Verhulst's Lagrangean and self-regulated systems

Á.G. Muñoz S.[a,b,*], D. Sierra Porta[a,b], T. Soldovieri[a], D. Montiel[a], R.O. Rodríguez[b], J. Toro-Mendoza[c], L. Rivero[a]

[a]*Grupo de Investigaciones de Física Teórica (GIFT) Departamento de Física, Facultad de Ciencias, La Universidad del Zulia. Maracaibo 4004, Venezuela.*

[b]*Postgrado en Física Fundamental, Centro de Astrofísica Teórica (CAT), Facultad de Ciencias, Universidad de Los Andes Mérida, 5101, Venezuela.*

[c]*Laboratorio de Fisicoquímica de Coloides, Centro de Física, Instituto Venezolano de Investigaciones Científicas (IVIC) Caracas, 1020A, Venezuela.*

[*]*e-mail: agmunoz@luz.edu.ve.*

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Many examples of systems presenting self-limiting behaviour exist in nature: population dynamics, structure engineering, Townsend's electron breakdown, nuclear decay in radioactive equilibrium, histeresis process, meteorological models, etcetera. Each case is treated, generally with a different theory, sometimes a phenomenological one. In this work, we call your attention to the advantages the use of a variational formulation should provide in the study of self-regulated systems, such as a unified description of the phenomena mentioned above, further comprehension of the internal structure and symmetries of the related equations, and the equilibria points obtained via the energy function. As a particular and useful case, we have the Lagrangean and Hamiltonian functions obtained from the logistic equation, studying some of its dynamical properties and applications.

Keywords: Logistic equation; Verhulst's Lagrangean; self-regulated systems.

Existen en la naturaleza múltiples ejemplos de sistemas que presentan un comportamiendo auto-limitante: dinámica de población, ingeniería de estructuras, cascada electrónica de Townsend, decaimiento radioactivo, procesos de histéresis, meteorología, etcetera. En este trabajo hacemos hincapié sobre las ventajas que brinde el uso de una formulación variacional en el estudio de sistemas autoregulados, tales como una descripción unificada de los fenómenos, mayor comprensión de la estructura interna, de las simetrías de las ecuaciones relacionadas y la obtención de los puntos de equilibrio por medio de la función de energía. Como caso particular se obtienen las funciones lagrangeana y hamiltoniana de la ecuación logística, tratando algunas aplicaciones.

Descriptores: Ecuación logística; Lagrangeana de Verhulst; sistemas auto-regulados.

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1. Introduction

The variational formalism is a very powerful tool in physics, presenting, in a single formula, all the dynamical information of a system. The usual problem is to find an adequate expression for the Lagrangean or Hamiltonian, when obtaining the equations of motion, conserved quantities and other relevant dynamical relationships. However, there is an interesting different approach, known as the inverse problem of the variational calculus (see, for example, [1,2]). It consists in studying the existence and uniqueness (or multiplicity) of Lagrangeans for systems of differential equations, meaning to find the Lagrangean, if it exists, from the equations of motion. One important result is that of Hojman et al. [3], who have proved that it is possible to construct the Lagrangean for any regular mechanical system as a linear combination of their own equations of motion. This particular construction is much wider than the traditional definition $L = T - V$, which is only true when the "forces" involved are derivable from position-dependent potentials (or very few cases of velocity-dependent potentials), therefore it may be used for general non-conservative systems.

Whereas the equations of the models employed to describe auto-regulated phenomena can be understood as equations of motion in the variational sense, the latter approach shows itself to be one of the most adequate to obtain the Lagrangean formulation of the problem. Its application to the study of self-limiting processes may provide additional understanding into the internal structure of these phenomena, and also enables the use of a well known mathematical machinery to find conserved quantities, equilibria and stability cases, as well as other dynamical properties.

In the following pages, we propose to deal with the problem of self-regulated systems, in particular with those described by the Verhulst's Logistic Equation (VLE, from now on) [4], by means of the Hojman proceedure, to illustrate how a variational formulation can be easily obtained once we know the equation of motion and some additional information about the system.

The VLE is a continuous non-linear population growth model with a self-limiting density dependent mechanism. If $n(t)$ is the population at a time t, $1/A$ is the representative time scale of response of the model to any change in the population, and B is the carrying capacity of the enviroment (the maximum size of the stable steady state population), then the logistic equation can be written as

$$\frac{dn}{dt} = An\left(1 - \frac{n}{B}\right) \tag{1}$$

The solution of equation (1), for $n(0) = n_0$ is:

$$n(t) = \frac{n_0 B e^{At}}{B + n_0\left(e^{At} - 1\right)}. \tag{2}$$

For $n_0 < B$ the profile is the characteristic sigmoid of the model, and for $n_0 > B$ the behavior is similar to an exponentially decay function (see Fig. 1).

2. Constructing the Verhulst's Lagrangean

For convenience, we shall write (1) as

$$\dot{q} \equiv \frac{dq}{dt} = kq\left(B - q\right), \tag{3}$$

with $k \equiv A/B$, and we shall consider the one-dimensional problem.

In our case it can be showed that the equation of motion is

$$\ddot{q} - k^2 q(B - q)(B - 2q) = 0. \tag{4}$$

The second order Hojman et al. method provides then

$$\widetilde{L_V} = \mu\left[\ddot{q} - k^2 q(B - q)(B - 2q)\right]. \tag{5}$$

We just need to determine the factor μ, which in this case can be written as

$$\mu = C_1 \frac{\partial C_2}{\partial \dot{q}}, \tag{6}$$

where C_2 is a constant of motion and C_1 is an arbitrary function whose argument is a constant of motion. Observe that from the initial conditions and (4) we can obtain the only constant of motion, C_2, the system possesses:

$$C_2 \equiv \frac{1}{2}\dot{q}^2 - \frac{1}{2}k^2 q^2\left(B - q\right)^2 = 0 \tag{7}$$

and a possible choice for C_1 is (see reference [1])

$$C_1 = cC_2^2, \tag{8}$$

where c is an arbitrary constant that multiplies the equation of motion. The Lagrangean reads then

$$\widetilde{L_V} = cC_2^2 \dot{q}\left[\ddot{q} - k^2 q(B - q)(B - 2q)\right]. \tag{9}$$

When inserted into the Euler-Lagrange equation, (9) provides the corresponding equation of motion (4) plus terms that are zero in virtue of the constant of motion. However, it has been shown that for the one-dimensional problem there exists an infinite number of Lagrangeans [2]. Thus, we can write a more simpler solution-equivalent Lagrangean by means of the total time derivative of a certain gauge. Given the constant of motion, let choose

$$\frac{d\Lambda}{dt} = \dot{q}^2 - \left(C_1 \frac{dC_2}{dt} + C_2\right) \tag{10}$$

Without loss of generality we set $c = 1$, and adding (10) and (9) we obtain

$$L_V = \frac{1}{2}\dot{q}^2 + \frac{1}{2}k^2 q^2\left(B - q\right)^2 \tag{11}$$

and the Hamiltonian can be written as

$$H_V = \frac{1}{2}P^2 - \frac{1}{2}k^2 q^2\left(B - q\right)^2 \tag{12}$$

where the generalized momentum is

$$P \equiv \frac{\partial L}{\partial \dot{q}} = \dot{q} \tag{13}$$

3. Energies and Equilibrium Conditions

Now we proceed to study some of the applications the Lagrangean formalism offers to the Verhulst system. The first term in (11) and (12) is identified as the usual traslational "kinetic" energy: quadratic and homogeneous in the first temporal derivative of the generalized coordinate. The second term on the right side of equation (12), explicitily time-independent, is identified as the potential energy of the system:

$$V = -\frac{1}{2}k^2 q^2\left(B - q\right)^2 \tag{14}$$

The behavior of this potential is qualitatively presented in Fig. 2 for fixed carrying capacity. Note that equation (14) can be interpreted as an impulsor-retardatrice potential, depending only on the generalized coordinate. In fact, observe that when $q_0 \leq B$, for the interval $(0, B/2)$ the acceleration associated impulses the movement, being conversely for the interval $(B/2, B)$; the point $(B/2)$ corresponds to a local minimum, where the acceleration instantaneously annuls itself. On the other hand, when $q_0 > B$, i.e. the interval (B, ∞), the character of the acceleration is always impulsive, even being $\dot{q} < 0$. Also, in virtue of (3) equation (12) is identically zero, and it is clear in Fig. 2 that the kinetic energy tends to annuls itself as q approachs B. As a consequence, the system takes an infinite time to reach the steady state. All this is consistent with the profiles sketched in Fig. 1 and with the standard knowledge about equation (3).

One important application is that of finding the stability cases. The usual derivative criteria provide as equilibrium solutions $q_e = \left\{0, \frac{B}{2}, B\right\}$. The first one corresponds to a point of instability, the second to minimal potential energy and the last one is, as expected, the state of stationary equilibrium. As before, this analysis is in accordance with the known behaviour of the VLE (see Fig. 1). It is important to remark here that usually the study of its equilibrium points is treated by means of a Taylor expansion of (3) about $q = 0$ and $q = B$ (see for example [4]). The variational formalism provides thus both a more elegant and complete treatment.

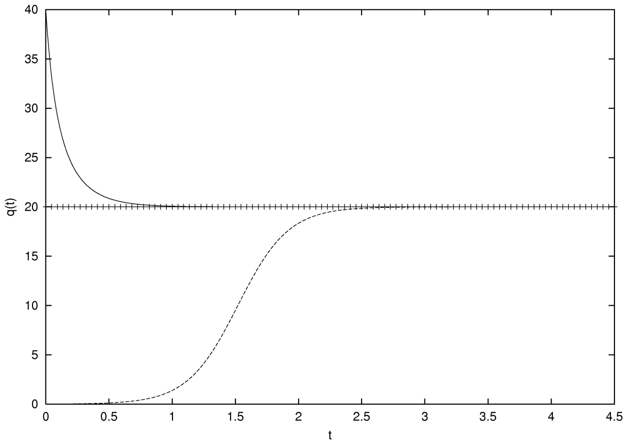

FIGURE 1. The Verhulst logistic equation ($B = 20$). The dash line corresponds to $n_0 < B$ and the solid one corresponds to $n_0 > B$.

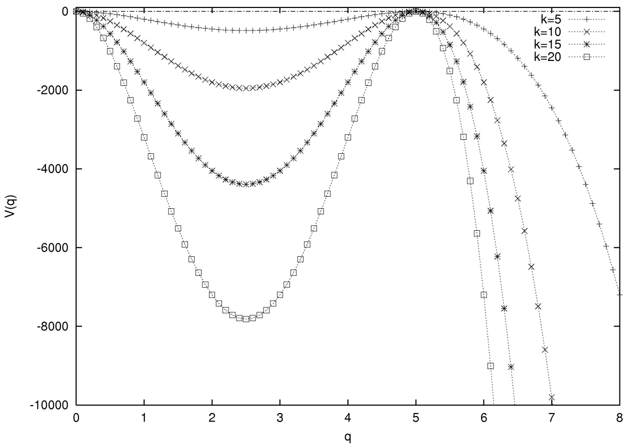

FIGURE 2. The Verhulst potential for different values of k ($B = 5$).

On the other hand, as far as the VLE can model various self-regulated phenomena, it should be useful for the description of the related statistical systems to write the total mean energy. We shall use the virial theorem for this porpouse. In our case,

$$\langle T\rangle = -\frac{1}{2}\left\langle \frac{\partial V}{\partial q} q\right\rangle = \frac{1}{2}k^2\left\langle q^2\left(B-q\right)\left(B-2q\right)\right\rangle \quad (15)$$

Then, the total mean energy is written as

$$\langle E\rangle \equiv \langle T+V\rangle = -\frac{1}{2}k^2\left\langle q^3(B-q)\right\rangle$$
$$= -\frac{k}{6}\left\langle \frac{d}{dt}\left(q^3\right)\right\rangle \quad (16)$$

or, explicitily, after evaluating the integral between $t = 0$ and $t = \infty$,

$$\langle E\rangle_\infty = \lim_{T\to\infty}\left\{-\frac{k}{6T}\left[q(T)^3 - q_0^3\right]\right\}$$
$$= 0 \quad (17)$$

Thus the total mean energy of the systems described by VLE is always null. But if as superior temporal bound we choose the characteristic time of the system, then

$$\langle E\rangle_{1/A} = -\frac{A^2}{6B}\left[q\left(\frac{1}{A}\right)^3 - q_0^3\right] \quad (18)$$

4. Concluding Remarks

By means of the inverse problem of the variational calculus, in this work we have found the Lagrangean (11) and Hamiltonian (12) corresponding to the logistic equation as an example of the way a variational formalism can be obtained for general self-regulated systems (up today there was no variational formalism for the VLE). This may be regarded as a proof that the VLE is an extremal, thus it is physically acceptable (it complies with the Hamilton's least-action principle) despite the fact Verhulst introduced it heuristically. We have also used the virial theorem to obtain the mean energies of the system.

Despite the fact the VLE describes multiple phenomena which one should think presents dissipation of energy (for example as in population dynamics, growth of living beings or meteorological models), it is easy to show that the Verhulst system is conservative (population systems with a first integral has been studied, see [4] and references therein): the Hamiltonian H_V equals the total energy E of the system and is the conserved quantity.

Finally, the self-regulated systems, not only those described by the VLE, are present everywhere: certain aspects in stability of structures, Townsend electron breakdown, histeresis and magnetization processes, the Amdhal law for scalability of computer programs and a long etcetera. As far as these phenomena are studied sometimes by quite different disciplines of science, it is interesting to explore if it is possible to provide a unified description for them in terms of families of Lagrangeans (or Hamiltonians), that also would help to classify the systems by their dynamical properties.

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