Propagation of linear MHD waves in a hydrogen plasma: the mode crossing problem

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Here we use linear analysis to investigate the propagation of small thermal and magnetohydrodynamic (MHD) disturbances in a heat-conducting, ionizing-recombining, hydrogen plasma threaded by an external uniform magnetic field. Linearization of the governing MHD equations for this model leads to a dispersion equation for the wavenumber k as a function of the frequency ω, which may be either quadratic or cubic in k^2, depending on the orientation of the magnetic field. In either case, the solution of the dispersion equation is such that crossing of the roots may happen at some frequencies, implying that they may not always correspond to the same particular physical wave. The crossing of modes is merely a mathematical property of the solution and must not be interpreted as an interchange of the thermal and MHD waves' physical nature at the crossing frequency. Here we find that mode crossing is a function of the wave frequency, plasma temperature, magnetic field strength and orientation.

Keywords: Magnetohydrodynamics and plasmas; magnetohydrodynamic waves; sound waves; wave propagation.

Mediante un análisis lineal se estudia la propagación de perturbaciones térmicas y magnetohidrodinámicas (MHD) en un plasma de hidrógeno sujeto a la acción de un campo magnético externo de intensidad uniforme, incluyendo los efectos de conducción de calor, fotoionización y fotorecombinación. A partir de la linealización de las ecuaciones MHD para este modelo se obtiene una ecuación de dispersión para el número de onda k en función de la frecuencia ω, que puede ser cuadrática o cúbica en k^2 dependiendo de la orientación del campo magnético. En ambos casos, la solución de la ecuación de dispersión es tal que las raices pueden cruzarse a determinadas frecuencias. De este modo, las raices no siempre corresponderán a la misma onda para todo el espectro de frecuencias. El cruce de modos es una propiedad matemática de la solución y no debe interpretarse como un intercambio de la naturaleza física de las ondas. Se encuentra que el cruce de modos es una función de la frecuencia, de la temperatura del plasma y de la orientación e intensidad del campo magnético.

Descriptores: Magnetohidrodinámica y plasmas; ondas magnetohidrodinámicas; ondas acústicas; propagación de ondas.

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1. Introduction

The study of the propagation and growth of small disturbances in an optically thin hydrogen plasma is fundamental for understanding the evolution of inhomogeneities appearing at different length and time scales in astrophysical plasmas. An approximate method for dealing with such problems relies on linear analysis, in which a perturbed equilibrium state is assumed and the gas-dynamic differential equations are solved by retaining only first-order terms. The linearization procedure leads to a complex polynomial (or dispersion equation) whose solution can be expressed in terms of the wavenumber k as a function of the frequency ω.

The propagation of linear sound and thermal waves in a heat-conducting hydrogen plasma, in which photoionization and photorecombination [$\mathrm{H}^+ + e^- \rightleftharpoons \mathrm{H} + h\nu(\chi)$, with χ denoting the ionization energy of hydrogen] processes are progressing was previously studied in Ref. 1. More recently, the same model was re-examined in Ref. 2, and extended in Ref. 3 to include the effects of an external uniform magnetic field in order to investigate the propagation of thermal and magnetohydrodynamic (MHD) waves for varying field strengths and orientations. Here we shall consider the same magnetized hydrogen model of [3] and derive from a heuristic point of view the dependence of mode crossing on the relevant physical parameters. For simplicity we shall assume that the magnetic field is oriented perpendicularly to the direction of wave propagation. In this case, the resulting dispersion equation is a complex quadratic polynomial in k^2 and its solution admits two distinct roots, say k_1 and k_2, which may cross at some given frequencies depending on the plasma temperature and field strength.

2. The Mode Crossing Problem

Linearization of the MHD equations for a heat-conducting, ionizing-recombining, hydrogen plasma threaded by an external magnetic field leads to a quadratic complex polynomial provided that the field is oriented perpendicularly to the direction of wave propagation ($\mathbf{B} \perp \mathbf{k}$). The same is also true in the limit of a vanishing magnetic field ($B_0 = 0$) [2]. Therefore, its solution admits two independent roots, say k_1 and k_2, as functions of the wave frequency ω. One of the roots corresponds to the thermal mode (k_T), while the other is a longitudinal magnetosonic wave propagating with a phase speed $v_{ms} = \sqrt{c^2 + v_{\mathrm{A}}^2}$, where $v_{\mathrm{A}} = B_0/\sqrt{4\pi\rho}$ is the Alfvén velocity and c denotes the sound speed. For the present model, c may be either the isothermal c_{iso} or isentropic c_{s} sound speed, where $c_{\mathrm{iso}}^2 = (\partial p/\partial\rho)_T = c_{\mathrm{s}}^2/\gamma$ (see Ref. 3). Note that in the absence of magnetic fields $v_{\mathrm{A}} = 0$, so the magnetosonic

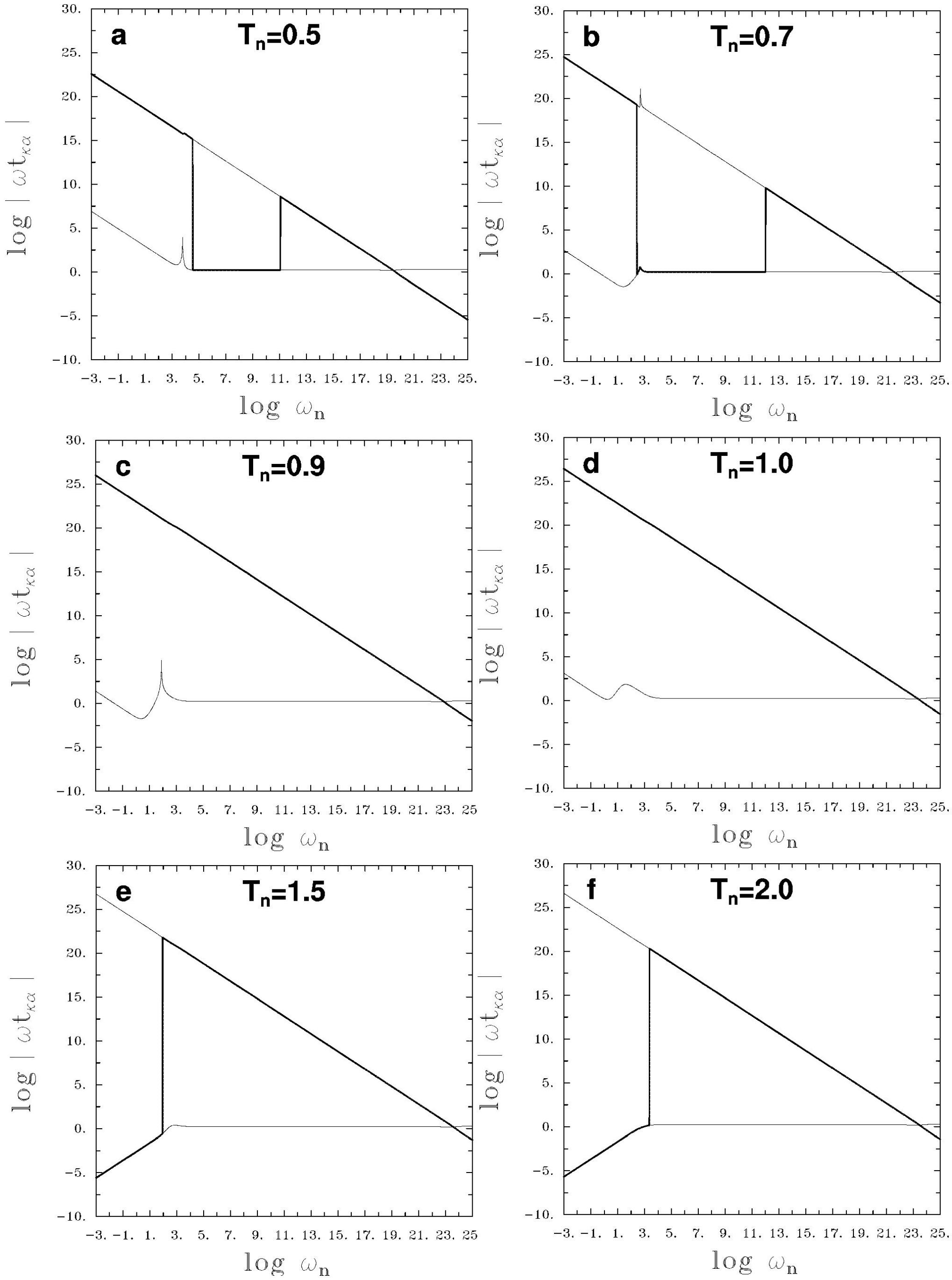

FIGURE 1. Logarithm of the absolute value of the dimensionless e-folding conducting times $\omega t_{\kappa T}$ (thin line) and $\omega t_{\kappa ms}$ (thick line) as functions of the normalized frequency ω_n for a sequence of temperatures from $T_n = 0.5$ (almost neutral hydrogen) to $T_n = 2.0$ (fully ionized hydrogen), when $\mathbf{B} \perp \mathbf{k}$ and $\beta \sim 1$. The vertical thick lines mark the crossing frequencies.

wave reduces to an ordinary sound wave. For a detailed account of the form of the MHD equations, the linearization procedure and assumptions made, the reader is referred to Ref. 3. A full description of the hydrogen plasma model and specialized expressions for the dissociation, recombination, and cooling rates can be found in Ref. 4.

For this specific model, the coefficients of the dispersion equation depends on the relevant physical parameters in such a way that crossing of the roots ($\mathrm{k}_1 \leftrightarrow \mathrm{k}_2$) may happen at some given frequency ω_{cross}. Thus, if at frequencies less than ω_{cross}, one root corresponds to the magnetosonic mode (k_{ms}) and the other to the thermal mode (k_T), we must then relabel the roots for all frequencies higher than the crossing value in order to get a correct representation of both waves. We call this mathematical property of the solution *mode crossing* and we emphasize that it should not be interpreted as an interchange of the thermal and magnetosonic waves' physical nature.

3. Dependence of Mode Crossing with Temperature and Field Strength

In order to keep consistency with the analysis of Refs. 2 and 3, we take an initial plasma temperature T_0 of $\approx$15062 K, as determined by assuming chemical equilibrium, a density ρ_0 of $4.12 \times 10^{-26} L_0$ g cm^{-5} s^3, fixed by assuming thermal equilibrium, and $\omega_0 = 1.5 \times 10^{-15} L_0$ cm^{-2} s^2, where $L_0 \approx 3.25 \times 10^{-4}$ cm^2 s^{-3} is the approximately constant galactic heating rate [5]. We further consider values of the initially uniform magnetic induction B_0 of 0, 2.5×10^{-8} and 2.0×10^{-6} gauss, corresponding to ratios $\beta = 6c_s^2/(5v_A^2)$ of the plasma pressure to the magnetic pressure of ∞ ($\beta \gg 1$), ≈ 1.0108 ($\beta \sim 1$) and $\approx 1.5795 \times 10^{-4}$ ($\beta \ll 1$), respectively. A value of 2.0×10^{-6} gauss is close to the strength of the interstellar magnetic field observed in our Galaxy [6].

The wave-frequency and plasma-temperature dependence of the phase velocity and scale-length for wave amplification (or damping) of the thermal and magnetosonic modes is fully described in reference [3] and will not be repeated here. The dependence of mode crossing on the relevant physical parameters is better illustrated in terms of the e-folding conducting times $\omega t_{\kappa T}$ and $\omega t_{\kappa ms}$ as defined in Ref. 3. Figure 1 shows these times as functions of the normalized frequency $\omega_n = \omega/\omega_0$, for a sequence of temperatures $T_n = T/T_0$, ranging from $T_n = 0.5$ (almost neutral hydrogen) to $T_n = 2.0$ (fully ionized hydrogen) when $B_0 = 2.5 \times 10^{-8}$ gauss (*i.e.*, $\beta \sim 1$). The vertical thick lines mark the exact frequency at which mode crossing occurs. We see that for this case mode crossing occurs only at temperatures where the plasma is weakly ($T_n = 0.5$ and 0.7) and highly ($T_n = 1.5$ and 2.0) ionized. At low frequencies, k_1 labels the thermal mode and k_2 the magnetosonic mode (Figs. 1a and b), with mode crossing ($\mathrm{k}_1 \leftrightarrow \mathrm{k}_2$) first occurring at $\log \omega_n = 4.54$ (for $T_n = 0.5$) and 2.43 (for $T_n = 0.7$). A second crossing happens at much higher frequencies: $\log \omega_n = 11.05$ and 12.01 for $T_n = 0.5$ and 0.7, respectively. Thus, rising the temperature in a weakly ionized plasma causes the first crossing to occur at lower frequencies and the second one at higher frequencies. At temperatures of $T_n = 0.9$ and 1.0, where the plasma is partially ionized, mode crossing is never seen to occur (see Figs. 1c and d). In contrast with the previous cases, when the plasma becomes highly ionized ($T_n = 1.5$ and 2.0), k_2 labels the thermal mode and k_1 the magnetosonic mode at very low frequencies (Figs. 1e and f). This time mode crossing occurs at $\log \omega_n = 1.95$ (for $T_n = 1.5$) and 3.36 (for $T_n = 2.0$), implying that the crossing frequency is shifted toward higher values as the temperature increases in a highly ionized plasma.

Similar trends to those shown in Fig. 1 are also seen in the limit when $B_0 \to 0$. In this case, for low-ionization temperatures, the first and second crossings occur at comparatively lower frequencies: $\log \omega_n = 4.29$ and 8.57 for $T_n = 0.5$ and $\log \omega_n = 2.14$ and 6.01 for $T_n = 0.7$. Evidently, decreasing the strength of the field causes mode crossing to occur at progressively lower frequencies in a weakly ionized plasma. However, the opposite is true at high-ionization temperatures, where crossing occurs at $\log \omega_n = 2.01$ for $T_n = 1.5$ and 3.45 for $T_n = 2.0$. In the opposite case of increased field strength ($B_0 = 2.0 \times 10^{-6}$ gauss), mode crossing is completely inhibited at low-ionization temperatures. Only for $T_n = 1.5$ and 2.0 mode crossing is observed at slightly lower frequencies ($\log \omega_n = 1.91$ and 3.12, respectively) compared to Figs. 1e and f. Thus, as the field strength increases mode crossing occurs at progressively lower frequencies in a highly ionized plasma. The picture of mode crossing described here applies only to the simple cases where $B_0 = 0$ and $\mathbf{B} \perp \mathbf{k}$. For arbitrary orientations of the field, including the case when $\mathbf{B} \parallel \mathbf{k}$, the dependence of mode crossing on the temperature and field strength complicates as the dispersion relation becomes cubic in k^2, admitting three distinct roots [3]. In this case, triple and double mode crossing occur for all plasma temperatures when $\beta \sim 1$ and only at high- and low-ionization temperatures as long as $\beta \ll 1$.

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