The Clifford structure of Nambu mechanics

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Recibido el 26 de noviembre de 2003; aceptado el 12 de octubre de 2004

The consequences of the Clifford structure of Nambu mechanics with more than one multiplet are presented. The only case considered is that in which S triplets are present.

Keywords: Nambu mechanics; Clifford algebra.

Se presentan las consecuencias de la existencia de una estructura de Clifford en la mecánica de Nambu con más de un multiplete. Sólo se presenta el caso de S tripletes.

Descriptores: Mecánica de Nambu; álgebra de Clifford.

PACS: 02.30.Ik; 45.05.+x

The study of the Nambu dynamical system has led to the conclusion that the Nambu tensor is decomposable if the Fundamental Identity is to be satisfied [1-9]. This has the immediate consequence that there is a single multiplet and therefore, if the original dynamical system is described by N variables, then the manifold is N dimensional and irreducible except for the singular points where it is zero-dimensional. This is expressed by stating that the leaves are either of maximal dimensionality or zero-dimensional. A complete classification of the Nambu tensor in the neighborhood of a singular point has been achieved [10]. Under the conditions that define a Nambu manifold, the study of this dynamical system is complete and this chapter of mathematical physics can be considered closed.

Substantial changes must be incorporated if the dynamical system is required to admit more than a single multiplet, so that a situation similar to the Hamiltonian scheme is conceived. It is by now well established [11-16] that if a description admits a group of S mutiplets of dimension K each so that $N = KS$ in an N dimensional manifold, a Clifford structure is necessary so that the powers of the K-eform are non-vanishing for any odd K. This leads to a set of $N = KS$ real variables $\mathbf{x} = (x_i^\alpha)$, $\alpha = 1, \ldots, S$, $i = 1, \ldots, K$ and S Clifford generators P^α that span the algebra C(S) over the reals so that the manifold is $F(R^{KS}) \otimes C(S)$, where $F(R^{KS})$ is the ring of smooth functions over R^{KS}.

The necessity of the Clifford structure is made evident when considering S triplets, so that there are $3S$ real variables and the evolution equation of a function $f(\mathbf{x})$ is given by

$$\frac{df}{dt} = \sum_{\alpha=1}^{S} \frac{\partial(f, H_1, H_2)}{\partial(x_1^\alpha, x_2^\alpha, x_3^\alpha)}, \tag{1}$$

where $\partial(\ldots)/\partial(\ldots)$ is a Jacobian determinant. If the evolution equations must be constructed from $X \rfloor \omega$, where ω is a 3-form, then it immediately follows that $\omega^2 = 0$, so that it is impossible to relate a certain power of ω to the volume form. If the square of ω must be non vanishing, a modification of the exterior product is needed. The new exterior product is defined for a couple of 1-eforms (extended form) by

$$\theta^\alpha \overline{\wedge} \theta^\beta = (-1)^{\delta(\alpha\,\beta)} \theta^\beta \overline{\wedge} \theta^\alpha, \tag{2}$$

where α and β are muliplet indices. This implies that the square of a 3-eform does not vanish. To achieve this relation, the usual 1-forms are modified so as to be the product of an ordinary 1-form and an object P^α, whose role is to take care of the necessary signs. The simplest generalization of the former real variables is to construct algebra valued objects $y^\alpha_{\beta\,i} = P^\alpha x_i^\beta$; the differential of this type of object is defined as

$$\hat{d} y^\alpha_{\beta\,i} = P^\alpha dx_i^\beta \tag{3}$$

so that if (2) is to be satisfied, the P^α are determined by

$$P^\alpha P^\beta + P^\beta P^\alpha = 2\delta_{\alpha\,\beta} I, \tag{4}$$

which shows that the P^α are the generators and I the identity of a Clifford algebra. In order to incorporate the full algebraic structure, the $y^\alpha_{\beta\,i}$ must be generalized to $y^{\overline{A}}_{\beta\,i} = P^{\overline{A}} x_{\beta\,i}$ where $\overline{A} = (a_1, \ldots, a_A)$ is a multi index in strict order and $P^{\overline{A}} = P^{a_1} \cdots P^{a_A}$; the number of $y^{\overline{A}}_{\beta\,i}$ is $3S2^S$. The manifold considered now is the set of generalized functions - called efunctions - with these objects ($\mathbf{y} = (y^\alpha_{\beta\,i})$) as arguments. An arbitrary efunction $f(\mathbf{y})$ belongs to the Clifford algebra

$$f(\mathbf{y}) = f_{\overline{A}}(\mathbf{x}) P^{\overline{A}}, \tag{5}$$

where $f_{\overline{A}}(\mathbf{x})$ is an ordinary smooth function on R^{3S}, and in (5) a sum over all subsets of A elements taken from

$1, \ldots, 3S$ and a sum over A is implicit. The differential of an efunction is required to satisfy the conditions

$$\hat{d}f(\mathbf{y}) = df_{\overline{A}}(\mathbf{x})P^{\overline{A}}, \quad \hat{d}(\hat{d}f(\mathbf{y})) = 0 \tag{6}$$

To be sure that the operation $\hat{d}$ is nilpotent of order two, the partial derivation must be modified slightly. Let us it as the operator $\hat{\partial}^{\beta\, i}_{\overline{A}}$ such that

$$\hat{\partial}^{\alpha\, i}_{\overline{A}} y^{\overline{B}}_{\beta\, j} = \hat{\partial}^{\alpha\, i}_{\overline{A}}(P^{\overline{B}} x_{\beta\, j}) = P^{\overline{A}} P^{\overline{B}} \delta^{\alpha}_{\beta} \delta^{i}_{j} \delta(\overline{A}\, \overline{B}) \tag{7}$$

where

$$\delta(\overline{A}\, \overline{B}) = \begin{cases} 0 & \text{if } \overline{A} \not\subset \overline{B} \\ (-1)^{A(A-1)/2} & \text{if } \overline{A} \subseteq \overline{B} \end{cases}$$

The inclusion of $\overline{A}$ in $\overline{B}$ means that $P^{\overline{A}}$ is a factor in $P^{\overline{B}}$. The differential of an efunction $f(\mathbf{y}) = f_{\overline{B}} P^{\overline{B}}$ is defined as

$$\begin{aligned} \hat{d}f(\mathbf{y}) =& \frac{1}{2^A} \hat{d}y^{\overline{A}}_{\alpha\, i} \hat{\partial}^{\alpha\, i}_{\overline{A}} (P^{\overline{B}} f_{\overline{B}}(\mathbf{x})) \\ =& \frac{1}{2^A} (P^{\overline{A}})^2 (-1)^{A(A-1)/2} P^{\overline{B}} dx_{\alpha\, i} \partial^{\alpha\, i} f_{\overline{B}} \\ =& P^{\overline{B}} df_{\overline{B}}(\mathbf{x}), \end{aligned} \tag{8}$$

which shows that the first of the requirements in (6) is satisfied and that the Clifford factors $P^{\overline{A}}$ are neutral as far as computation of the differential. The definition of the partial derivative shows that two of them commute if $P^{\overline{A}} P^{\overline{B}} = P^{\overline{B}} P^{\overline{A}}$, and ar anti-commuting otherwise; its permutation properties and the definition of the differential imply that $\hat{d}(\hat{d}f(\mathbf{y})) = 0$, as required.

A vector field is a first order differential operator $U = U^{\overline{A}}_{\alpha\, i}(\mathbf{x}) \hat{\partial}^{\alpha\, i}_{\overline{A}}$. A subalgebra of the algebra of vector fields is a Lie algebra if $A = 0$, which means those vector fields proportional to the Clifford identity. This is the only Lie subalgebra. If $P_S \equiv P^1 \cdots P^S$, then the transformation generated by P_S is an endomorphism of the Clifford algebra that relates C_A with C_{S-A}, so that as vector spaces these are isomorphic. As algebra they are homomorphic if both A and $S - A$ are even and if $(P^{\overline{A}})^2 = (P^{\overline{S-A}})^2$.

The generalized bracket naturally arising in this scheme corresponds to the generalized Lie derivative of a vector field

$$\mathcal{L}_X(Y) = [X, Y] \tag{9}$$

The contraction - denoted $\hat{\imath}$ - is defined by

$$\begin{aligned} \hat{i}_v f(\mathbf{x}) &= 0 \\ \hat{i}_v \hat{d}x^{\overline{A}}_{\alpha\, j} &= v^{\overline{A}}_{\alpha\, j} \\ \hat{i}_v(\hat{d}x^{\overline{A}}_{\alpha\, i} \overline{\wedge} \hat{d}x^{\overline{B}}_{\beta\, j}) &= v^{\overline{A}}_{\alpha\, j} \hat{d}x^{\overline{B}}_{\beta\, j} + (-1)^{\sigma(\overline{AB})} \hat{d}x^{\overline{A}}_{\alpha\, i} v^{\overline{B}}_{\beta\, j} \end{aligned} \tag{10}$$

so that it is an anti-derivation when acting on 2-forms with the same multiplet indices and a derivation on 2-forms with different multiplet indices. As is clear from (10), $\hat{\imath}$ is completely determined by its action on 0, 1 and 2-eforms. In (10) $\sigma(\overline{AB})$

$$\sigma(\overline{AB}) = \sum_{i=1}^{A} \sum_{j=1}^{B} \delta_{a_i\, b_j} \tag{11}$$

Now we turn our attention to the dynamical system. A minimum set of requirements is the following: given a vector field V, the evolution equations are obtained from the contraction of the vector field V and the canonical 3-eform Ω^3 as $V \rfloor \Omega^3$ and, to ensure that the structure is stable, the Lie derivative of the 3-eform must vanish.

The problem then is the determination of the 3-eform and vector fields so that the following conditions are satisfied:

$$\begin{aligned} \mathbf{V} \rfloor \Omega^{(3)} &= \hat{d}H \,\overline{\wedge}\, \hat{d}G \\ \mathcal{L}_V \Omega^{(3)} &= 0. \end{aligned} \tag{12}$$

The vector field, the 3-eform and the functions H and G considered are

$$\begin{aligned} V = V_0 I + V_S P_S\,; &\quad \Omega^{(3)} = \Omega^{(3)}_0 I + \Omega^{(3)}_S P_S \\ H = H_0 I + H_S P_S\,; &\quad G = G_0 I + G_S P_S, \end{aligned} \tag{13}$$

where $\Omega^{(3)}_K = \Omega^{rsu}_{\rho\sigma\tau\; K} dx^{\rho}_r \wedge dx^{\sigma}_s \wedge dx^{\tau}_u$.

The Lie derivative $\mathcal{L}_V$ associated with a vector field $V{=}V_{\overline{A}} P^{\overline{A}}$ when acting on a p-eform $\psi = \psi_{\overline{A}} P^{\overline{A}}$ is defined by

$$\mathcal{L}_V \psi = P^{\overline{A}} P^{\overline{B}} \mathcal{L}_{V_{\overline{A}}} \psi_{\overline{B}} + d(V_{\overline{A}} \rfloor \psi_{\overline{B}}) \Phi^{\overline{A}\, \overline{B}}_{\overline{C}} P^{\overline{C}}, \tag{14}$$

which must satisfy $\mathcal{L}_V(\rho \overline{\wedge} \sigma) = (\mathcal{L}_V \rho) \overline{\wedge} \sigma + \rho \overline{\wedge} (\mathcal{L}_V \sigma)$, and the functions $\Phi^{\overline{AB}}_{\overline{C}}$ are still unknown. If ψ is the 0-eform $\psi = P^{(\overline{A})}$, where the parentheses indicate that a single product of A factors of Clifford generators is present, then $\mathcal{L}_V P^{(\overline{A})} = 0$.

The total number of equations and efunctions in the set is

$$\begin{aligned} e &= 2^S \left\{ \frac{3S(3S-1)}{2} + S(3S-1)(3S-2) + 1 \right\} \\ f &= 2^S \{ S(3S-1)(3S-2) + 3S + 2^{2S} + 2 \} \end{aligned} \tag{15}$$

so that

$$f - e = 2^S \{ 2^{2S} + 1 - \frac{9S}{2}(S-1) \}, \tag{16}$$

which is always positive.

The expansion of the contraction of the vector field with the 3-eform, where use has been made of $(P_S)^2 = (-1)^{S(S-1)/2} I$, is as follows:

$$\begin{aligned} &\left(v_0 \rfloor \Omega^{(3)}_0 + (-1)^{S(S-1)/2}\, v_S \rfloor \Omega^{(3)}_S \right) I \\ &\quad + \left(v_0 \rfloor \Omega^{(3)}_S \;\; v_0 \rfloor \Omega^{(3)}_S \right) P_S \\ &= (dH_0 \wedge dG_0 + (-1)^{S(S-1)/2}\, dH_S \wedge dG_S \Big) I \\ &\quad + (dH_0 \wedge dG_S + dH_S \wedge dG_0)\, P_S \end{aligned} \tag{17}$$

from which

$$v^{\alpha}_{i\,0}\Omega^{isu}_{\alpha\sigma\tau\;0} + (-1)^{S(S-1)/2} v^{\alpha}_{i\,S}\Omega^{isu}_{\alpha\sigma\tau\;S} = \frac{\partial(H_0, G_0)}{\partial(x^{\sigma}_{s}, x^{\tau}_{u})} + (-1)^{S(S-1)/2}\frac{\partial(H_S, G_S)}{\partial(x^{\sigma}_{s}, x^{\tau}_{u})} \quad (18)$$

and

$$v^{\alpha}_{i\,S}\Omega^{isu}_{\alpha\sigma\tau\;0} + v^{\alpha}_{i\,S}\Omega^{isu}_{\alpha\sigma\tau\;0} = \frac{\partial(H_0, G_S)}{\partial(x^{\sigma}_{s}, x^{\tau}_{u})} + \frac{\partial(H_S, G_0)}{\partial(x^{\sigma}_{s}, x^{\tau}_{u})}. \quad (19)$$

To make contact with the Nambu dynamical system, it is important to recall that the basic input when dealing with triplets is a pair of functionally independent functions (the Nambu functions). These will be taken to be H_0 and G_0 and $H_S = H_S(H_0, G_0)$, $G_S = G_S(H_0, G_0)$. This leads to

$$v^{\alpha}_{i\,0}\Omega^{isu}_{\alpha\sigma\tau\;0} + (-1)^{S(S-1)/2} v^{\alpha}_{i\,S}\Omega^{isu}_{\alpha\sigma\tau\;S} = \left[1 + (-1)^{S(S-1)/2}\right]\frac{\partial(H_S, G_S)}{\partial(H_0, G_0)}\frac{\partial(H_0, G_0)}{\partial(x^{\sigma}_{s}, x^{\tau}_{u})}$$

$$v^{\alpha}_{i\,0}\Omega^{isu}_{\alpha\sigma\tau\;S} + v^{\alpha}_{i\,S}\Omega^{isu}_{\alpha\sigma\tau\;0} = \left[\frac{\partial G_S}{\partial G_0} + \frac{\partial H_S}{\partial H_0}\right]\frac{\partial(H_0, G_0)}{\partial(x^{\sigma}_{s}, x^{\tau}_{u})} \quad (20)$$

with at least one of the Jacobian $\partial(H_0, G_0)/\partial(x^{\sigma}_{s}, x^{\tau}_{u})$ different from zero, and at least one of the factors of these Jacobian non-vanishing.

The $6S$ equations with $\sigma = \tau$ make the computation of the components of the vector field possible, if the determinant of this system is different from zero. Consider the very particular case in which the components of the 3-eform satisfy

$$\Omega^{iru}_{\alpha\sigma\sigma\;K} = \delta_{\sigma\alpha}\Omega^{iru}_{\alpha\alpha\alpha\;K}, \quad K = 0, S. \quad (21)$$

Then the $6S$ equations simplify to include only two terms on the left hand side since (i, r, u) must be all different. These components of the vector field appear in two of the equations and can easily be determined easily with the result

$$v^{\alpha}_{i\,K} = L_K \frac{\partial(H_0, G_0)}{\partial(x^{\alpha}_{r}, x^{\alpha}_{u})}, \quad (22)$$

with (i, r, u) in cyclic order and L_K function coefficient. The Nambu form of the vector fields is recovered if at least one of the L_K is not zero so that it can be removed by a time reparametrization.

Since the vector fields have components along the identity and P_S, the composition of two vector fields with the generalized product $[X, Y]$ is a vector field along the identity and P_S, so that this subset is invariant [17]. The samething happens with a similar subset of p-eforms and efunctions, so that the action of any of these vector fields will leave some subset invariant. They correspond to the leaves of this dynamical system with the important result that the Clifford structure allows more leaves than in the case of a single multiplet, where it is known that the leaves have either a dimension equal to the dimension of the manifold, or zero.

Acknowledgment

Financial support of the Vicerrectorado de Investigación y Posgrado de la Universidad Pedagógica Experimental Libertador is gratefully acknowledged.

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