

Clebsch-Gordan coefficients for $U(8) \supset O(8) \supset SU(3)$: the first steps

I. Sánchez Lima and P.O. Hess

*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Apartado Postal 70-543, Circuito Exterior, C.U., 04510 México, D.F., Mexico.*

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The group chain $U(8) \supset O(8) \supset SU(3)$ plays an important role in systems of many gluons and in a schematic model for QCD at low energy. In order to calculate decay probabilities one has to calculate the Clebsch-Gordan coefficients of this group chain. In this contribution we present the basic idea in the example of $SU(3) \supset SO(3)$. Afterwards, the polynomial states of the $U(8)$ chain are constructed and the procedure to obtain the Clebsch-Gordan coefficients is outlined. Partial results are presented.

Keywords: Clebsch-Gordan coefficients; gluons; QCD.

La cadena de grupos $U(8) \supset O(8) \supset SU(3)$ juega un papel importante en sistemas de muchos gluones y en un modelo esquemático para QCD a bajas energías. Para poder calcular probabilidades de decaimiento se necesita calcular los coeficientes de Clebsch-Gordan para esa cadena de grupos. En esta contribución presentamos la idea básica en el ejemplo de $SU(3) \supset SO(3)$. Posteriormente, se construye los estados polinomiales de la cadena $U(8)$ y se indica como obtener los coeficientes de Clebsch-Gordan. Presentamos resultados parciales.

Descriptores: Coeficientes de Clebsch-Gordan; gluones; QCD.

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1. Introduction

The group $U(8)$ appears always when eight degrees of freedom are involved. For example, a gluon has eight color degrees of freedom [1] and thus the color part can be described by a $U(8)$ group. Because the gluon has spin one (three mathematical degrees of freedom) a many gluon system can be described by $U(24) \supset U(8) \otimes U(3)$ [2], where the $U(3)$ refers to the spin part and a many gluon state has to be in the complete symmetric irreducible representation (irrep) $[N]$ of $U(24)$, which relates the color and the spin part. Of course, the $U(8)$ group is reduced to the color $SU(3)$ group and $U(3)$ is reduced to the spin $SO(3)$ group, with integer spins only. The reduction is well known and given in [2,3]. However, the construction of states is less known. The first attempts have been made in [4, 5], though the $SU(3)$ subgroup considered in [5] is *not* the color group and only symmetric irreps of $U(8)$ were considered. Using a symmetric irrep in $U(8)$ is a first valid step towards a more general construction of states and it will be followed also in this contribution.

In [7–9] many quark and antiquark states were considered. A general classification for many quark and antiquark states in only the s orbital level was given. For simplifying the calculations, pairs of quark-antiquark were mapped to bosons [10]. There are four different types of bosons corresponding to quark-antiquark pairs with flavor (λ, λ) ($\lambda = 0, 1$) and spin S ($S = 0, 1$) denoted by $[\lambda, S]$. The cases $[0,0]$ and $[0,1]$ correspond to a one- and three-dimension harmonic oscillator, known from text books. The case $[1,0]$ corresponds to the eight dimensional harmonic oscillator, *i.e.* to the $U(8)$ group with a symmetric irrep. The last case $[1,1]$ is mathematically identical to the many gluon problem.

Why does one need the explicit form of states of the $U(8)$ group? One answer is the calculation of Clebsch-Gordan co-

efficients needed in order to determine transition ratios and decay properties of hadrons. Normally, the algebraic properties of a group and its algebra are used, as illustrated in [11] for the $SU(2)$ group and in [12–14] for the $SU(3)$ group. However, for higher rank groups these methods get more involved and unpractical. In [15, 16] a more practical procedure was proposed for the $U(5) \supset SO(5) \supset SO(3)$, playing an important role for the geometric model of the nucleus [17]. There, the polynomial expressions of the $U(5)$ states were constructed explicitly using basic couplings in terms of boson creation operators. The Clebsch-Gordan Coefficients were obtained by direct calculation of the integrals involving the polynomial states. The basic idea for the construction of the polynomials were borrowed from [18, 19].

Calculation of decay properties in the schematic model of QCD at low energy [7, 9] might be very important in order to decide whether the pentaquark exists [20–24] or not. In a first estimate within the schematic model, where we have information about the distribution of quark-antiquark pairs in the pentaquark and in the residual particles, indicate that the pentaquark is just the sum of a nucleon and a kaon, *i.e.* the width of the state should be very large and a peak should not be seen, confirmed in part by other the experiments and also criticized in Ref. [26] (and references therein). Whether or not this is the case, can only be decided through an explicit calculation.

A first and important step forward towards the construction of many gluon and quark-antiquark states is therefore constricting the states to the symmetric irrep of $U(8)$. The Clebsch-Gordan coefficients (CGC) are then obtained as integrals over a product of three polynomials.

In the second section we illustrate the basic ideas on how to obtain the CGC's for the example of $SU(3) \supset SO(3)$. In the third section we give the expressions of the polynomi-

als of symmetric states in $U(8) \supset SU(3)$, on how to obtain states with good seniority and how to obtain the CGC's of this problem. Only a report on the current status is given. Further work has still to be done.

2. The Example $SU(3) \supset SO(3) \supset SO(2)$

This is a well known example with important implications in nuclear physics [25, 27]. The group chain under consideration is

$$\begin{aligned} SU(3) &\supset SO(3) \supset SO(2) \\ (\lambda, \mu)\kappa &\quad L \quad M, \end{aligned} \quad (1)$$

where (λ, μ) denote the $SU(3)$ irrep, κ is a multiplicity label, L the angular momentum and M its projection. Here, we will only consider irreps of the type $(N, 0)$ which are symmetric and the multiplicity label $\kappa=1$.

The first step consists in the construction of polynomials which are in the highest weight state of $SO(3)$, i.e. $M = L$. For that, the elementary permissible diagrams (epd's) [28] have to be constructed, or in other words a complete set of basic couplings in terms of the boson creation operator b_m^\dagger ($m = 0, \pm 1$) such that all states of highest weight for a fixed angular momentum can be obtained for a given total number of oscillation quanta N [29, 30]. There are two epd's [29, 30], *i.e.*

$$\begin{aligned} A &= b_{+1}^\dagger \\ B &= -\sqrt{3} \left[b^\dagger \otimes b^\dagger \right]_0^{[0]} = (b \cdot b). \end{aligned} \quad (2)$$

The maximum weight state in terms of these operators is given by

$$P_{(N,0)lM=l} \left(\mathbf{b}^\dagger \right) | 0 \rangle = A_{nl} \left(\mathbf{b}^\dagger \cdot \mathbf{b}^\dagger \right)^{\frac{N-l}{2}} (b_{+1})^l | 0 \rangle, \quad (3)$$

with $n = (N - l)/2$ and A_{nl} given by

$$A_{nl} = \left[\frac{(2l+2)!!}{(2n)!!(2l+2n+1)!!l!} \right]^{\frac{1}{2}}. \quad (4)$$

A state with lower weight can be obtained by applying a certain number of times the lowering operator L_- [11, 29] and the result is

$$\begin{aligned} &\left[\frac{(l+m)!}{(l-m)!(2l)!} \right]^{\frac{1}{2}} \left(\mathbf{b}^\dagger \cdot \mathbf{b}^\dagger \right)^{\frac{N-l}{2}} L_-^{l-m} b_{+1}^l | 0 \rangle \\ &= B_{nl} \left(\mathbf{b}^\dagger \cdot \mathbf{b}^\dagger \right)^{\frac{N-l}{2}} Y_{lm} \left(\mathbf{b}^\dagger \right) | 0 \rangle, \end{aligned} \quad (5)$$

where the $Y_{lm}(\mathbf{b}^\dagger)$ is the spherical harmonics in terms of the boson operators [11] and B_{nl} given by [30]

$$B_{nl} = \left[\frac{4\pi}{(2n+2l+1)!!(2n)!!} \right]^{\frac{1}{2}}, \quad (6)$$

which differs in the sign factor $(-1)^n$ to Ref. [30] for convenience.

The CGC's of $SU(3) \supset SO(3) \supset SO(2)$ are related to the following integration over three polynomials,

$$\begin{aligned} \langle (N_3, 0)l_3l_3 | P_{(N_1,0)l_1l_1}(\mathbf{b}^\dagger) | (N_2, 0)l_2l_3 - l_1 \rangle &= (l_2l_3 - l_1, l_1l_1 | l_3l_3) \langle (N_3, 0)l_3 | | P_{(N_1,0)l_1}((b)) | | (N_2, 0)l_2 \rangle \\ &= \langle (N_2, 0)l_2l_3 - l_1; (N_1, 0)l_1l_1 | (N_3, 0)l_3l_3 \rangle_1 \langle (N_3, 0) | | | P_{(N_1,0)}(\mathbf{b}^\dagger) | | | (N_2, 0) \rangle_1, \end{aligned} \quad (7)$$

where we have chosen for convenience the highest weight in $SO(3)$ for the polynomials with indices 1 and 3 and where $\langle (N_3, 0)l_3l_3 | P_{(N_1,0)l_1l_1}(\mathbf{b}^\dagger) | (N_2, 0)l_2l_3 - l_1 \rangle$ can be calculated algebraically, using the commutation relations of the creation and annihilation boson operators and the expression of L_- in terms of the boson operators. The matrix element can be obtained by either using the results for the spherical harmonics, given in [11], or by applying directly the polynomial $\langle 0 | P_{(N_3,0)l_3l_3}(b)$ to the right hand side, using $b^m = (\partial/\partial b_m^\dagger)$ and algebraic manipulations through a numerical algebraic program like MATHEMATICA [31]. The algebraic program will be the option used in the determination of the matrix elements.

In Eq. (7) the second line determines the reduced matrix elements $\langle (N_3, 0) | | P_{(N_1,0)}((b)) | | (N_2, 0) \rangle$, supposing that the $SU(2)$ CGC's are known. Once the reduced matrix elements are known, with the help of the $SU(2)$ CGC's all

matrix elements of Eq. 7 are obtained. The next step is to determine the triple reduced matrix elements of $SU(3)$ in the third line. They are obtained by choosing the particular values $l_k = N_k$. For that case the $SU(3)$ CGC is 1 and the triple reduced matrix elements are obtained. Knowing the reduced and the triple reduced matrix elements and the $SU(2)$ CGC's enables us to calculate the $SU(3)$ CGC's. Because the $SU(3)$ CGC can be written as a product of an isoscalar factor and an $SU(2)$ CGC [13], we obtain

$$\begin{aligned} &\langle (N_2, 0)l_2; (N_1, 0)l_1 | | (N_3, 0)l_3 \rangle_1 \\ &= \frac{\langle (N_3, 0)l_3 | | P_{(N_1,0)l_1}((b)) | | (N_2, 0)l_2 \rangle}{\langle (N_3, 0) | | | P_{(N_1,0)}(\mathbf{b}^\dagger) | | | (N_2, 0) \rangle_1}. \end{aligned} \quad (8)$$

This procedure leads to the analytic expression of

$$\langle (N_2, 0)1l_2; (N_1, 0)1l_1 || (N_3, 0)1l_3 \rangle_1 = (-1)^{\frac{l_1+l_2-l_3}{2}} \frac{\left(\frac{l_1+l_2+l_3}{2}\right)!}{\left(\frac{l_2+l_3-l_1}{2}\right)! \left(\frac{l_1+l_3-l_2}{2}\right)! \left(\frac{l_1+l_2-l_3}{2}\right)!} \\ \times \left[\frac{N_1!N_2!}{N_3!} \right]^{\frac{1}{2}} \left[\frac{(N_3+l_3+1)!!(N_3-l_3)!!(2l_1+1)(2l_2+1)(l_2+l_3-l_1)!(l_1+l_2-l_3)!(l_1+l_3-l_2)!}{(N_1+l_1+1)!!(N_1-l_1)!!(N_2+l_2+1)!!(N_2-l_2)!!(l_1+l_2+l_3+1)!} \right]^{\frac{1}{2}}, \quad (9)$$

which can be compared to the list given in [12] and with the analytical formulae given in Ref. [14] (there, the more elegant method of vector coherent states was used).

Of course, there are more general procedures to obtain this result as mentioned in the introduction, e.g. using the commutation properties of the algebra of the group. However, the method described here is practical for higher rank groups (especially for the example discussed in the next section) and leads to the CGC's needed. In general, not all possible irreps of a high rank group are needed. For example, in the $U(8)$ group only the completely symmetric irrep is needed in this work and for gluons at most three rows are requested. For such restricted cases, the method presented in this work, might be sufficiently effective.

3. The chain $U(8) \supset O(8) \supset SU(3)$

The bosons under consideration have eight degrees of freedom and belong to the flavor irrep $(1,1)$.

The group chain under consideration is

$$U(8) \supset O(8) \supset SU(3) \supset U(1) \otimes SU(2) \\ [N] \quad (\nu 0000) \quad (\lambda, \mu) \quad Y \quad T, \quad T_3, \quad (10)$$

where Y is the hypercharge, T the isospin and T_3 its third component. The N is the total number of bosons, ν the seniority (number of bosons *not coupled* in pairs with flavor $(0,0)$). CGC's of the chain $SU(3) \supset U(1) \otimes SU(2)$ are well known and available [32, 33].

The generators of $U(8)$ are given by

$$C_{YTT_3}^{\rho\Gamma} = \left[\mathbf{b}^\dagger \otimes \mathbf{b} \right]_{YTT_3}^{\rho\Gamma}, \quad (11)$$

where Γ is a short hand notation for (λ, μ) and ρ is a multiplicity label which is normally 1 except for the irrep $\Gamma=(1,1)$ where it obtains values 1 and 2. The value $\rho = 1$ refers to the antisymmetric coupling and $\rho = 2$ to the symmetric one [34]. The possible values of Γ are $(0,0)$, $(2,2)$ and $(1,1)_2$ for the symmetric and $(3,0)$, $(0,3)$ and $(1,1)_1$ for the antisymmetric coupling. One can linearize the index (YTT_3) to f where the association is given in Table I.

The generators of $O(8)$ are given by the antisymmetric couplings and thus the algebra of $O(8)$ contains 28 generators. The generators of the $SU(3)$ subgroup are given by the coupling $\Gamma = (1,1)_1$.

The algebra for $SU(3)$, the only one needed for the calculation, is given by

$$\left[C_{f'}^{1(1,1)}, C_f^{1(1,1)} \right] \\ = \sum_{f_0} \langle (1,1)f', (1,1)f | (1,1)f_0 \rangle_1 \left[\mathbf{b}^\dagger \otimes \mathbf{b} \right]_{f_0}^{(1,1)_a}. \quad (12)$$

In terms of the standard notation of the $SU(3)$ generators [6], the relation to the $C_\mu^{1(1,1)_1}$ of Eq. 11 is given by

$$\mathbf{T}_\pm = \pm \sqrt{6} \left[\mathbf{b}^\dagger \otimes \mathbf{b} \right]_{01\pm 1}^{(1,1)_a}, \quad \mathbf{T}_0 = -\sqrt{3} \left[\mathbf{b}^\dagger \otimes \mathbf{b} \right]_{010}^{(1,1)_a} \\ \mathbf{V}_\pm = \pm \sqrt{6} \left[\mathbf{b}^\dagger \otimes \mathbf{b} \right]_{\pm 1\frac{1}{2} \pm \frac{1}{2}}^{(1,1)_a}, \\ \mathbf{U}_\pm = \sqrt{6} \left[\mathbf{b}^\dagger \otimes \mathbf{b} \right]_{\pm 1\frac{1}{2} \mp \frac{1}{2}}^{(1,1)_a}, \\ \mathbf{Y} = -2 \left[\mathbf{b}^\dagger \otimes \mathbf{b} \right]_{000}^{(1,1)_a}. \quad (13)$$

The next step is to construct the epd's (elementary couplings) of $U(8)$. In [4] the following epd's (also denoted as *integrity basis*) where obtained using the method of generating functions (see Eq. (16) of Ref. [4]).

$$\mathbf{A} = \mathbf{b}_{011}^\dagger \\ \mathbf{B} = \left[\mathbf{b}^\dagger \otimes \mathbf{b}^\dagger \right]_{000}^{(0,0)} \\ \mathbf{C} = \left[\mathbf{b}^\dagger \otimes \mathbf{b}^\dagger \right]_{011}^{(1,1)} \\ \mathbf{D} = \left[\mathbf{b}^\dagger \otimes \left[\mathbf{b}^\dagger \otimes \mathbf{b}^\dagger \right]^{(1,1)} \right]_{000}^{(0,0)} \\ \mathbf{E} = \left[\mathbf{b}^\dagger \otimes \left[\mathbf{b}^\dagger \otimes \mathbf{b}^\dagger \right]^{(1,1)} \right]_{1\frac{3}{2} \frac{3}{2}}^{(3,0)} \\ \mathbf{F} = \left[\mathbf{b}^\dagger \otimes \left[\mathbf{b}^\dagger \otimes \mathbf{b}^\dagger \right]^{(1,1)} \right]_{-1\frac{3}{2} \frac{3}{2}}^{(0,3)}. \quad (14)$$

A state is a polynomial in these epd's, where according to [4] the epd \mathbf{C} can appear only in powers of 0, 1 or 2. Instead of choosing \mathbf{C} as a dependent epd, one can take instead \mathbf{E} and \mathbf{F} as dependent epd's. These two epd's satisfy a relation which permits the appearance of only one of the couplings \mathbf{E} or \mathbf{F} as a power in the polynomial. The relation is

$$\mathbf{EF} = -\frac{1}{6} \sqrt{\frac{15}{2}} \mathbf{C}^3 + \sqrt{\frac{3}{5}} \mathbf{A}^2 \mathbf{BC} - \frac{2}{\sqrt{15}} \mathbf{A}^3 \mathbf{D}, \quad (15)$$

TABLE I. Linearization of the index (YTT_3) to μ .

f	Y	T	T_3
1	0	1	1
2	1	1/2	1/2
3	-1	1/2	1/2
4	0	1	0
5	0	0	0
6	1	1/2	-1/2
7	-1	1/2	-1/2
8	0	1	-1

This relation can also be used to express third powers of \mathbf{C} in terms of the other epd's, as is suggested in [4]. We though take the second choice as also was done in Ref. [5] (there, however, the subgroup $SU(3)$ is not the color group, whose generators have to be antisymmetric).

Choosing \mathbf{E} and \mathbf{F} as dependent epd's, there are two types of polynomials, one with $(\lambda + 3k, \lambda)$ and the other with $(\lambda, \lambda + 3k)$ respectively, i.e.

$$\mathbf{E}^{n_5} \mathbf{D}^{n_4} \mathbf{C}^{n_3} \mathbf{B}^{n_2} \mathbf{A}^{n_1} |0\rangle \mathbf{F}^{n_5} \mathbf{D}^{n_4} \mathbf{C}^{n_3} \mathbf{B}^{n_2} \mathbf{A}^{n_1} |0\rangle. \quad (16)$$

Fixing the total number of quanta N , the λ and μ for the first case (only powers of \mathbf{E} appear), we obtain the following relation between the powers on the monomial

$$\begin{aligned} N &= n_1 + 2n_2 + 2n_3 + 3n_4 + 3n_5 \\ \lambda &= n_1 + n_3 + 3n_5 \\ \mu &= n_1 + n_3. \end{aligned} \quad (17)$$

Similar relations hold when only powers of \mathbf{F} appear

$$\begin{aligned} N &= n_1 + 2n_2 + 2n_3 + 3n_4 + 3n_5 \\ \lambda &= n_1 + n_3 \\ \mu &= n_1 + n_3 + 3n_5. \end{aligned} \quad (18)$$

Up to now, the polynomials have no good seniority ν , the quantum number of the $O(8)$ group. This is achieved requiring that the application of

$$\bar{\mathbf{B}} = [\mathbf{b} \otimes \mathbf{b}]_{000}^{(0,0)} \quad (19)$$

on a polynomial in terms of the above monomials, give zero (no pairs are contained). This leads to the polynomials which have a given seniority and where $N = \nu$. Explicitly, the condition reads

$$\bar{\mathbf{B}} P_{N=\nu(\lambda, \mu)}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}(\mathbf{F})) |0\rangle = 0 \quad (20)$$

where the polynomial has the following structure when only powers of \mathbf{E} appear

$$P = \mathbf{E}^{\frac{\lambda-\mu}{3}} \times \sum_{n_1 n_2} c_{n_1 n_2} \mathbf{A}^{n_1} \mathbf{B}^{n_2} \mathbf{C}^{\mu-n_1} \mathbf{D}^{\frac{N-\lambda-\mu+n_1-2n_2}{3}}, \quad (21)$$

and similar when only powers of \mathbf{F} appear

$$P = \mathbf{F}^{\frac{\mu-\lambda}{3}} \times \sum_{n_1 n_2} c_{n_1 n_2} \mathbf{A}^{n_1} \mathbf{B}^{n_2} \mathbf{C}^{\lambda-n_1} \mathbf{D}^{\frac{N-\mu-\lambda+n_1-2n_2}{3}}. \quad (22)$$

In Table II we give a list of all polynomial up to four bosons. The result up to 12 bosons is also available. In realistic calculations [2, 7, 9] no more than six bosons were needed.

States of lower weight but with the same hypercharge Y in $SU(3)$ are obtained by applying the lowering operators of $SU(3)$ in a particular way to the highest weight state (see Ref. [6]). Restricting, for the moment, to irreps of the type $(\lambda, \mu = \lambda)$, i.e. no powers of \mathbf{E} nor \mathbf{F} appear, the state of lower weight with the same Y as in the maximum weight state is given by

$$\mathcal{N} \mathbf{T}_-^\alpha \mathbf{U}_+^\beta \mathbf{V}_-^\beta P_{N=\nu(\lambda, \mu)} |0\rangle, \quad (23)$$

TABLE II. List of the polynomial with definite seniority up to $N = 4$.

N	ν	(λ, μ)	polynomial
0	0	(0,0)	1
1	1	(1,1)	\mathbf{A}
2	0	(0,0)	$(1/4)\mathbf{B}$
2	2	(1,1)	$(1/\sqrt{2})\mathbf{C}$
2	2	(2,2)	$(1/2)\mathbf{A}^2$
3	1	(1,1)	$(1/2\sqrt{5})\mathbf{AB}$
3	3	(0,0)	$(1/2\sqrt{5})\mathbf{D}$
3	3	(3,0)	$(\sqrt{5}/3\sqrt{2})\mathbf{E}$
3	3	(0,3)	$(\sqrt{5}/3\sqrt{2})\mathbf{F}$
3	3	(2,2)	$\sqrt{(5/14)}\mathbf{AC}$
3	3	(3,3)	$(1/\sqrt{6})\mathbf{A}^3$
4	0	(0,0)	$(1/8\sqrt{10})\mathbf{B}^2$
4	2	(2,2)	$(1/4\sqrt{3})\mathbf{A}^2\mathbf{B}$
4	2	(1,1)	$(1/4\sqrt{3})\mathbf{BC}$
4	4	(1,1)	$\sqrt{(2/55)}\mathbf{AD}$
4	4	(2,2)	$(5/4\sqrt{19})\mathbf{C}^2$
4	4	(3,3)	$(\sqrt{5}/6)\mathbf{A}^2\mathbf{C}$
4	4	(4,4)	$(1/2\sqrt{6})\mathbf{A}^4$
4	4	(4,1)	$(\sqrt{5}/6)\mathbf{AE}$
4	4	(1,4)	$(\sqrt{5}/6)\mathbf{AF}$

TABLE III. Some quadruple reduced matrix elements up to $N_k = 4$.

N_3	ν_3	N_1	ν_1	N_2	ν_2	$\langle N_3\nu_3 P_{N_1\nu_1} N_2\nu_2 \rangle$
2	2	1	1	1	1	1
3	1	1	1	2	0	1
3	3	1	1	2	2	$\sqrt{3}$
4	0	2	1	2	0	$8\sqrt{10}$
4	2	2	1	2	2	$24/\sqrt{10}$
4	4	2	2	2	2	$\sqrt{6}$

where the normalization of the state is given by

$$\mathcal{N}^{-2} = \frac{\alpha!^2 \beta!^4 (2\lambda - 2\beta)!}{(\lambda - \beta)!^2} \times \sum_{k=0}^{\alpha} \frac{(\lambda - \beta + \alpha - k)!(\lambda + \beta - \alpha + k)!}{k!(\alpha - k)!^2 (2\lambda - 2\beta - k)!(\beta - \alpha + k)!^2}. \quad (24)$$

This equation can be used as a check for the algebraic calculation using MATEMATICA. The restriction to irreps (λ, λ) is sufficient for most cases of interest in hadron physics, at least at low energy. The other irreps might however be important for the description of pentaquark states.

Now, we are in the same position as in the simple example of the second section. Suppose, we know

$$\langle N_3\nu_3(\lambda_3, \mu_3) f_{m3} | P_{N_1\nu_1\nu_1(\lambda_1, \mu_1)f_{m1}} | N_2\nu_2(\lambda_2, \mu_2) f_2 = f_{m3} - f_{m1}, \quad (25)$$

where f_m is a short hand notation of the maximum weight $(YTT_3)_m$ as defined in [6], *i.e.* $(Y)_m = (\lambda - \mu)/3$ and $(T)_m = (\lambda + \mu)/2$. Through the additional knowledge of the $SU(3)$ CGC's [32, 33] we can obtain the CGC's of the chain given in (10), *i.e.*

$$\begin{aligned} & \langle N_3\nu_3(\lambda_3, \mu_3) f_{m3} | P_{N_1\nu_1\nu_1(\lambda_1, \mu_1)f_{m1}} | N_2\nu_2(\lambda_2, \mu_2) f_{m3} - f_{m1} \rangle \\ &= \sum_{\rho} \langle (\lambda_2, \mu_2) f_2 = f_{m3} - f_{m1}; (\lambda_1, \mu_1) f_{m1} | (\lambda_3, \mu_3) f_{m3} \rangle_{\rho} \langle N_3\nu_3(\lambda_3, \mu_3) ||| P_{N_1\nu_1\nu_1(\lambda_1, \mu_1)} ||| N_2\nu_2(\lambda_2, \mu_2) \rangle_{\rho} \\ &= \sum_{\xi} \langle N_2\nu_2(\lambda_2, \mu_2); N_1\nu_1(\lambda_1, \mu_1) ||| N_3\nu_3(\lambda_3, \mu_3) \rangle_{\xi} \langle (\lambda_2, \mu_2) f_2 = f_{m3} - f_{m1}; (\lambda_1, \mu_1) f_{m1} | (\lambda_3, \mu_3) f_{m3} \rangle_{\xi} \langle N_3\nu_3 ||| P_{N_1\nu_1} ||| N_2\nu_2 \rangle_{\xi}, \end{aligned} \quad (26)$$

where the CGC of $U(8) \supset O(8) \supset SU(3) \supset U(1) \otimes SU(2)$ has been written as a product of an isoscalar factor of $U(8) \supset O(8) \supset SU(3)$ and the CGC of $SU(3) \supset U(1) \otimes SU(2)$.

In a first step, the triple reduced matrix elements of the polynomials is obtained in the second row through a set of linear equations involving different α and β and knowing the expressions of Eq. (25) and the $SU(3)$ CGC's. The next step is to obtain the quadruple reduced matrix elements, which are obtained choosing $\nu_k = N_k$ and $(\lambda, \mu) = (N_k, N_k)$. The calculation of the expression (25) is done using the code MATEMATICA [31].

Using the first and the last two lines in Eq. (26) we can determine certain quadruple reduced matrix elements. Up to now, we used highest weight states in $O(8)$ only, for $SU(3)$ irreps (λ, λ) , *i.e.* $(\lambda_k, \lambda_k) = (\nu_k, \nu_k)$. This implies that also $\nu = 3 = \nu_1 + \nu_2$ and, by construction $N_3 = N_1 + N_2$. In Table III we list some of these values. The corresponding $U(8) \supset O(8) \supset SU(3)$ isoscalar factors are all equal to one because the couplings considered are all stretched. In order to obtain non trivial values one has to use lower weight state for the polynomial with indices 2.

Up to now we have only calculated the expressions in Eq. (25) with $f_{m3} - f_{m1} = f_{m2}$, *i.e.* within the maximum weight states in all polynomials. The lowering operators (Eq. (23)) have not been applied yet. This is the next step we plan to do.

4. Conclusions

In this contribution we have indicated the procedure on how to obtain the Clebsch-Gordan coefficients of the chain $U(8) \supset O(8) \supset SU(3) \supset U(1) \otimes SU(2)$ for symmetric irreducible representations in $U(8)$. The procedure was illustrated for the group chain $SU(3) \supset SO(3) \supset SO(2)$ for the $SU(3)$ Clebsch-Gordan coefficients, in the case of symmetric representations of $SU(3)$. The obtained analytical expression can be compared to existing ones.

The importance of the Clebsch-Gordan coefficients of the chain starting with $U(8)$ lies in the possibility to obtain via their use branching ratios of hadron decays involving gluons and quark-antiquark pairs, but it can also be used in any other problem related to a $U(8)$ group.

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1. K. Huang, *Quarks, Leptons and Gauge Fields*, 2nd edition, (World Scientific, Singapore, 1992).
2. P.O. Hess, S. Lerma, J.C. López, C.R. Stephens and A. Weber, *Eur. Phys. Jour. C* **9** (1999) 121.
3. R. López, P.O. Hess, P. Rochford and J.P. Draayer, *J. Phys. A* **23** (1990) L229.
4. R. Gaskell, A. Peccia and R.T. Sharp, *J. Math. Phys.* **19** (1978) 727.
5. E. Chacón, in *Anales de Física. Monografías*, eds. M. A. Olmo, M. Santander and J. Mateos Guilarte, CIEMAT, Proceedings of the *XIX International Colloquium Salamanca*, (Spain, June 29 - July 4, 1992) 331.
6. W. Greiner and B. Müller, *Quantum Mechanics: Symmetries*, (Springer, Heidelberg, 1989).
7. S. Lerma, S. Jesgarz, P.O. Hess, O. Civitarese and M. Reboiro, *Phys. Rev. C* **67** (2003) 055209.
8. S. Jesgarz, S. Lerma, P.O. Hess, O. Civitarese and M. Reboiro, *Phys. Rev. C* **67** (2003) 055210.
9. M. Nuñez *et al.*, *Phys. Rev. C* **70** (2004) 025201.
10. A. Klein and E.R. Marshalek, *Rev. Mod. Phys.* **63** (1991) 375.
11. A.R. Edmonds, *Angular Momentum in Quantum Mechanics*, (Princeton Univ. Press, Princeton, 1957).
12. J.D. Vergados, *Nucl. Phys. A* **111** (1968) 681.
13. J.P. Draayer and Y. Akiyama, *J. Math. Phys.* **14** (1973) 1904.
14. D.J. Rowe and C. Bahri, *J. Math. Phys.* **41** (2000) 6544.
15. D.J. Rowe, P.S. Turner and J. Repka, *J. Math. Phys.* **45** (2004) 2761.
16. D.J. Rowe, *Nucl. Phys. A* **735** (2004) 372.
17. J.M. Eisenberg and W. Greiner, *Nuclear Theory I: Nuclear Models*, (North-Holland, Amsterdam, 1987).
18. E. Chacón, M. Moshinsky and R.T. Sharp, *J. Math. Phys.* **17** (1976) 668.
19. E. Chacón and M. Moshinsky, *J. Math. Phys.* **18** (1977) 870.
20. T. Nakano *et al.*, *Phys. Rev. Lett.* **91** (2003), 012002.
21. V.V. Barmin *et al.* (DIANA collaboration), preprint hep-ex/0304040, *Yed. Fis.* (in press).
22. S. Stepanyan *et al.* (CLAS collaboration), preprint hep-ex/0307018.
23. J. Barth *et al.* (SAPHIR collaboration), preprint hep-ex/307083, *Nucl. Phys. B* (in press).
24. A.E. Asratyan, A.G. Dolgolenko and M.A. Kubantsev, preprint hep-ex/0309042, submitted to *Yad. Fis.* (2003).
25. J.P. Elliott, *Proc. R. Soc. London, A* **245** (1958) 128; *ibid.* **245** (1958) 562.
26. A.R. Dzierba, D. Krop, M. Swat, S. Teige and A.P. Szczepaniak, *Phys. Rev. D* **69** (2004) 051901.
27. M. Harvey, *Adv. Nucl. Phys.* **1** (1968) 67.
28. R. Gaskell, A. Peccia and R.T. Sharp, *J. Math. Phys.* **19** (1978) 727.
29. M. Moshinsky, *Group Theory and the Many-Body Problem*, (Gordon and Breach, New York, 1967).
30. M. Moshinsky and Yu. F. Smirnov, *The Harmonic Oscillator in Modern Physics*, (Harwood academic publishers, New York, 1996).
31. MATEMATICA-5 package, Wolfram Research, (2004).
32. Y. Akiyama and J.P. Draayer, *Comp. Phys. Comm.* **5** (1973) 405.
33. C. Bahri, D.J. Rowe and J.P. Draayer, *Comp. Phys. Comm.* **159** (2004) 121.
34. J. Escher and J.P. Draayer, *J. Math. Phys.* **39** (1998) 5123.