) = \mathrm{Tr} \int d^dx L(x).$$

Thus, for any commutative field theory, we can write a corresponding noncommutative field theory, instead of going to the noncommutative spacetime, by substituting the commutative function product by the Moyal product all over.

3. The Seiberg-Witten Map

3.1. Noncommutative Gauge Theories

In order to define noncommutative gauge theories, noncommutative gauge fields which make derivatives covariant must be introduced. If we have a field which transforms as $\delta_\lambda \Phi = i\lambda * \Phi$, then $D_\mu\Phi = \partial_\mu\Phi - iA_\mu * \Phi$ is its covariant derivative if the gauge field A_μ transforms as

$$\delta_\lambda A_\mu = \partial_\mu \lambda + i[\lambda \stackrel{*}{,} A_\mu]. \tag{8}$$

Similarly, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{*}{,} A_\nu]$, the field strength, has the transformation law $\delta_\lambda F_{\mu\nu} = i[\lambda \stackrel{*}{,} F_{\mu\nu}]$. Therefore, in principle, starting from any commutative gauge theory, we could construct a noncommutative one by means of the substitution of the usual product of functions by the Moyal product. The invariant action for the gauge sector is

$$S = \int d^4x F^{\mu\nu} * F_{\mu\nu}.$$

However, for a Lie group $\mathbf{G}$, with Lie algebra $\mathcal{G}$ generated by $\{T_a\}$, which satisfies $[T_a, T_b] = f_{ab}{}^c T_c$, for the noncommutative fields we have

$$\begin{aligned}[\lambda \stackrel{*}{,} A_\mu] &= \left(\lambda^a * F^b_{\mu\nu} - F^a_{\mu\nu} * \lambda^b\right) T_a T_b \\ &= \frac{1}{2}\left(\lambda^a * F^b_{\mu\nu} + F^b_{\mu\nu} * \lambda^a\right)[T_a, T_b] \\ &+ \frac{1}{2}\left(\lambda^a * F^b_{\mu\nu} - F^b_{\mu\nu} * \lambda^a\right)\{T_a, T_b\},\end{aligned} \tag{9}$$

where $A_\mu = A^a_\mu T_a$ and $\lambda = \lambda^a T_a$. These gauge transformations will generate, for the gauge fields, components in the enveloping algebra $\mathcal{U}$ of $\mathcal{G}$, which is the algebra obtained from all the products of $\mathcal{G}$, that is, by taking besides the commutators, the anticommutators, as can be seen from

$$T_a T_b = \frac{1}{2}[T_a, T_b] + \frac{1}{2}\{T_a, T_b\}.$$

Thus, the enveloping algebra can be obtained by repeatedly computing all commutators and anticommutators, until it closes. That is, its general form will be

$$[T_I, T_J] = i f_{IJ}{}^K T_K, \quad \{T_I, T_J\} = d_{IJ}{}^K T_K. \tag{10}$$

In particular, the Lie algebras of the groups $U(n)$, in the fundamental representation, coincide with their enveloping algebras. In fact, this construction depends on the representation. For instance, for $SU(2)$ in the fundamental representation, the generators are the Pauli matrices, which satisfy $[\sigma_a, \sigma_b] = i\varepsilon_{abc}\sigma_c$ and $\{\sigma_a, \sigma_b\} = 2\delta_{ab} I$. Thus, the enveloping algebra contains the unity matrix besides the Pauli matrices, and corresponds to $U(2)$. For the vector representation, the generators are $(T_a)_b{}^c = i\varepsilon_{ab}{}^c$, and it can be easily shown that its enveloping algebra is given by $U(3)$.

Therefore, the number of degrees of freedom of a noncommutative theory will increase. However, the number of gauge parameters will increase, which means that it will also be possible to gauge away at least one part of the new degrees of freedom. For the particular case of noncommutative gauge theories constructed through the Seiberg-Witten map from a usual (commutative) gauge theory, the number of degrees of freedom are the same.

3.2. The Seiberg-Witten map

If we go back to string theory, we have seen that there must be a relation between the usual gauge theories and the noncommutative ones, given by the Seiberg-Witten map

$$\widehat{A}_\mu(A + \delta_\lambda A) = \widehat{A}_\mu(A) + \widehat{\delta}_{\widehat{\lambda}} \widehat{A}_\mu(A), \tag{11}$$

that is,

$$\delta_\lambda \widehat{A}_\mu(A) = \widehat{\delta}_{\widehat{\lambda}} \widehat{A}_\mu. \tag{12}$$

This relation can be generalized for any linear representation, by $\delta_\lambda \widehat{\Phi}(\Phi, A) = \widehat{\delta}_{\widehat{\lambda}} \widehat{\Phi}(\Phi, A) = i\widehat{\lambda} * \widehat{\Phi}(\Phi, A)$, or for the adjoint representation, $\delta_\lambda \widehat{\Phi}(\Phi, A) = \widehat{\delta}_{\widehat{\lambda}} \widehat{\Phi}(\Phi, A) = i[\widehat{\lambda} \stackrel{*}{,} \widehat{\Phi}(\Phi, A)]$.

The Seiberg-Witten map permits the formulation of noncommutative gauge theories, with an explicit dependence on the commutative fields and their derivatives, without the need to include new degrees of freedom, by means of a sort of effective theory. If we have a noncommutative gauge theory, constructed by means of the Seiberg-Witten map, we will have

$$\delta_\lambda S = \widehat{\delta}_{\widehat{\lambda}} S = 0. \tag{13}$$

That is, the invariance of this action can be seen in terms of the noncommutative fields, under the noncommutative gauge transformations (8), as well as directly in terms of the transformations of the commutative fields, on which the noncommutative ones depend in a complicated nonlinear way, by the Seiberg-Witten map.

This map can be written for any gauge group [7], and its solution can be obtained iteratively. In order to do that, the fields are written in a power series expansion in the noncommutativity parameters,

$$\widehat{A} = A + \theta^{\mu\nu} A^{(1)}_{\mu\nu} + \theta^{\mu\nu}\theta^{\rho\sigma} A^{(2)}_{\mu\nu\rho\sigma} + \cdots. \tag{14}$$

As a first step, the map for the transformation parameters is obtained. In general, the commutative parameters in a linear representation $\delta_\alpha \Phi = i\alpha\Phi = i\alpha^a T_a \Phi$ satisfy the following consistency conditions:

$$[\delta_\alpha, \delta_\beta]\Phi = -[\alpha, \beta]\Phi = \delta_{i[\alpha,\beta]}\Phi. \tag{15}$$

It can be seen that the noncommutative parameters must depend on the gauge fields, $\widehat{\lambda} = \widehat{\lambda}(\lambda, A)$; that is, the previous condition is in the noncommutative case given by

$$\begin{aligned}\delta_\alpha \delta_\beta \widehat{\Phi} &= \delta_\alpha\left(\widehat{\delta}_{\widehat{\beta}} \widehat{\Phi}\right) = i\delta_\alpha \widehat{\lambda}(\beta) * \widehat{\Phi} + i\widehat{\lambda}(\beta) * \delta_\alpha \widehat{\Phi} \\ &= i\delta_\alpha \widehat{\lambda}(\beta) * \widehat{\Phi} - \widehat{\lambda}(\beta) * \widehat{\lambda}(\alpha)\widehat{\Phi};\end{aligned} \tag{16}$$

that is,

$$[\delta_\alpha, \delta_\beta]\widehat{\Phi} = (i\delta_\alpha\widehat{\lambda}(\beta) - i\delta_\beta\widehat{\lambda}(\alpha) + [\widehat{\lambda}(\alpha) \stackrel{*}{,} \widehat{\lambda}(\alpha)]) * \widehat{\Phi} = \delta_{i[\alpha,\beta]}\widehat{\Phi}, \quad (17)$$

from which we get the transformation law for the noncommutative parameters,

$$i\delta_\alpha\widehat{\lambda}(\beta) - i\delta_\beta\widehat{\lambda}(\alpha) + [\widehat{\lambda}(\alpha) \stackrel{*}{,} \widehat{\lambda}(\alpha)] = \widehat{\lambda}(i[\alpha,\beta]). \quad (18)$$

In order to solve this equation, we write

$$\widehat{\lambda} = \lambda + \theta^{\mu\nu}\lambda^{(1)}_{\mu\nu} + \theta^{\mu\nu}\theta^{\rho\sigma}\lambda^{(2)}_{\mu\nu\rho\sigma} + \cdots,$$

which, substituted back into the preceding equation gives an equation for the first order term $\lambda^{(1)}_{\mu\nu}$, which has the solution [6],

$$\widehat{\lambda}(\lambda, A) = \lambda + \frac{1}{4}\theta^{\mu\nu}\{\partial_\mu\lambda, A_\nu\} + \mathcal{O}(\theta^2). \quad (19)$$

Now with this solution at hand, for the connection we proceed as well Eq. (14) substituted into (12), with the solution to first order given by,

$$\widehat{A}_\mu(A) = A_\mu - \frac{1}{4}\theta^{\rho\sigma}\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\} + \mathcal{O}(\theta^2). \quad (20)$$

Substitution of this equation into the field strength gives

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4}\theta^{\rho\sigma}\Big(2\{F_{\mu\rho}, F_{\nu\sigma}\} - \{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\}\Big) + \mathcal{O}(\theta^2). \quad (21)$$

For matter fields, the same procedure is followed; in this case, the map is given by

$$\widehat{\Phi} = \Phi + \frac{1}{2}\theta^{\mu\nu}\left(-A_\mu\partial_\nu\Phi + \frac{1}{2}A_\mu A_\nu\Phi\right).$$

However, for the adjoint representation, the equation to be solved is $\delta_\lambda\Phi = [\widehat{\lambda} \stackrel{*}{,} \widehat{\Phi}]$, and in this case the map is

$$\widehat{\Phi}(\Phi, A) = \Phi - \frac{1}{4}\theta^{\mu\nu}\{A_\mu, (D_\nu + \partial_\nu)\Phi\} + \mathcal{O}(\theta^2). \quad (22)$$

The higher order terms can be obtained in the same way, or by an equation given by Seiberg and Witten:

$$\frac{\partial}{\partial\theta^{\mu\nu}}\widehat{\Phi} = \widehat{\Phi^{(1)}}_{\mu\nu}, \quad (23)$$

where $\widehat{\Phi^{(1)}}$ is obtained from the first order term of the map by substituting the fields by their noncommutative counterparts, all of them multiplied by the Moyal product. For instance, for the gauge fields we have

$$\widehat{A}^{(1)}_\mu = -\frac{1}{4}\theta^{\rho\sigma}\left\{\widehat{A}_\rho \stackrel{*}{,} \partial_\sigma\widehat{A}_\mu + \widehat{F}_{\sigma\mu}\right\}. \quad (24)$$

Note that the general solution of the Seiberg-Witten map has an infinity of free parameters, and the solutions given here are the simplest ones, but not always the most suitable, depending on the problem [7]. Solutions (19) and (20) given by Seiberg and Witten have the nice property that the corrections to the field strength (21) vanish if the commutative field strength vanishes.

The higher terms in Eq. (14) can be obtained from the observation that the Seiberg-Witten map preserves the operations of the commutative function algebra; hence the following differential equation can be written [6]

$$\delta\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}\widehat{A}(\theta) = \delta\theta^{\mu\nu}\widehat{A^{(1)}_{\mu\nu}}(\theta), \quad (25)$$

where $\widehat{A^{(1)}_{\mu\nu}}$ is obtained from $A^{(1)}_{\mu\nu}$ in Eq. (14) by substituting the commutative fields by the noncommutative ones under the $*$-product.

4. Topological Gravity: Preliminaries

In this section, we briefly review four-dimensional topological gravity. Let R be the field strength, corresponding to a $SO(3,1)$ connection ω:

$$R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_\nu{}_c{}^b - \omega_\mu{}^{bc}\omega_\nu{}_c{}^a, \quad (26)$$

and let $\widetilde{R}$ be the dual of R with respect to the group (not with respect to spacetime) given by

$$\widetilde{R}_{\mu\nu}{}^{ab} = -\frac{i}{2}\varepsilon^{ab}{}_{cd}R_{\mu\nu}{}^{cd}. \quad (27)$$

We start from the following $SO(3,1)$ invariant action

$$I_{TOP} = \frac{\Theta^P_G}{2\pi}\mathrm{Tr}\int_X R\wedge R + i\frac{\Theta^E_G}{2\pi}\mathrm{Tr}\int_X R\wedge\widetilde{R}, \quad (28)$$

where X is a four-dimensional, closed, pseudo-Riemannian manifold and the coefficients are the gravitational analogs of the $\Theta-$vacuum in QCD [27–29].

In this action, the connection satisfies the first Cartan structure equation, which relates it to a given tetrad. This action can be written as the integral of a divergence, and a variation of it with respect to the tetrad vanishes; hence it is metric independent, and therefore topological.

The action (28) arises naturally from the MacDowell-Mansouri type action. A similar construction can be made for $(2+1)$-dimensional Chern-Simons gravity; keeping this philosophy in mind, action (28) can be rewritten in terms of the self-dual and anti-self-dual parts, $R^\pm = \frac{1}{2}(R \pm \widetilde{R})$, of the Riemann tensor as follows:

$$I_{TOP} = \mathrm{Tr}\int_X \left(\tau R^+\wedge R^+ + \overline{\tau}R^-\wedge R^-\right) = \mathrm{Tr}\int_X \left(\tau R^+\wedge R^+ + \overline{\tau}\overline{R^+}\wedge\overline{R^+}\right), \quad (29)$$

where

$$\tau = \left(\frac{1}{2\pi}\right)\left(\Theta_G^E + i\Theta_G^P\right),$$

and the bar denotes complex conjugation. In local coordinates on X, this action can be rewritten as

$$I_{TOP} = 2\mathrm{Re}\left(\tau \int_X d^4x\, \varepsilon^{\mu\nu\rho\sigma} R^{+\ ab}_{\mu\nu} R^{+}_{\rho\sigma ab}\right). \tag{30}$$

Therefore, it is sufficient to study the complex action

$$I = \int_X d^4x\, \varepsilon^{\mu\nu\rho\sigma} R^{+\ ab}_{\mu\nu} R^{+}_{\rho\sigma ab}. \tag{31}$$

Further, the self-dual Riemann tensor satisfies $\varepsilon^{ab}{}_{cd} R^{+\ cd}_{\mu\nu} = 2iR^{+\ ab}_{\mu\nu}$. This tensor has the useful property that it can be written as a usual Riemann tensor, but in terms of the self-dual components of the spin connection,

$$\omega^{+\ ab}_{\mu} = \frac{1}{2}\left(\omega^{ab}_{\mu} - \frac{i}{2}\varepsilon^{ab}{}_{cd}\omega^{cd}_{\mu}\right),$$

as

$$R^{+\ ab}_{\mu\nu} = \partial_\mu \omega^{+\ ab}_{\nu} - \partial_\nu \omega^{+\ ab}_{\mu} + \omega^{+\ ac}_{\mu}\omega^{+\ b}_{\nu\ c} - \omega^{+\ bc}_{\mu}\omega^{+\ a}_{\nu\ c}. \tag{32}$$

In this case, action (30) can be rewritten as

$$I = \int_X d^4x\, \varepsilon^{\mu\nu\rho\sigma}\left[2R_{\mu\nu}{}^{0i}(\omega^+)R_{\rho\sigma 0i}(\omega^+) + R_{\mu\nu}{}^{ij}(\omega^+)R_{\rho\sigma ij}(\omega^+)\right]. \tag{33}$$

Now, we define $\omega_\mu{}^i = i\omega^{+0i}_\mu$, from which we obtain, by means of the self-duality properties, $\omega^{+ij}_\mu = -\varepsilon^{ij}{}_k\omega_\mu{}^k$. Then it turns out that

$$R^{oi}_{\mu\nu}(\omega^+) = -i(\partial_\mu\omega^i_\nu - \partial_\nu\omega^i_\mu + 2\varepsilon^i_{jk}\omega^j_\mu\omega^k_{\nu c}) = -i\mathcal{R}_{\mu\nu}{}^i(\omega) \tag{34}$$

$$R^{ij}_{\mu\nu}(\omega^+) = \partial_\mu\omega^{+ij}_\nu - \partial_\nu\omega^{+ij}_\mu - 2(\omega_\mu{}^i\omega_\nu{}^j - \omega_\nu{}^i\omega_\mu{}^j) = -\varepsilon^{ij}{}_k\mathcal{R}_{\mu\nu}{}^k(\omega). \tag{35}$$

This amounts to the decomposition between the complex orthogonal Lie group $SO(3,1)$ and the product of two complex Lie groups $SL(2,\mathbb{C})$ given by the isomorphism $SO(3,1) \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, such that $\omega_\mu{}^i$ is a complex $SL(2,\mathbb{C})$ connection. If we choose the algebra $s\ell(2,\mathbb{C})$ to satisfy $[T_i,T_j] = 2i\varepsilon_{ij}^{\ k}T_k$ and $\mathrm{Tr}(T_iT_j) = 2\delta_{ij}$, then we can write

$$I = \mathrm{Tr}\int_X \widetilde{\mathcal{R}} \wedge \star\widetilde{\mathcal{R}} = \mathrm{Tr}\int_X d^4x\, \varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu}(\omega)\mathcal{R}_{\rho\sigma}(\omega), \tag{36}$$

where, $\mathcal{R}_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu,\omega_\nu]$ is the field strength, $\star$ is the usual Hodge star operation with respect to the underlying spacetime metric, $\mathcal{R}$ is the two-form field strength, and $\widetilde{\mathcal{R}}$ is the dual of $\mathcal{R}$ with respect to the group. This action is invariant under the $SL(2,\mathbb{C})$ transformations, $\delta_\lambda\omega_\mu = \partial_\mu\lambda + i[\lambda,\omega_\mu]$.

In the case of a Riemannian manifold X, the signature and the Euler topological invariants of X are the real and imaginary parts of (36):

$$\sigma(X) = -\frac{1}{24\pi^2}\mathrm{Re}\left(\mathrm{Tr}\int_X d^4x\, \varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu}(\omega)\mathcal{R}_{\rho\sigma}(\omega)\right), \tag{37}$$

$$\chi(X) = \frac{1}{32\pi^2}\mathrm{Im}\left(\mathrm{Tr}\int_X d^4x\, \varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu}(\omega)\mathcal{R}_{\rho\sigma}(\omega)\right). \tag{38}$$

5. Noncommutative topological gravity

We wish to have a noncommutative formulation of the $SO(3,1)$ action (28). Its first term can straightforwardly made noncommutative, in the same way as for the usual Yang-Mills theory,

$$\mathrm{Tr}\int_X \widehat{R} \wedge \widehat{R}. \tag{39}$$

If the $SO(3,1)$ generators are chosen to be hermitian, for example in the spin $1/2$ representation given by $\gamma^{\mu\nu}$, then from the discussion at the end of the second section, it turns out that $\widehat{R}_{\mu\nu}$ is hermitian and consequently (39) is real.

If we now turn to the second term of (28), such an action cannot be written, because it involves the Levi-Civita symbol, an invariant Lorentz tensor, but which is not invariant under the full enveloping algebra. However, as mentioned at the end of the preceding section, this term can be obtained from Eq. (36).

Thus, in general we will consider as the noncommutative topological action of gravity, the $SL(2,\mathbb{C})$ invariant action,

$$\widehat{I} = \mathrm{Tr}\int_X d^4x\, \varepsilon^{\mu\nu\rho\sigma}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}_{\rho\sigma}, \tag{40}$$

where $\widehat{\mathcal{R}}_{\mu\nu} = \partial_\mu\widehat{\omega}_\nu - \partial_\nu\widehat{\omega}_\mu - i[\widehat{\omega}_\mu \overset{*}{,} \widehat{\omega}_\nu]$ is the $SL(2,\mathbb{C})$ noncommutative field strength. This action does not depend on the metric of X. Indeed, as well as the commutative one, it is given by a divergence,

$$\widehat{I} = \mathrm{Tr}\int_X d^4x\, \varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\widehat{\omega}_\nu * \partial_\rho\widehat{\omega}_\sigma + \frac{2}{3}\widehat{\omega}_\nu * \widehat{\omega}_\rho * \widehat{\omega}_\sigma\right). \tag{41}$$

Thus, a variation of (40) with respect to the noncommutative connection will vanish identically because of the noncommutative Bianchi identities,

$$\delta_{\widehat{\omega}}\widehat{I} = 8\mathrm{Tr}\int \varepsilon^{\mu\nu\rho\sigma}\delta\widehat{\omega}_\mu * \widehat{D}_\nu\widehat{R}_{\rho\sigma} \equiv 0, \tag{42}$$

where $\widehat{D}_\mu$ is the noncommutative covariant derivative.

At this stage, we can make use of the first Cartan structure equation; then the SO(3,1) connection, and thus its $SL(2,\mathbb{C})$ projection ω_μ^i, can be written in terms of the tetrad and the torsion. Furthermore, from the Seiberg-Witten map, the noncommutative connection can be written as well as $\widehat{\omega}(e)$. Therefore, a variation of the action (40) with respect to the tetrad of the action, can be written as

$$\delta_e \widehat{I} = 8\mathrm{Tr}\int \varepsilon^{\mu\nu\rho\sigma}\delta_e\widehat{\omega}_\mu(e) * \widehat{D}_\nu \widehat{R}_{\rho\sigma} \equiv 0; \quad (43)$$

hence it is topological, as the commutative one.

As we will show later, the explicit expansion of the action (40) in the noncommutative parameter θ, gives terms that one does not expect to vanish identically. Thus, we see from (41) that, in a $\theta-$power expansion of the action, each one of the resulting terms will be independent of the metric, as well as being be given by a divergence. Therefore, these terms will be topological. (For the case of Euler characteristic, compare with the noncommutative nontrivial generalization of it given by Connes in pp. 64-69 from Ref. 2).

Furthermore, the whole noncommutative action, expressed in terms of the commutative fields by the Seiberg-Witten map, is invariant under the SO(3,1) transformations. Thus, each term of the expansion will also be invariant. Thus these terms will be topological invariants.

The action (40) is not real, as well as the limiting commutative action. Hence, it is not obvious that the signature (39) will be precisely its real part. In this case, we could not say that $\widehat{\chi}(X)$ is given by its imaginary part. In fact we could only say that $\widehat{\chi}(X)$ could be obtained from the difference of (40) and (39). However, the real and the imaginary parts of (40) are invariant under SL$(2,\mathbb{C})$ and consequently under SO(3,1), and thus they are the natural candidates for $\widehat{\sigma}(X)$ and $\widehat{\chi}(X)$, as in (37) and (38). In order to write down these noncommutative actions as an expansion in θ, we will take as generators for the algebra of $SL(2,\mathbb{C})$, the Pauli matrices. In this case, to the second order in θ, the Seiberg-Witten map for the Lie algebra valued commutative field strength $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\nu}{}^i(\omega)\sigma_i$, is given by

$$\widehat{\mathcal{R}}_{\mu\nu} = \mathcal{R}_{\mu\nu} + \theta^{\alpha\beta}\mathcal{R}^{(1)}_{\mu\nu\alpha\beta} + \theta^{\alpha\beta}\theta^{\gamma\delta}\mathcal{R}^{(2)}_{\mu\nu\alpha\beta\gamma\delta} + \cdots , \quad (44)$$

where, from Eq. (21) we get

$$\theta^{\rho\sigma}\mathcal{R}^{(1)}_{\mu\nu\rho\sigma} = \frac{1}{2}\theta^{\rho\sigma}\big[2\mathcal{R}_{\mu\rho}{}^i\mathcal{R}_{\nu\sigma i} - \omega_\rho{}^i\left(\partial_\sigma\mathcal{R}_{\mu\nu i} + D_\sigma\mathcal{R}_{\mu\nu i}\right)\big]\mathbf{1}, \quad (45)$$

where $\mathbf{1}$ is the unity 2×2 matrix. Further, by means of Eq. (25), we get,

$$\begin{aligned}\theta^{\rho\sigma}\theta^{\tau\theta}\mathcal{R}^{(2)}_{\mu\nu\rho\sigma\tau\theta} = \frac{1}{4}\theta^{\rho\sigma}\theta^{\tau\theta}\Big(&\varepsilon^i_{jk}\big[i\partial_\tau\mathcal{R}^j_{\mu\rho}\partial_\theta\mathcal{R}^k_{\nu\sigma} + \partial_\tau\omega^j_\rho\partial_\theta(\partial_\sigma + D_\sigma)\mathcal{R}^k_{\mu\nu}\big] - \omega^i_\rho\partial_\tau\omega^j_\sigma\partial_\theta\mathcal{R}_{\mu\nu j}\\ &+ \mathcal{R}^i_{\mu\rho}[2\mathcal{R}^j_{\nu\tau}\mathcal{R}_{\sigma\theta j} - \omega^j_\tau(\partial_\theta + D_\theta)\mathcal{R}_{\nu\sigma j}] - \mathcal{R}^i_{\nu\rho}\big[2\mathcal{R}^j_{\mu\tau}\mathcal{R}_{\sigma\theta j} - \omega^j_\tau(\partial_\theta + D_\theta)\mathcal{R}_{\mu\sigma j}\big]\\ &+ \frac{1}{2}\omega^j_\tau(\partial_\theta\omega_{\rho j} + \mathcal{R}_{\theta\rho j})(\partial_\sigma + D_\sigma)\mathcal{R}^i_{\mu\nu} - 2\omega^i_\rho\left\{2\partial_\sigma\mathcal{R}^j_{\mu\tau}\mathcal{R}_{\nu\theta j} - \partial_\sigma[\omega^j_\tau(\partial_\theta + D_\theta)\mathcal{R}_{\mu\nu j}]\right\}\Big)\sigma_i. \end{aligned} \quad (46)$$

Therefore, to the second order in θ, the action (40) will be given by,

$$\widehat{I} = \mathrm{Tr}\int_X d^4x\,\varepsilon^{\mu\nu\rho\sigma}\Big[\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma} + 2\theta^{\tau\vartheta}\mathcal{R}_{\mu\nu}\mathcal{R}^{(1)}_{\rho\sigma\tau\vartheta} + \theta^{\tau\theta}\theta^{\vartheta\zeta}\left(2\mathcal{R}_{\mu\nu}\mathcal{R}^{(2)}_{\rho\sigma\tau\theta\vartheta\zeta} + \mathcal{R}^{(1)}_{\mu\nu\tau\theta}\mathcal{R}^{(1)}_{\rho\sigma\vartheta\zeta}\right)\Big]. \quad (47)$$

Taking into account (45), we get that the first order term is proportional to Tr(σ_i) and thus vanishes identically. Further using (46), we finally get

$$\begin{aligned}\widehat{I} = \int_X d^4x\,\varepsilon^{\mu\nu\rho\sigma}\Big\{&2\mathcal{R}^i_{\mu\nu}\mathcal{R}_{\rho\sigma i} + \frac{1}{4}\theta^{\tau\theta}\theta^{\vartheta\zeta}\Big[-\varepsilon_{ijk}R^i_{\mu\nu}\big[\partial_\vartheta R^j_{\rho\tau}\partial_\zeta R^k_{\sigma\theta} - \partial_\vartheta\omega^j_\tau\partial_\zeta(\partial_\theta + D_\theta)R^k_{\rho\sigma}\big]\\ &+ [R^i_{\mu\tau}R_{\nu\theta i} - \frac{1}{2}\omega^i_\tau(\partial_\theta + D_\theta)R_{i\mu\nu}][R^j_{\rho\vartheta}R_{\sigma\zeta j} - \frac{1}{2}\omega^j_\vartheta(\partial_\zeta + D_\zeta)R_{\rho\sigma j}]\\ &+ R^i_{\mu\nu}\big\{R_{i\sigma\theta}[2R^j_{\rho\vartheta}R_{\tau\zeta j} - \omega^j_\vartheta(\partial_\zeta + D_\zeta)R_{\rho\tau j}] + \frac{1}{4}(\partial_\theta + D_\theta)R_{\rho\sigma i}\omega^j_\vartheta(\partial_\zeta\omega_{\tau j} + R_{\zeta\tau j})\\ &+ \omega_{\theta i}[\partial_\tau(R^j_{\rho\vartheta}R_{\sigma\zeta j}) - \frac{1}{2}\partial_\tau\omega^j_\vartheta(\partial_\zeta + D_\zeta)R_{\rho\sigma j}]\big\} - \frac{1}{2}R^i_{\mu\nu}\omega_{\tau i}\partial_\vartheta\omega^j_\theta\partial_\zeta R_{\rho\sigma j}\Big]\Big\}, \end{aligned} \quad (48)$$

where the second order correction does not identically vanish.

Similarly to the second order term (46), the third order term for $\widehat{\mathcal{R}}$ can be computed by means of Eq. (25). The result is given by a rather long expression, which however is proportional to the unity matrix $\mathbf{1}$, like (45). Thus the third order term in (47), given by

$$2\theta^{\tau_1\theta_1}\theta^{\tau_2\theta_2}\theta^{\tau_3\theta_3}\mathrm{Tr}\int_X \varepsilon^{\mu\nu\rho\sigma}\left(\mathcal{R}_{\mu\nu}\mathcal{R}^{(3)}_{\rho\sigma\tau_1\theta_1\tau_2\theta_2\tau_3\theta_3} + \mathcal{R}^{(1)}_{\mu\nu\tau_1\theta_1}\mathcal{R}^{(2)}_{\rho\sigma\tau_2\theta_2\tau_3\theta_3}\right), \quad (49)$$

vanishes identically, because $\mathcal{R}^{(2)}$ is proportional to σ_i. Thus, (48) is valid to the third order. In fact, it seems that all its odd order terms vanish.

6. Self-dual Gravity: An Overview

In this section, we briefly overview the formulation of Self-dual gravity in the approach by Plebański [23] and of great interest in the Ashtekar's hamiltonian description of Self-dual gravity [24] and in loop quantum gravity (for a review, see [25]).

One of the main features of the tetrad formalism of the theory of gravitation is that it introduces local Lorentz SO(3,1) transformations. In this case, the generalized Hilbert-Palatini formulation is written as

$$\int e_a^{\ \mu} e_b^{\ \nu} R_{\mu\nu}^{\ \ ab}(\omega) d^4x,$$

where $e_a^{\ \mu}$ is the inverse tetrad, and $R_{\mu\nu}^{\ \ ab}(\omega)$ is the so(3,1) valued field strength. The decomposition of the complex Lorentz group as SO(3,1)=SL$(2,\mathbb{C})\otimes$SL$(2,\mathbb{C})$, and the geometrical structure of four-dimensional space-time, makes it possible to formulate gravitation as a complex theory, as in Refs. 23 and 24. These formulations take advantage of the properties of the fundamental or spinorial representation of SL$(2,\mathbb{C})$, which allows a simple separation of the action in the fields of both factors of SO(3,1), as shown in great detail in Ref. 23. All the Lorentz Lie algebra-valued quantities, in particular the connection and the field strength, decompose into the self-dual and anti-self-dual parts, in the same way as the Lie algebra so(3,1)=s$\ell(2,\mathbb{C})\oplus$s$\ell(2,\mathbb{C})$. However, Lorentz vectors, like the tetrad, transform under mixed transformations of both factors and so this formulation cannot be written as a chiral SL$(2,\mathbb{C})$ theory. Various proposals in this direction have been made (for a review, see [25]). In an early formulation, this problem was solved by Plebański [23], where by means of a constrained Lie algebra valued two-form Σ, the theory can be formulated as a chiral SL$(2,\mathbb{C})$ invariant BF-theory,

$$Tr\int \Sigma\wedge R(\omega).$$

In this formulation, Σ has two SL$(2,\mathbb{C})$ spinorial indices, and it is symmetric on them, $\Sigma^{AB}=\Sigma^{BA}$, as any such s$\ell(2,\mathbb{C})$ valued quantity. The constraints are given by

$$\Sigma^{AB}\wedge\Sigma^{CD}=\frac{1}{3}\delta^C_{(A}\delta^D_{B)}\Sigma^{EF}\wedge\Sigma_{EF}$$

and, as shown in Ref. 23, their solution implies the existence of a tetrad one-form, which squared gives the two-form Σ. In the language of SO(3,1), this two-form is a second rank, antisymmetric, self-dual two-form, $\Sigma^{+ab}=\Pi^{+ab}_{\ \ \ cd}\Sigma^{cd}$, where

$$\Pi^{+ab}_{\ \ \ cd}=\frac{1}{4}\left(\delta^{ab}_{cd}-i\varepsilon^{ab}_{\ \ cd}\right).$$

In this case, the constraints can be recast into the equivalent form

$$\Sigma^{+ab}\wedge\Sigma^{+cd}=-\frac{1}{3}\Pi^{+abcd}\Sigma^{+ef}\wedge\Sigma^{+}_{\ ef},$$

with solution

$$\Sigma^{ab}=2e^a\wedge e^b.$$

For the purpose of the noncommutative formulation, we will consider self-dual gravity in a somewhat different way as in the papers [23,24]. In this section we will fix our notations and conventions.

We start from the SL$(2,\mathbb{C})$ complex *BF*-action

$$I=-4i\,Tr\int B\wedge F, \quad (50)$$

where $B=B_iT^i$ and the connection $\Omega=\Omega_iT^i$, is $s\ell$(2,C) Lie algebra-valued and $F=d\Omega+\Omega^2$ is the field strength. We choose a hermitian representation for the algebra generators such that $[T^i,T^j]=2i\varepsilon^{ij}_{\ \ k}T^k$ and $Tr(T^iT^j)=2\delta^{ij}$. Further, we decompose the fields into their real and imaginary parts, $\Omega=(1/2)(\omega+i\widetilde{\omega})$, and $B=(1/2)(\Sigma+i\widetilde{\Sigma})$. Thus we get $F=(1/2)(R+i\widetilde{R})$, where

$$R_{\mu\nu}^{\ \ \ i}=\partial_\mu\omega^i_\nu-\partial_\nu\omega^i_\mu-\varepsilon^i_{\ jk}(\omega^j_\mu\widetilde{\omega}^k_\nu+\widetilde{\omega}^j_\mu\omega^k_\nu), \quad (51)$$

$$\widetilde{R}^i_{\mu\nu}=\partial_\mu\widetilde{\omega}^i_\nu-\partial_\nu\widetilde{\omega}^i_\mu-\varepsilon^i_{\ jk}(\widetilde{\omega}^j_\mu\widetilde{\omega}^k_\nu+\omega^j_\mu\omega^k_\nu). \quad (52)$$

For the action we get,

$$I=-2i\int\left[\Sigma^iR_i-\widetilde{\Sigma}^i\widetilde{R}_i+i(\Sigma^i\widetilde{R}_i+\widetilde{\Sigma}^iR_i)\right]. \quad (53)$$

Now let us define $\omega^{ij}=\varepsilon^{ij}_{\ \ k}\widetilde{\omega}^k$ and $R_{\mu\nu}^{\ \ \ ij}=\varepsilon^{ij}_{\ \ k}\widetilde{R}^k_{\mu\nu}$. In this case we get,

$$R_{\mu\nu}^{\ \ \ ij}=\partial_\mu\omega_\nu^{\ ij}+\omega_\mu^{\ il}\omega_{\nu l}^{\ \ j}+\omega_\mu^{\ i}\omega_\nu^{\ j}-(\mu\leftrightarrow\nu). \quad (54)$$

Further, if we define $\omega_\mu^{\ 0i}=-\omega_\mu^{\ i0}=\omega_\mu^{\ i}$, $\omega_\mu^{\ 00}=0$ and $R_{\mu\nu}^{\ \ \ 0i}=R_{\mu\nu}^{\ \ \ i}$, then after putting together (51) and (54) we get,

$$R_{\mu\nu}^{\ \ \ ab}=\partial_\mu\omega_\nu^{\ ab}-\partial_\nu\omega_\mu^{\ ab}+\omega_\mu^{\ ac}\omega_{\nu c}^{\ \ b}-\omega_\nu^{\ ac}\omega_{\mu c}^{\ \ b}, \quad (55)$$

where the indices a,b,c run from 0 to 3. In this case, if we define also

$$\Sigma^{0i}_{\mu\nu}=-\Sigma^{i0}_{\mu\nu}=\Sigma^i_{\mu\nu},$$

it turns out that

$$I=-\frac{i}{2}\int \varepsilon^{\mu\nu\rho\sigma}\left(-\Sigma^{ab}_{\mu\nu}R_{\rho\sigma ab}+\frac{i}{2}\varepsilon_{abcd}\Sigma^{ab}_{\mu\nu}R_{\rho\sigma}{}^{cd}\right). \quad (56)$$

If we now vary over Ω in (50), we have

$$\delta_\Omega I = 4\delta_\Omega Tr\int(-dB\Omega + B\Omega^2)$$
$$= -4Tr\int DB\delta\Omega = 0, \quad (57)$$

where $DB = dB + [\Omega, B]$ is the covariant derivative of the gauge group $\mathrm{SL}(2,\mathbb{C})$. This is a complex equation, whose real and imaginary parts vanish separatedly, giving us the following equations,

$$\varepsilon^{\mu\nu\rho\sigma}D_\mu\Sigma^{0i}_{\nu\rho} = 0, \quad (58)$$

and

$$\frac{1}{2}\varepsilon^i{}_{jk}\varepsilon^{\mu\nu\rho\sigma}D_\mu\Sigma^{jk}_{\nu\rho} = 0, \quad (59)$$

where

$$\varepsilon^{\mu\nu\rho\sigma}D_\mu\Sigma^{0i}_{\nu\rho} = \varepsilon^{\mu\nu\rho\sigma}\partial_\mu\Sigma^i_{\nu\rho} - \varepsilon^i{}_{jk}(\omega_\mu{}^j\widetilde{\Sigma}^k_{\nu\rho} + \widetilde{\omega}^j_\mu\Sigma_{\nu\rho}{}^k)$$

and

$$\frac{1}{2}\varepsilon^i{}_{jk}\varepsilon^{\mu\nu\rho\sigma}D_\mu\Sigma^{jk}_{\nu\rho} = \varepsilon^{\mu\nu\rho\sigma}\partial_\mu\widetilde{\Sigma}^i_{\nu\rho}$$
$$+ \varepsilon^i{}_{jk}(\omega_\mu{}^j\Sigma_{\nu\rho}{}^k - \widetilde{\omega}^j_\mu\widetilde{\Sigma}^k_{\nu\rho}).$$

That is, gathering all that, we get

$$\varepsilon^{\mu\nu\rho\sigma}D_\mu\Sigma^{ab}_{\nu\rho} = 0. \quad (60)$$

Let us now make the usual ansatz

$$\Sigma^{ab}_{\mu\nu} = e^a_\mu e^b_\nu - e^a_\nu e^b_\mu;$$

in this case, the action turns out to be

$$I = \int(\det e\, R_{\mu\nu}{}^{\mu\nu} + i\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\, d^4x. \quad (61)$$

Moreover, the Eq. (60) resulting from the variation of Ω turns into

$$\varepsilon^{\mu\nu\rho\sigma}\left(T_{\mu\nu}{}^a e^b_\rho - T_{\mu\nu}{}^b e^a_\rho\right) = 0, \quad (62)$$

from which it turns out that the torsion vanishes, $T_{\mu\nu}{}^\rho = 0$, resulting in the fact that the second term in (61) vanishes identically, due to the Bianchi identities. Moreover, we obtain the second Cartan structure equation,

$$\omega_{\mu\nu\rho} = \frac{1}{2}\big[e_{\mu a}(\partial_\nu e^a_\rho - \partial_\rho e^a_\nu) - e_{\nu a}(\partial_\rho e^a_\mu - \partial_\mu e^a_\rho)$$
$$- e_{\rho a}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)\big], \quad (63)$$

which, put into the action (61), gives us the Einstein-Hilbert action.

7. Noncommutative Self-dual Gravity

Now we intend to formulate the self-dual gravity from the above section on noncommutative grounds. The noncommutative action can be obtained following the Seiberg-Witten construction [6], further developed in [7]. The same symmetry principle is applied, however now with the field dependent enveloping algebra-valued transformation parameters, given the first order in the noncommutativity parameter by

$$\widehat{\lambda} = \widehat{\lambda}_A T^A = \lambda + \frac{1}{4}\theta^{\mu\nu}\{\partial_\mu\lambda, \Omega_\nu\} + \ldots. \quad (64)$$

In order to obtain the enveloping algebra, a choice of the representation of generators of $s\ell$(2,C) has to be made. If we take for instance the Pauli matrices σ^i, then the enveloping algebra will be given by the four generators

$$\{T^A\} = \{\sigma^0 = 1, \sigma^i\}.$$

Thus, the action, invariant under the noncommutative transformations,

$$\widehat{\delta}\widehat{\Omega}_\mu = \partial_\mu\widehat{\lambda} + [\widehat{\Omega}_\mu \stackrel{*}{,} \widehat{\lambda}], \quad (65)$$

$$\widehat{\delta}\widehat{B} = [\widehat{\lambda} \stackrel{*}{,} \widehat{B}], \quad (66)$$

will be,

$$\widehat{I} = -4iTr\int \widehat{B}\wedge\widehat{F}, \quad (67)$$

where $F_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - i[\Omega_\mu, \Omega_\nu]$ is the field strength and $\widehat{F}_{\mu\nu} = \partial_\mu\widehat{\Omega}_\nu - \partial_\nu\widehat{\Omega}_\mu - i[\widehat{\Omega}_\mu \stackrel{*}{,} \widehat{\Omega}_\nu]$ is the noncommutative field strength. The noncommutative fields $\widehat{B}$ and $\widehat{\Omega}$ are given by the Seiberg-Witten map,

$$\widehat{\Omega}_\mu(\Omega) = \Omega_\mu - \frac{1}{4}\theta^{\nu\rho}\{\Omega_\nu, \partial_\rho\Omega_\mu + F_{\rho\mu}\} + \mathcal{O}(\theta^2),$$
$$\widehat{B} = B - \frac{1}{2}\theta^{\mu\nu}\left([\Omega_\mu, \partial_\nu B] - \frac{i}{2}[\Omega_\mu, [\Omega_\nu, B]]\right)$$
$$+ \mathcal{O}(\theta^2), \quad (68)$$

Further, the action can be written as,

$$I = -4i\int \widehat{B}\wedge\widehat{F}$$
$$= -16i\int \varepsilon^{\mu\nu\rho\sigma}\widehat{B}^i_{\mu\nu}\left(\partial_\rho\widehat{\Omega}_{\sigma i} + i\varepsilon_{ijk}\widehat{\Omega}^j_\rho * \widehat{\Omega}^k_\sigma\right). \quad (69)$$

Although we are taking the commutative fields as the fundamental ones, the action is written in terms of the noncommutative ones. Furthermore, the relation between the commutative and the noncommutative degrees of freedom is one to one [6], so the variation of the action with respect to ω will be equivalent to the variation with respect to $\widehat{\Omega}$. We have

$$\delta_{\widehat{\Omega}} I = 16\int \varepsilon^{\mu\nu\rho\sigma}\bigg(\partial_\rho\widehat{B}^i_{\mu\nu}$$
$$+ i\varepsilon^i_{jk}\left(\widehat{\Omega}^j_\mu * \widehat{B}^k_{\nu\rho} + \widehat{B}^j_{\nu\rho} * \widehat{\omega}^k_\mu\right)\bigg) * \delta\widehat{\Omega}_{\sigma i}. \quad (70)$$

That is, we obtain the noncommutative version of (60) which is given by

$$\varepsilon^{\mu\nu\rho\sigma}\widehat{D}_\mu\widehat{B}=\varepsilon^{\mu\nu\rho\sigma} \times\left[\partial_\rho\widehat{\Sigma}^i_{\mu\nu}+i\varepsilon^i_{jk}\left(\widehat{\omega}^j_\mu*\widehat{\Sigma}^k_{\nu\rho}+\widehat{\Sigma}^j_{\nu\rho}*\widehat{\omega}^k_\mu\right)\right]=0, \quad (71)$$

which substituted back into the action, gives us

$$I=-16i\int\varepsilon^{\mu\nu\rho\sigma}\varepsilon^i{}_{jk}\widehat{\Omega}^j_\mu*\widehat{B}^k_{\nu\rho}*\widehat{\Omega}_{\sigma i}. \quad (72)$$

Equation (71) is covariant under the non-commutative transformations (68), which means that their Seiberg-Witten expansion will correspond to the one of a matter field, as the one of the field $\widehat{B}$. Moreover, as can be seen at the first order in θ in (68), if the commutative field vanishes, so will the noncommutative one, as can be also verified at all orders. Thus, it is enough to set equal to zero the $\theta=0$ part of (71), $\varepsilon^{\mu\nu\rho\sigma}D_\mu B=0$. In order to obtain noncommutative gravity, the expansion in real and imaginary parts of the commutative fields $B^i_{\mu\nu}$ and Ω^i_μ must be performed inside of the Seiberg-Witten map then the identification of these fields with the Lorentz covariant ones ${\omega_\mu}^{ab}$ and ${\Sigma_{\mu\nu}}^{ab}$, and then we must go back to the vierbein,

$${\Sigma_{\mu\nu}}^{ab}={e_\mu}^a{e_\nu}^b-{e_\nu}^a{e_\mu}^b.$$

After that we have the solution to the Eqs. (71), given by the connection ${\omega_\mu}^{ab}$ in terms of the vierbein (63).

We can now write (68) as

$$\widehat{\Omega}_\mu=\Omega_\mu+\theta^{\rho\sigma}\Omega^{(1)}_{\mu\rho\sigma}+\mathcal{O}\left(\theta^2\right),$$

and

$$\widehat{B}_{\mu\nu}=B_{\mu\nu}+\theta^{\rho\sigma}B^{(1)}_{\mu\nu\rho\sigma}+\mathcal{O}\left(\theta^2\right),$$

where, as well as for the commutative fields Ω_μ and $B_{\mu\nu}$, the corrections $\Omega^{(1)}_{\mu\rho\sigma}$ and $B^{(1)}_{\mu\nu\rho\sigma}$, depend on the vierbein and its derivatives.

At the first order, the action will be given by,

$$I=-16i\int\varepsilon^{\mu\nu\rho\sigma}\Bigg(\varepsilon_{ijk}B^i_{\nu\rho}\Omega^j_\mu\Omega^k_\sigma +\theta^{\tau\theta}Tr\left\{\Omega_\mu\left([B_{\nu\rho},\Omega^{(1)}_{\sigma\tau\theta}]+\Omega_\nu B^{(1)}_{\rho\sigma\tau\theta}\right)\right\}\Bigg)d^4x. \quad (73)$$

8. Final Comments

In this article, we have reviewed a noncommutative version for topological gravity with quadratic actions and noncommutative self-dual gravity discussed in Refs. 14 and 15. On the side of the noncommutative topological gravity, our proposal is based on the complex action (40), in terms of the self-dual and anti-self-dual connections, and from which we found that the noncommutative natural generalization of the (37) and Euler (38) topological invariants can be extracted. More precisely, it is shown that the corresponding noncommutative versions of signature and the Euler topological invariants are given by the real and imaginary parts of (40) respectively. This proposed action can be written as an $\mathrm{SL}(2,\mathbb{C})$ action, whose noncommutative counterpart can be obtained in the same way as in the Yang-Mills case, by means of the Seiberg-Witten map. We compute this action up to the third θ-order, and we obtain that the first and the third order vanish, but the second order is different from zero. The action to this order is given by (48). It seems that all odd θ-orders vanish identically. Thus we found that these natural generalizations for the topological invariants are modified non-trivially by the noncommutative deformation.

Some comments are in order: on a (commutative) Riemannian manifold, signature and Euler topological invariants characterize gravitational instantons. Thus the study of noncommutative topological invariants should allow us, through the Seiberg-Witten map, to deform gravitational instantons into noncommutative versions for them. In order to make explicit computations, specific gravitational (noncommutative) metrics need to be chosen. In this context, it seems necessary to implement a noncommutative formulation for dynamical gravity, following the lines of this work.

In this direction we have proposed a noncommutative generalization for the Plebanski-Ashtekar gravitation [15], which is invariant under the noncommutative gauge transformations. It is torsionless and can be written perturbatively in a straightforward way in terms of the vierbein and its derivatives. Still we have the challenge of constructing a noncommutative gravity closer to Einstein gravity and of providing predictions of this theory concerning the classical tests of general relativity, including black holes, cosmological solutions, gravitational waves and weak field approaches to this noncommutative Einstein theory. We would like to pursue some of these topics in the near future.

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