On the existence of conformal Killing vectors for ST-homogeneous Gödel type space-times

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Tsamparlis *et al.* [3] have developed a systematic method for computing of the conformal algebra of 1+3 space-times. The proper CKV's are found in terms of gradient CKVs of the 3-space. In this paper we apply Tsamparlis' results to the study CKVs of the Gödel ST-Homogeneous type spacetimes. We find that the only space-time admitting proper CKV's is the ST-Homogeneous Gödel type with $m^2 = 4\Omega^2$ (RT).

Keywords: General relativity; conformal symmetries.

Tsamparlis *et al.* [3] han encontrado un método sistemático para calcular el álgebra conforme de espacio-tiempos 1+3. Los CKV's propios son encontrados en términos de los gradientes CKV's del 3-espacio. En este trabajo aplicamos los resultados de Tsamparlis para estudiar los CKVs del espacio-tiempo tipo Gödel ST-Homogeneo. Encontramos que el único espacio-tiempo que admite CKVs propios es el tipo Gödel ST-Homogeneo con $m^2 = 4\Omega^2$ (RT).

Descriptores: Relatividad general; simetrías conformes.

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1. Introduction

A space-time is called 1+3 decomposable along a non-null vector field $\zeta^a = \delta^a_1$ if ζ^a is covariantly constant, equivalently a gradient KV. The metric of a 1+3 space-time can be written as:

$$ds^2 = \varepsilon(dx^1)^2 + g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta \tag{1}$$

where $\alpha, \beta = 2, 3, 4$ and $\varepsilon =$sign$(\partial/\partial x^1)$. For $\varepsilon = 1$ (-1) we refer to a 1+3 spacelike (timelike) space-time.

The 1+3 space-times are important because many solution of interest belong to this class (*i.e.* Robertson-Walker). Therefore, there has been considerable interest in the study of the conformal algebra of 1+3 decomposable space-times. Coley and Tupper [1] have determined all decomposable space-times which admit conformal Killing vectors (CKV) and they have given general expressions for the CKVs in special coordinate systems. A similar but more qualitative study of CKVs in decomposable space-times has been given by Capocci and Hall [2]. Finally Tsamparlis *et al.* [3] have developed a systematic method for computing the conformal algebra in terms of the gradient CKVs of 3-space. In this work we study conformal symmetries of the Gödel-type solutions.

In 1949 Gödel [4] published the following result: ”there exist only two types of solutions to Einstein's equations corresponding to a space-time which is homogeneous (ST-Homogeneous), with incoherent matter and with rigid rotation (no shear and no expansion). They are the Gödel solution (with rotation, $\Omega \neq 0$) and the static Einstein universe ($\Omega = 0$).” Even though this result was obtained for an energy tensor corresponding to dust and with cosmological constant ($\rho, \Lambda, p = 0$), it is easy to verify that with a change of variables ($\rho' = \rho + p, \Lambda' = \Lambda + kp$), it can be reinterpreted as a perfect fluid solution. Other studies have revealed the existence of families of isometrically non-equivalent Gödel-type solutions [5]. In 1980, Raychaudhuri and Thakurta [6] found the necessary condition for a space-time to be homogeneous (ST Gödel) for arbitrary values of Ω. Rebouças and Tiomno [5] obtained all the ST-Homogeneous Gödel type isometrically non equivalent space-times (written in cylindrical coordinates) and characterized by vorticity Ω and parameter m $(-\infty \leq m^2 \leq +\infty)$. The case $m^2 = 2\Omega^2$ corresponds to the Gödel solution. They found that $m^2 = 4\Omega^2$ is the first completely causal, exact Gödel-type solution (from now on RT solution). Teixeira *et al.* [7] computed the isometries of the ST-Homogeneous Gödel type metrics and showed that the RT solution admits a G_7 isometry group while all the others admit a G_5,with the exception of the $\Omega = 0$ case, which admits a G_6.

In this paper we use Tsamparlis *et al.* results to compute proper conformal vectors of the Gödel type space-times and find that they exist only for the RT solution. Therefore also from the point of view of conformal symmetry, this solution acquires a particular interest. This is done in Sec. 3 while in Sec. 2 we give a brief summary of Tsamparlis results [3].

2. CKV of 1+3 space-times

We are interested in the CKV of Gödel-type solutions and, since these space-times belong to the 1+3 class of metrics, we will use the results obtained in Ref. 3, where it is shown that the CKVs of a 1+3 decomposable metric g_{ab}

$$ds^2 = \varepsilon (dx^1)^2 + g_{\alpha\beta} dx^\alpha dx^\beta$$

where $\alpha, \beta = 2, 3, 4$ and $\varepsilon = sign(\partial/\partial x^1)$, are computed in terms of the CKVs of the 3-metric $g_{\alpha\beta}$ by means of the following proposition:

Proposition 1.1
All proper CKVs of a 4-metric g_{ab} along and normal to ς_a is $X_a = f(x^b)\varsigma_a + K_\alpha \delta^\alpha_a$, where K^α is CKVs of the 3-metric $g_{\alpha\beta}$ and has the following form:

$$K^\alpha = \frac{1}{p} m(x^1)\xi^\alpha + L^a(x^\beta) \tag{2}$$

where:

(a) $f(x^b)$ is obtained from CKVs K^α :

$$\frac{\partial f(x^b)}{\partial x^\alpha} = -\varepsilon \frac{\partial K_\alpha}{\partial x^1} \tag{3}$$

and

$$\frac{\partial f(x^b)}{\partial x^1} = \lambda(K) \tag{4}$$

(b) $\xi^\alpha = \lambda(\xi)^{,\alpha}$ is a proper gradient conformal vector field (if it is a KV, then space-time brakes further in 1+1+2, which we exclude as a trivial case) of the 3- metric satisfying the relation:

$$\lambda(\xi)_{|\alpha\beta} = p\lambda(\xi) g_{\alpha\beta} \qquad (p \neq 0). \tag{5}$$

(c) The function $m(x^1)$ satisfies the equation $(p \neq 0)$:

$$\frac{d^2 m}{d^2 x^1} + \varepsilon p m = 0. \tag{6}$$

(d) $L^a(x^\beta)$ is a non-gradient Killing vector (KV) or Homothetic Killing Vector (HKV) of the 3-metric such that:

$$L_{\alpha|\beta} = F_{\alpha\beta}(L) = F_{\alpha\beta}(K). \tag{7}$$

The KVs of the three-metric are identical with those of the 1+3 metric. The homothetic vector of the 1+3 metric (which exists only when the 3-metric admits one) is given by:

$$X^a = b\delta^a_1 + H^\mu \delta^a_\mu \tag{8}$$

where b is the conformal factor of the homothetic KV H^α of the 3-metric.

3. The geometric structure of Gödel type space-times

Now we apply the results of Sec. 2 to the computing of the CKV of Gödel-type space-times. In Cartesian coordinates (t, x, y, z), the metric of a Gödel type space-time is:

$$ds^2 = -dt^2 - 2H(x)dtdy + dx^2 - [D^2(x) - H^2(x)]dy^2 + dz^2 \tag{9}$$

where $H(x), D(x)$ are arbitrary functions.

The Gödel metric follows for $H(x){=}e^{mx}$, $D(x){=}(1/\sqrt{2})e^{mx}$.

In cylindrical coordinates t, r, ϕ, z, the metric of these spacetimes becomes:

$$ds^2 = -dt^2 - 2H(r)dtd\phi + dr^2 + [D^2(r) - H^2(r)]d\phi^2 + dz^2 \tag{10}$$

Gödel's metric is defined in these coordinates by: $H(r) = (2\sqrt{2}/m)\sinh^2(mr/2)$, $D(r) = (1/m)\sinh mr$.

The Gödel-type spacetimes are 1+3 decomposable (the gradient KV being ∂_z) with 3-metric:

$$ds_3^2 = -dt^2 - 2H(x)dtdy + dx^2 + [D^2(x) - H^2(x)]dy^2 \tag{11}$$

Therefore the theorem applies.

ST-Gödel-type metrics satisfy the following conditions [5]:

$$\frac{D''}{D} = const = 2\Omega \tag{12}$$

$$\frac{H'^2}{D^2} = -m^2 \tag{13}$$

We will solve Eq. (5) for the following cases:

- Case I $m = 0$

$$D(r) = ar + b, \tag{14}$$

$$H(r) = -\frac{\Omega}{a}(ar + b)^2 + c \tag{15}$$

where a, b, c are constants.

For $a{=}1, b{=}c{=}0$, we have: $D(r){=}r$, $H(r){=}{-}\Omega r^2$, which is the solution given by Reboucas and Tiomno (see Eq. (3.20) p. 1257 in Ref. 5).

Then we have that the only solution to Eq. (5) is

$$\lambda = 0$$

Therefore this space-times does not admit proper CKVs.

- Case II $m^2 \equiv -\mu^2 < 0$

$$D(r) = a \sin mr + b \cos mr, \tag{16}$$

$$H(r) = \frac{2\Omega}{m}(b \sin mr - a \cos mr) + c \tag{17}$$

The solution found by Reboucas and Tiomno follows for $a = 1/m, b = 0, c = (2\Omega^2)/m^2$ (see Eq. (3.25), p. 1258) and it is:

$$D(r) = \frac{1}{m} \sin mr, \qquad H(r) = \frac{2\Omega}{m^2} \sin^2 \frac{mr}{2}.$$

In this case, a solution to Eq. (5) with $\lambda \neq 0$ is possible only if we demand that

$$\mu^2 = -4\Omega^2$$

Thus, these ST-Gödel-type space-times do not admit CKVs.

- Case III $m^2 > 0$

$$D(r) = a \sinh mr + b \cosh mr, \tag{18}$$

$$H(r) = -\frac{2\Omega}{m}(b \sinh mr + a \cosh mr) + c \tag{19}$$

The solution found by Reboucas and Tiomno follows for $a = 1/m, b = 0, c = -(2\Omega)/m^2$ (see Eq. (3.4), p. 1256) and it is:

$$D(r) = \frac{1}{m} \sinh mr, \qquad H(r) = -\frac{4\Omega}{m^2} \sinh^2 \frac{mr}{2}.$$

For this case, we have that a solution to Eq.(5) with $\lambda \neq 0$ is obtained for $m = \pm 2\Omega$.

Then Proposition 1 implies that it definitely admits CKVs.

From Eq. (5) with $m = \pm 2\Omega$, we find that a non-trivial solution is obtained iff $p > 0$. that is

$$p = \Omega^2$$

We have, then, the following solutions:

$$\lambda = (c_1 \sin \Omega t + c_2 \cos \Omega t) \sinh \Omega r$$
$$+ [(-c_3 \cos \phi + c_4 \sin \phi) \sin \Omega t$$
$$+ (c_3 \sin \phi + c_4 \cos \phi) \cos \Omega t] \cosh \Omega r$$

If we perform the following coordinate transformation

$$\phi = \phi' + \Omega t \tag{20}$$

we arrive at:

$$\lambda = (c_1 \sin \Omega t + c_2 \cos \Omega t) \sinh \Omega r$$
$$+ (c_3 \sin \phi' + c_4 \cos \phi') \cosh \Omega r$$

which is the same result obtained by Tsamparlis for the metric RT [5], following a different approach:

$$ds_3^2 = -\cosh^2 \Omega r dt^2 + dr^2 + \frac{1}{\Omega^2} \sinh^2 \Omega r d\phi'^2 \tag{21}$$

The only ST-Gödel-type space-time which accepts CKVs is the conformally flat RT space-time. The KVs of the RT space-time have been determined in Ref. 7, and the CKVs are given in Ref. 3.

4. Conclusions

In this paper we have carried out a detailed study of the conformal symmetries of ST-Homogeneous Gödel-type space-times, and have shown that the only solution of this type which admits proper CKVs, is the RT solution, given by Eq. (21). As mentioned before, the RT solution is the only completely causal ST-Homogeneous Gödel-type space-time; furthermore, this is also the metric which exhibits the largest isometry group (G_7). It is, therefore, interesting that the requirement of the existence of a more general symmetry (conformal symmetry) singles out, once again, this solution among the whole family. Since this metric admits 15 CKV's: 7 KV and 8 CKV's, it is conformally flat.

The approach followed in this work requires that one solve the differential equations which derive from Eq. (5). This has been done successfully in this case; nevertheless, for more general 1+3 cases, it can be a difficult and awkward task. To overcome this difficulty, we are currently working on a criterion which will enable us to easily establish wether a given 1+3 metric might admit proper CKV's.

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