Time dependent problems in relativistic quantum mechanics

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The most effective procedure for dealing with time dependent problems is through the Feynman propagator. In this note we indicate the explicit expression for this propagator in relativistic problems using spectral decomposition. We then take as an initial state one of a Dirac oscillator and consider the behaviour of the wave function when the interaction is suddenly supressed.

Keywords: Transient phenomena; propagators; wave functions.

La manera más efectiva para analizar problemas dependientes del tiempo en mecánica cuántica es a través del propagador de Feynman. En esta nota indicamos la forma explícita de este propagador en problemas relativistas, usando la decomposición espectral. Posteriormente tomamos un estado del oscilador de Dirac como condición inicial y estudiamos el comportamiento de la función de onda cuando la interacción desaperece repentinamente.

Descriptores: Fenómenos transitorios; propagadores; funciones de onda.

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1. Introduction

From the beginning of quantum mechanics, in its Schroedinger or Heisenberg version, the formulation was dynamic in the sense that time dependent problems were also dicussed. Thus, if initially we had a definite wave function its evolution in time could be explicitly derived. Another dynamical version of quantum mechanics was suggested by Feynman in the nineteen forties, when he introduced the concept of propagator and a procedure to derive it. If we then have an initial wave packet, its evolution can be obtained with the help of such propagator. While this concept was used first in non relativistic quantum mechanics, it is possible to extend it to the relativistic case. We shall illustrate how to use it with the example of an initial stationary state of the Dirac oscillator and then seeing how it evolves if the oscillator interaction is supressed. We shall conclude by indicating how our procedure can be applied in general.

2. General Problem

As we mentioned in the introduction, we are interested in the time evolution of Dirac oscillator wave functions when the interaction is supressed at $t = 0$. Let H_{free} be the Dirac hamiltonian for a free particle. Consider a system described by a hamiltonian

$$H = H_{\text{free}} + \theta(-t)V \tag{1}$$

so that for negative times the system is in some bound state Ψ_n produced by the potential V and at $t = 0$ such interaction disappears suddenly. In order to study the behaviour of the initial condition at positive times we have but to employ the free propagator [1] corresponding to H_{free} to find

$$\Psi(\mathbf{r}, t) = \int d^3r' K_{\text{free}}(\mathbf{r}, \mathbf{r}'; t)\Psi_n(\mathbf{r}', 0) \tag{2}$$

Now, $K_{\text{free}}(\mathbf{r}, \mathbf{r}'; t)$ represents also the spatial matrix elements of the evolution operator $U = e^{-iH_{\text{free}}t}$ and therefore satisfies

$$\left(H_{\text{free}} - i\frac{\partial}{\partial t}\right) K_{\text{free}}(\mathbf{r}, \mathbf{r}'; t) = -i\delta^3(\mathbf{r} - \mathbf{r}')\delta(t) \tag{3}$$

furthermore, it is possible to write K_{free} in terms of the Feynman propagator for the Klein Gordon equation [2], *i.e.*

$$K_{\text{free}}(\mathbf{r}, \mathbf{r}'; t) = \left(H_{\text{free}} + i\frac{\partial}{\partial t}\right) K_{\text{KG}}(\mathbf{r}, \mathbf{r}'; t) \tag{4}$$

where K_{KG} can be given, for instance, in terms of plane waves

$$K_{\text{KG}}(\mathbf{r}, \mathbf{r}'; t) = \int d^4k \frac{e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')+ik_0 t}}{k_\mu k^\mu + m^2} \tag{5}$$

In order to restore Lorentz invariance of our K_{free} we observe that it satisfies the invariant equation

$$\left(-i\gamma^\mu \partial_\mu + m\right) \beta K_{\text{free}}(\mathbf{r} - \mathbf{r}'; t - t') = -i\delta^4(r_\mu - r'_\mu) \tag{6}$$

with $\beta = \gamma_0$. Therefore βK_{free} stands for the usual definition of the Dirac propagator of a free particle. Note, however, that β is a unitary matrix and any probability density we compute will be unaffected.

3. Free evolution of the Dirac oscillator ground state

Consider a Dirac oscillator of mass m and frequency ω and define $\lambda = \sqrt{m\omega}$ for later use. The wave functions of the Dirac Oscillator were obtained since the time it was proposed by Moshinsky and Szczepaniak [3] and then refined in [4] by showing their completeness. We extract from those references the expression for the ground state of the Dirac oscillator in spherical coordinates (meaning the lowest positive energy state with finite degeneracy)

$$\Psi_{0,1,\frac{1}{2}}(\mathbf{r},0)=\frac{1}{r}\begin{pmatrix} P_{01}(r)\Omega_{1\frac{1}{2}}(\hat{r}) \\ iQ_{01}(r)\Omega_{-1\frac{1}{2}}(\hat{r}) \end{pmatrix} \tag{7}$$

corresponding to principal number $n=0$, angular momentum $j=1/2$ and angular momentum projection $m_j=1/2$. The functions are given by

$$P_{01}=\sqrt{\frac{\lambda(E_{01}+m)}{\Gamma(5/2)E_{01}}}(\lambda r)^2 e^{-\lambda^2 r^2/2},$$

$$\Omega_{1\frac{1}{2}}=\begin{pmatrix} -\frac{1}{\sqrt{3}}Y_1^0 \\ \sqrt{\frac{2}{3}}Y_1^1 \end{pmatrix} \tag{8}$$

$$iQ_{01}=\sqrt{\frac{\lambda(E_{01}-m)}{\Gamma(3/2)E_{01}}}\lambda r e^{-\lambda^2 r^2/2},$$

$$\Omega_{-1\frac{1}{2}}=\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

with eigenvalue

$$E_{01}=m\sqrt{1+6\omega/m} \tag{10}$$

Let our initial condition be $\Psi(\mathbf{r})=\Psi_{0,1,1/2}(\mathbf{r},0)$. Since using (4,5) the free propagator of the Dirac equation can be written as

$$K_{\text{free}}(\mathbf{r},\mathbf{r}';t)=\left(-i\boldsymbol{\alpha}\cdot\nabla+m\beta-i\frac{\partial}{\partial t}\right)K_{\text{KG}}(\mathbf{r},\mathbf{r}';t) \tag{11}$$

equation (2) contains the three relevant integrals

$$I_1=\int_{-\infty}^{\infty}d^3r'\int_{-\infty}^{\infty}d^4k\frac{e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')+ik_0t-\lambda^2r'^2/2}}{k^2-k_0^2+m^2}, \tag{12}$$

from the first component of $\Omega_{-1,\frac{1}{2}}$,

$$I_2=\int_{-\infty}^{\infty}d^3r'\int_{-\infty}^{\infty}d^4k\frac{r'\cos\theta' e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')+ik_0t-\lambda^2r'^2/2}}{k^2-k_0^2+m^2}. \tag{13}$$

from P_{01} and the first component of $\Omega_{1,\frac{1}{2}}$,

$$I_3=\int_{-\infty}^{\infty}d^3r'\int_{-\infty}^{\infty}d^4k\frac{r'\sin\theta' e^{i\phi'}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')+ik_0t-\lambda^2r'^2/2}}{k^2-k_0^2+m^2} \tag{14}$$

from P_{01} and the second component of $\Omega_{1,1/2}$, and the multiplicative operator in parenthesis of (11) will be introduced later.

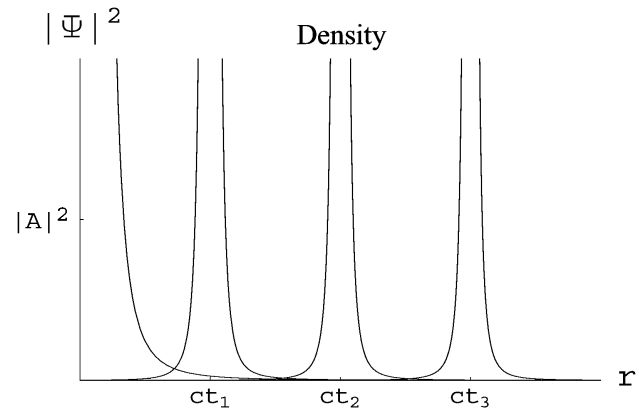

FIGURE 1. Wave function at different times.

It can be easily proven that

$$I_2=-\frac{1}{\lambda^2}\frac{\partial I_1}{\partial z},\qquad I_3=-\frac{1}{\lambda^2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)I_1$$

(see the appendix). Therefore we will focus on computing I_1.

From (12) we can see that integrating over $\mathbf{r}'$, k_0 (taking the positive energy pole) and the solid angle of $\mathbf{k}$ (in spherical coordinates) gives

$$I_1=\frac{(2\pi)^{5/2}}{\lambda^3 r}\int_0^{\infty}dk\frac{k\sin(kr)e^{-k^2/(2\lambda^2)-i\sqrt{k^2+m^2}t}}{\sqrt{k^2+m^2}} \tag{15}$$

At first sight this integral looks unfamiliar. Nevertheless we can expand the gaussian $e^{-k^2/(2\lambda^2)}$ in a power series of λ^{-1} in order to find an approximate result by truncating the series. Such an approximation corresponds to an initial wave function which is highly concentrated at the origin being the high order terms corrections to a delta distribution. We also notice that the parity of the integrand in (15) allows to extend the limits of integration from $-\infty$ to ∞

$$I_1=\frac{(2\pi)^{3/2}\pi}{\lambda^3 r}\times\sum_{n=0}^{\infty}\frac{(-)^n}{(2\lambda^2)^n n!}\int_{-\infty}^{\infty}dk\frac{k^{2n+1}\sin(kr)e^{-i\sqrt{k^2+m^2}t}}{\sqrt{k^2+m^2}} \tag{16}$$

A change of variables $\sqrt{k^2+m^2}=(m/2)(\zeta+\zeta^{-1})$, $k=(m/2)(\zeta-\zeta^{-1})$ and the integral representation for the modified Bessel function K [5]

$$K_\nu(xy)=\tfrac{1}{2}\left(\frac{y}{2}\right)^{\nu}\int_0^{\infty}d\zeta\frac{e^{-\frac{x}{2}(\zeta+y^2/\zeta)}}{\zeta^{\nu+1}} \tag{17}$$

gives the result

$$I_1=\frac{\pi^{5/2}m}{i\sqrt{2}r\lambda^3}\sum_{n=0}^{\infty}\left(\frac{-m^2}{2\lambda^2}\right)^n\times\sum_{l=0}^{2n+1}\frac{(-)^l(2n+1)!2^{2l}}{l!(2n+1-l)!}\frac{K_{2(l-n)-1}(im\sqrt{t^2-r^2})}{(t^2-r^2)^{l-n-1/2}}\times\left((t-r)^{2(l-n)-1}-(t+r)^{2(l-n)-1}\right) \tag{18}$$

The numerical evaluation of the complete wave function can be carried out to any order in λ^{-1}, but the most interesting case is taking only the first term in the series, which implies that the initial wave function is concentrated near the origin. In that case we find that only I_1 contributes with

$$I_1 \approx \frac{5\sqrt{2}\pi^{5/2}m}{i\lambda^3}\frac{K_1(im\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} \tag{19}$$

and the wave function at time t is given by

$$\Psi(\mathbf{r},t) = \frac{5\sqrt{2}\pi^{5/2}m}{i\lambda^{3/2}}\sqrt{\frac{E_{01}-m}{\Gamma(3/2)E_{01}}}\begin{pmatrix} -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} \\ -m+i\frac{\partial}{\partial t} \\ 0 \end{pmatrix} \times \frac{K_1(im\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} \tag{20}$$

Note that we have used units $c=1, \hbar=1$, but by restoring to c.g.s. we have the non relativistic limit when $E_{01} \ll mc^2$. From (20) the probability density can be directly computed. We have sketched Ψ at different times in the figure. Define A as the factor in front of the differential operator in the last expression

It is clearly observed that the integrity of the distribution is preserved around the light cone, time is "frozen" and there is no dissipation, contrary to what we see in the non relativistic case where the light cone is infinitely wide. The higher order corrections are anisotropic contributions to the evolution of the wave function and they deal with the non zero angular momentum. We have analyzed a specific relativistic problem but the procedure can be extended to any other case, though some times the integrals can not be evaluated analitically.

A Integrals

Let us prove the two identities

$$I_2 = -\frac{1}{\lambda^2}\frac{\partial I_1}{\partial z}, \quad I_3 = -\frac{1}{\lambda^2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)I_1.$$

The first follows from expressing I_2 cartesian coordinates

$$I_2 = \int_{-\infty}^{\infty} d^4k \frac{e^{-i\mathbf{k}\cdot\mathbf{r}+ik_0t}}{k^2-k_0^2+m^2}\int_{-\infty}^{\infty} dz' z' e^{ik_3z'-\lambda^2z'^2/2} \times \int_{-\infty}^{\infty} dx' e^{ik_1x'-\lambda^2x'^2/2}\int_{-\infty}^{\infty} dy' e^{ik_2y'-\lambda^2y'^2/2} \tag{21}$$

the spatial integrals are trivial and we obtain

$$I_2 = \int_{-\infty}^{\infty} d^4k \frac{e^{-i\mathbf{k}\cdot\mathbf{r}+ik_0t}}{k^2-k_0^2+m^2}\left(\frac{-i(2\pi)^{3/2}k_3}{\lambda^5}e^{-k^2/(2\lambda^2)}\right) \tag{22}$$

A comparison of (22) with (12) after performing the spatial integrals leads to the sought result. For I_3 we proceed analogously but using $r' sin\theta' e^{i\phi'} = x' + iy'$.

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