Symmetries of the Super Heat Kernel $N = 1$ and SKdV Hierarchy

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A new heat operator with $N = 1$ supersymmetry is proposed. We study the symmetries of the corresponding heat kernel , which generalizes the bosonic one in a natural way. We used these symmetries to obtain, in the asymptotic limit, the super KdV hierarchy. Finally, a supersymmetric tree version associated with this hierarchy is presented .

Keywords: Supersymmetry; supersymmetric models; symmetry and conservation laws.

Se propone un nuevo operador del calor con $N = 1$ supersimetrías. Se estudian las simetrías del núcleo del calor correspondiente, las cuales generalizan de manera natural la del caso bosónico. Usamos estas simetrías para obtener, en el límite asintótico, la jerarquía super KdV. Finalmente presentamos una versión diagramática asociada con esta jerarquía.

Descriptores: Supersimetrías; modelos supersimétricos; simetrías y leyes de conservación.

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1. The Super Heat Kernel N=1 and the SKdV hierarchy

The heat operator is highly relevant in quantum mechanics, in addition to its interest in thermodynamics. It is known that by considering the analytic continuation from $t \rightarrow it$, the heat equation with potential gives rise to the Schröedinger equation [1]. In particular, the rigorous treatment of the Feynman path integral follows for the heat equation and, thus by analytic continuation, the corresponding formula for the Schröedinger equation may be obtained. Supersymmetric quantum mechanical systems are relevant for several reasons, in particular at high energy physics. Then the corresponding analysis of the supersymmetric Green's function is of interest. However, there is no rigorous Feynman path integral formulation for supersymmetric models [2]. In this paper, by using the symmetries of the super heat kernel, we deduce from its asymptotic expansion when $t \rightarrow 0^+$ all the hierarchy of the Super KdV equation.

It is known that the Green's function $G_t(x - x')$ of the operator with potential $u(x)$:

$$L_u \equiv \partial_t - \Delta + u(x) \tag{1}$$

has an asymptotic expansion as $t \rightarrow 0^+$ of the form

$$G_t(x - x') = g_t(x - x') \sum_{k=0}^{\infty} \frac{1}{k!} a_k(x, x') t^k \tag{2}$$

where

$$g_t(x - x') = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4t}}$$

and $a_k(x, x')$ satisfy the recursive differential equations

$$\left(1 + \frac{1}{k+1} D\right) a_{k+1} = [\Delta - u(x)] a_k, \qquad k = 0, 1, \ldots \quad a_0 = 1, \tag{3}$$

in which

$$D = \sum_{p=1}^{n} (x_p - x'_p) \frac{\partial}{\partial x_p}.$$

The value of $a_k(x, x')$ when $x = x'$ is a finite polynomial in $u(x)$ and its derivatives. Passing from $\mathbb{R}^{n+1}$ to R^{1+1} we may write

$$a_k(x, x) = G_k \left(u(x), \frac{du(x)}{dx}, \ldots, \left(\frac{d}{dx}\right)^i u(x), \ldots \right), \qquad k = 1, 2, \ldots \tag{4}$$

Then, the equations of the KdV hierarchy [3], for an unknown function $w(x, t)$ are

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} G_k \left(w, \frac{\partial w}{\partial x}, \ldots \right), \quad k = 1, 2, \ldots$$

In this section, we obtain the supersymmetric extension of this construction. The super heat operator is given by

$$L = \frac{\partial}{\partial t} - \left(D^4 - D(\Phi \cdot) \right), \tag{5}$$

where $\Phi = \xi(x) + \theta u(x)$ is the superpotential with values on the odd part of a given Grassmann algebra, and

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$$

is the superderivation asociated with it.

It turns out [4] that the Green's function of this operator $K_t(x, \theta; x', \theta')$ has the following symmetry. If we expand it as

$$K_t(x, \theta; x', \theta') = A_t(x, x') + \theta B_t(x, x') - \theta' C_t(x, x') + \theta\theta' D_t(x, x'), \tag{6}$$

then, it satisfies

$$K_t(x, \theta; x', \theta') = K_t(x', \theta'; x, \theta)$$

which in terms of the components yields

$$A_t(x, x') = A_t(x', x)$$
$$B_t(x, x') = C_t(x', x)$$
$$D_t(x, x') = -D_t(x', x). \quad (7)$$

The existence of the Green's function and its properties are extensively studied in [4] where it is proven that this symmetry for the exact Green's function is also a symmetry for the coefficients of asymptotic expansion as $t \to 0^+$. This is a relevant and non-trivial step in our construction since the asymptotic expansion is not convergent in general.

We may now extend (2) and (3) in terms of the following asymptotic expansion for $K_t(x, x', \theta, \theta')$

$$K_t(x, x', \theta, \theta') = g_t(x - x' - \theta\theta') \times \sum_{k=0}^{\infty} \Big[a_k(x - x' - \theta\theta', x, \theta) + (\theta - \theta')b_k(x - x' - \theta\theta', x, \theta)\Big] \frac{t^k}{k!}. \quad (8)$$

After replacing this expression on the supersymmetric equation

$$LK_t = \delta_t \delta(x - x' - \theta\theta') \quad (9)$$

and using the symmetry (7), we obtain the following iterative procedure to evaluate the coefficients of K_t when $x = x'$ and $\theta = \theta'$:

$$\partial_z a_{k+1}(0, x, \theta) = D\mathcal{M}a_k(0, x, \theta) - \Phi c_k(0, x, \theta)$$
$$\partial_z c_k(0, x, \theta) = \mathcal{M}a_k(0, x, \theta) \quad (10)$$

where $c_k = \partial_y b_k(0, x, \theta)$ and

$$\mathcal{M} = D^5 - 3a_1 D^2 - a_2 D - 2a_3,$$

a_0 must be equal to 1.

To see this in more detail, we replace the expression given by (8) in (9) and obtain:

$$\left(1 - y\frac{\partial_x}{k+1}\right) a_{k+1} = La_k + \Phi b_k,$$
$$L = \partial_x^2 - D\Phi + \Phi D, \quad (11)$$
$$\left(1 - \frac{y\partial_x}{k+1}\right) b_{k+1} + \frac{1}{2}\Phi y a_{k+1} = Lb_k + \Phi\partial_y a_k, \quad (12)$$

where $y = x - x' - \theta\theta'$, the new supersymmetric variable.

The symmetry given by Eq. (7) implies now the following relations between a_k and b_k:

$$a_k(y, x, \theta) = a_k(-y, y + x, \theta),$$
$$b_k(y, x, \theta) - b_k(-y, y + x, \theta) = -Da_k(-y, y + x, \theta).$$

Taking derivatives with respect to y in these relations and evaluating at $x = x'$, $\theta = \theta'$, we obtain

$$\partial_y a_k(0, x, \theta) = \frac{1}{2}\partial_z a_k(0, x, \theta)$$
$$\partial^3 a_k(0, x, \theta) = \frac{3}{2}\partial_z\partial_y^2 a_k(0, x, \theta) - \frac{1}{4}\partial_z^3 a_k$$

and

$$b_k(0, x, \theta) = -\frac{1}{2}Da_k(0, x, \theta)\partial_y^2 b_k(0, x, \theta) = -\frac{1}{2}\partial_y^2 Da_k(0, x, \theta) + \partial_z\partial_y b_k(0, x, \theta) - \frac{1}{2}\partial_z^2 b_k(0, x, \theta). \quad (13)$$

The evaluation of Eqs. (11) and (12) at $x = x'$, $\theta = \theta'$ yields

$$\partial_y^2 a_k(0, x, \theta) - D(\Phi a_k) + \Phi b_k, \quad (14)$$
$$b_{k+1}(0, x, \theta) = (-\partial_y + \partial_z)(-\partial_y + \partial_z)b_k(0, x, \theta) - D\left(\Phi b_k(0, x, \theta)\right) + \frac{1}{2}\Phi\partial_z a_k(0, x, \theta). \quad (15)$$

We now take derivative of Eq. (11) with respect to y, evaluate at $x = x'$, $\theta = \theta'$, and replace $\partial_y^3 a_k$, $\partial_y^2 a_k$, $\partial_y a_k$ by the expressions obtained in the previous relations. We finally get the system given by Eq. (10).

The operator $D\mathcal{M}$ may be expressed as

$$D\mathcal{M} = \mathcal{P}(a_2) + 3a_1 D^3 - a_3 D,$$

where $\mathcal{P}$ is the bosonic operator

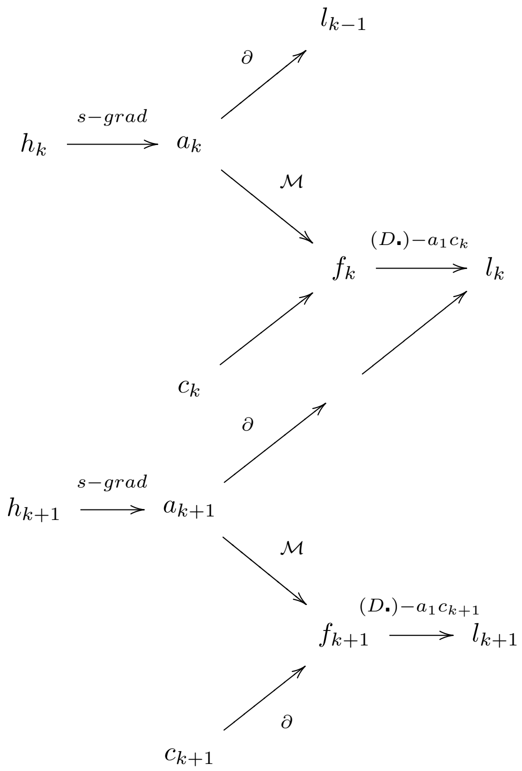
$$\mathcal{P}(u) = D^6 - 4uD^2 - 2D^2 u\mathbf{1}.$$

In the bosonic limit, when $\xi = 0$ and $\theta = 0$, the system of Eqs. (10) decouples, and the equation for $a_k(0, x, 0)$ describes the bosonic iterative procedure satisfied by the coefficients of the asymptotic expansion, when $t \to 0^+$, of the bosonic Green's function $G_t(x - x')$.

The existence of the supersymmetric Green's function and its asymptotic expansion ensures the existence of a solution for the system (10) with the initial condition $a_0 = 1$.

It follows from (10) that $\mathcal{M}a_k(0, x, \theta)$ are the right hand members of the super KdV hierarchy, and in particular a_2 corresponds to the super KdV equations of Mathieu [5].

The construction of the supersymmetric “tree” is then as follows



2. Conclusions

We have proposed a new supersymmetric $N = 1$ heat operator. Using the natural symmetries of this operator we obtained in an explicit way the super KdV hierarchy $N = 1$. This generalizes the approach which yields for the bosonic heat operator the KdV hierarchy. We think that this would be of great help in understanding the $N > 1$ models, for example, the quantum behavior of the supermembrane and super D-brane theories.

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