

# Analysis of some atmospheric mesoscale models

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Computational mesoscale models have become an important tool for air pollution studies and operational weather analyses. In this work, we analyze some of these mesoscale models. The results show that some of these models have inconsistencies that limit their use in current applications in Mexico. Some modifications are suggested to improve these models.

**Keywords:** Mesoscale meteorology; Weather analysis and prediction; Air quality and air pollution.

Los modelos computacionales de mesoescala se han convertido en una herramienta importante para estudios de la calidad del aire y para el análisis meteorológico operacional. En este trabajo se analizan algunos modelos usados en México. Los resultados muestran que tales modelos tienen inconsistencias que limitan su uso en algunas aplicaciones. Se sugieren algunas modificaciones para mejorar los modelos.

**Descriptores:** Meteorología de mesoescala; análisis del tiempo y predicción numérica; calidad del aire y contaminación atmosférica.

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## 1. Introduction

Mesoscale atmospheric flows are described as having a temporal and horizontal spatial scale smaller than the rawinsonde network but significantly larger than individual cumulus clouds. This implies that the horizontal scale is on the order of a few kilometers to several hundred kilometers [1, p. 1]. It is generally accepted that if the horizontal scale  $L$  is on the order of  $10^3$  km or smaller [2, p. 10], atmospheric flows can be located in a system of cartesian coordinates  $xyz$  with the plane  $xy$  tangent to the earth at a point; the  $z$  axis and its unit vector  $\mathbf{k}$  are oriented in the opposite direction from the gravity  $\mathbf{g}$ . In this reference system  $\mathbf{g}$  is approximated by  $-\mathbf{g}\mathbf{k}$  with  $g = 9.8 \text{ ms}^{-2}$  and the resulting momentum equation used in the mesoscale literature [1-16] is

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p - \mathbf{g}\mathbf{k} - 2\tilde{\boldsymbol{\Omega}} \times \mathbf{v} + \mathbf{f}. \quad (1)$$

The simplicity of Eq. (1) has been particularly useful for theoretical analysis [2,3] and several computational mesoscale models solve it to analyze and simulate problems with complex topography on a small domain  $\mathcal{D}(L) = 2L \times 2L$  of the  $xy$  plane [1,4-16]. However, some problems have motivated the use of computational mesoscale models on domains  $\mathcal{D}(L)$  larger than  $1000 \times 1000 \text{ km}^2$ . In fact, some authors [17] consider that, in order to realistically simulate mesoscale flows, it is necessary to represent the local terrain influences and simultaneously simulate large-scale synoptic influences. According to Pielke [17], this motivates the use of a domain of at least  $5000 \text{ km}$  on one side, which in turn reduces the boundary errors (which are unavoidable in limited-area numerical prediction models). Although important theoretical results have been obtained from Eq. (1), there has been a computational abuse of it since the results of a recent work [18] show that Eq. (1) is valid on a domain  $\mathcal{D}(L)$  bounded by  $200 \times 200 \text{ km}^2$ .

Reference 18 is not the first one that considers the validity region of Eq. (1). In 1949 MacVitte [19] pointed out that Eq. (1) is valid on a small region  $\mathcal{D}(L)$  of the tangent plane  $xy$ , but the region was not estimated. Unfortunately, MacVitte's work has not been considered in several texts on mesoscale meteorology [1,2,4-7] and the documentation of some computational mesoscale models [8-16]. For instance, Zeytounian comments in Ref. 2, p. 10, that it is generally accepted that if the horizontal scale  $L$  is on the order of  $10^3$  km or smaller, atmospheric flows can be studied with the coordinate system  $xyz$  and the approximation  $\mathbf{g} \sim -\mathbf{g}\mathbf{k}$ . Pielke [17] analyzed seven computational mesoscale models, some of which use the coordinate system  $xyz$  and  $\mathbf{g} \sim -\mathbf{g}\mathbf{k}$  and the models are considered to be valid on a domain  $\mathcal{D}(L)$  of several hundred kilometers [17, p. 459]. These inconsistencies motives for the analysis of some computational mesoscale models.

In Sec. 2, we give a short deduction of the correct momentum equation in the coordinate system  $xyz$  and a summary of the results of Ref. 18 which show that the exact  $\mathbf{g}$  yields a momentum equation valid on any domain  $\mathcal{D}(L)$  whereas the Eq. (1) is valid on a domain  $\mathcal{D}(L)$  bounded by  $200 \times 200 \text{ km}^2$ . In Secs. 2.2-4 we analyze the mesoscale models RAMS [1,8-10], HOTMAC [11-15] and ARPS [16], which have been used for air-quality studies and meteorological research (see, *e.g.*, [1,12-14,17,20-22]). These models solve momentum equations obtained from Eq. (1) but some applications use a domain significantly larger than  $200 \times 200 \text{ km}^2$ .

Map projections have been used in atmospheric modeling with the aim of including the earth's sphericity into model equations [4,7,23,24]. Map projections generate a curvilinear coordinate system  $x_py_pz_p$  which is formally defined in Sec. 3, and a deduction of the momentum equations in coordinates  $x_py_pz_p$  is given in Sec. 3.1. In Sec. 3.2 we give a

summary of the problems generated by the use of map projections to define the topography in models such as ARPS [16] and HOTMAC [15]. In Secs. 3.3-3.5 we analyze some inconsistencies in the models RAMS [10] and MM5 [25-28] with the use of map projections. The model MM5 is of special interest since it was adopted by the Mexican meteorological service to carry out the operational analysis [29,30] and it has been used in several meteorological studies in Mexico [31-35]. Section 4 is devoted to some concluding remarks.

## 2. Momentum equations

Let  $X^i$  denote an inertial cartesian system with its origin at the center of the earth, and the earth's rotation axis coincides with the  $X^3$  axis. The momentum equation of an air particle is

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}$$

where  $p$ ,  $\rho$ ,  $\mathbf{V}$  are the pressure, density and velocity vector of the particle,  $\mathbf{g}$  is the gravity acceleration and  $\mathbf{F}$  is a frictional force. Hereafter we assume that the earth is a sphere with radius  $a$ ; then  $\mathbf{g}$  is given by

$$\mathbf{g} = -g\frac{a^2}{r^3}\mathbf{R}$$

where  $g \equiv GMa^{-2}$ ,  $M$  is the earth mass,  $G$  is the gravitational constant, and  $\mathbf{R}$  is the vector from the earth's center to the air particle,  $r = \|\mathbf{R}\|$  [36]. If  $\hat{\mathbf{X}}^i$  denote the unit vectors and  $X^i$  are the coordinates of a particle at time  $t$ , we have  $\mathbf{R} = \hat{\mathbf{X}}^i X^i$  and  $\mathbf{V} = \hat{\mathbf{X}}^i V^i$  with  $V^i \equiv \dot{X}^i (\equiv dX^i/dt)$ . Repeated indices in one term indicate summation.

Let  $Y^i$  denote a cartesian coordinate system fixed to the earth with  $Y^3 = X^3$ , and  $Y^1, Y^2$  are the axes  $X^1, X^2$  rotated, respectively. If  $\hat{\mathbf{Y}}^i$  are the corresponding unit vectors, then

$$\hat{\mathbf{Y}}^i = \mathbb{P}_{ij}(t)\hat{\mathbf{X}}^j \quad (2)$$

where  $\mathbb{P}_{11}=\mathbb{P}_{22}=\cos\Lambda$ ,  $\mathbb{P}_{12}=-\mathbb{P}_{21}=\sin\Lambda$ ,  $\mathbb{P}_{3i}=\mathbb{P}_{i3}=\delta_{i3}$  (the Kronecker delta) with  $\Lambda = \Omega t + \Lambda_0$ ,  $\Omega$  is the angular velocity of the earth and  $\Lambda_0$  is a constant. If  $Y^i$  are the coordinates of an air particle at time  $t$ , we have  $\mathbf{R} = \hat{\mathbf{X}}^i X^i = \hat{\mathbf{Y}}^i Y^i$  and using (2) we get  $Y^i = \mathbb{P}_{ij}(t)X^j$ . This relationship together with Eq. (2) yields

$$\mathbf{V} = \mathbf{V}_Y + \vec{\Omega} \times \mathbf{R}$$

$$\frac{d\mathbf{V}}{dt} = \mathbf{A}_Y + 2\vec{\Omega} \times \mathbf{V}_Y + \vec{\Omega} \times (\vec{\Omega} \times \mathbf{R}),$$

where  $\mathbf{V}_Y$  and  $\mathbf{A}_Y$  are velocity and acceleration with respect to the earth,

$$\mathbf{V}_Y = \hat{\mathbf{Y}}^i V_Y^i \quad \mathbf{A}_Y = \hat{\mathbf{Y}}^i \dot{V}_Y^i \quad (3)$$

with  $V_Y^i \equiv \dot{Y}^i$  and  $\vec{\Omega} = \Omega \hat{\mathbf{Y}}^3$ . The centripetal acceleration  $\vec{\Omega} \times (\vec{\Omega} \times \mathbf{R})$  is usually neglected. Thus the momentum equation is

$$\mathbf{A}_Y = -\rho^{-1}\nabla p - ga^2r^{-3}\mathbf{R} - 2\vec{\Omega} \times \mathbf{V}_Y + \mathbf{F}, \quad (4)$$

where the original gradient  $\nabla = \hat{\mathbf{X}}^i(\partial/\partial X^i)$  is replaced by  $\nabla = \hat{\mathbf{Y}}^i(\partial/\partial Y^i)$ . The standard literature [1-7] denotes  $\mathbf{A}_Y$  by  $d\mathbf{V}_Y/dt$ , although the latter means

$$d\mathbf{V}_Y/dt = \dot{V}^i \hat{\mathbf{Y}}^i + V^i d\hat{\mathbf{Y}}^i/dt$$

with  $d\hat{\mathbf{Y}}^i/dt = \vec{\Omega} \times \hat{\mathbf{Y}}^i$  [Eq. (2)].

The main coordinate system used in mesoscale meteorology [1-7] is a cartesian system  $x^i$  with its origin at a point on the earth with latitude  $\phi_c$  and longitude  $\lambda_c$ . Let us suppose that the plane  $x^1x^2$  is tangent to the earth at  $(\lambda_c, \phi_c)$  and the axis  $x^3$  is opposite to  $\mathbf{g}$  at  $(\lambda_c, \phi_c)$ . The general relation between the coordinates  $Y^i$  and  $x^i$  of an air particle is

$$x^i = \mathbb{R}_{ij}^c Y^j - a\delta_{i3} \quad (5)$$

with constant  $\mathbb{R}_{ij}^c$ . The orthogonality of the system  $x^i$  implies that the vectors  $\hat{\mathbf{x}}^i = \partial\mathbf{R}/\partial x^i$  are orthonormal,  $\mathbb{R}^c$  is orthogonal and  $\hat{\mathbf{x}}^i = \mathbb{R}_{ij}^c \hat{\mathbf{Y}}^j$ . From this and (2.2,4) we get the transformation of  $\mathbf{V}_Y$  and  $\mathbf{A}_Y$ , namely,

$$\mathbf{V}_Y = \hat{\mathbf{x}}^i u^i \equiv \mathbf{u} \quad \mathbf{A}_Y = \hat{\mathbf{x}}^i \dot{u}^i \equiv \mathbf{a}$$

with  $u^i \equiv \dot{x}^i$ . The gravity acceleration is  $\mathbf{g} = \hat{\mathbf{x}}^i g^i$  with

$$g^i = -ga^2r^{-3}(x^i + \delta_{i3}a). \quad (6)$$

Thus the momentum equation (4) is

$$\mathbf{a} = -\rho^{-1}\nabla p + \hat{\mathbf{x}}^i g^i - 2\vec{\Omega} \times \mathbf{u} + \mathbf{F} \quad (7)$$

with  $\nabla = \hat{\mathbf{x}}^i(\partial/\partial x^i)$ . In scalar form we have

$$\frac{du^i}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x^i} - ga^2r^{-3}(x^i + \delta_{i3}a) - 2\varepsilon_{ijk}\Omega_j u^k + F^i, \quad (8)$$

where  $\varepsilon_{ijk} = \hat{\mathbf{x}}^i \cdot (\hat{\mathbf{x}}^j \times \hat{\mathbf{x}}^k)$ . These equations will be referred to as the *exact momentum equations* since they have the exact components (6) of  $\mathbf{g}$  while the standard mesoscale literature [1-16] uses the approximation  $\mathbf{g} \sim -g\hat{\mathbf{x}}^3$  and the resulting momentum equations

$$\frac{du^i}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x^i} - g\delta_{i3} - 2\varepsilon_{ijk}\Omega_j u^k + F^i. \quad (9)$$

### 2.1. Validity region of momentum equations

Hereafter  $\mathcal{D}(L) = 2L \times 2L$  denotes a rectangular region of the tangent plane  $x^1x^2$  with center at the origin  $x^i = 0$  and  $|x^1|, |x^2| \leq L$ . This section gives a summary of Ref. 18, where the validity region of Eqs. (8) and (9) is estimated. A first approach is given by the magnitude of the terms in (9). Table I yields the magnitude of the terms in the  $u^1$ -equation from (9) as reported by Atkinson [5], where we have added a column with the term  $ga^2r^{-3}x^1$  [omitted in (9)] for a flow with horizontal scale  $L$  (km). We observe that the term of  $ga^2r^{-3}x^1$  is one order of magnitude larger than the largest term of the  $u^1$ -equation from (9) for  $L = 10^2, 10^3$  km. For flows with  $L = 10$  km, the magnitude of  $ga^2r^{-3}x^1$  is equal to that of the Coriolis terms and  $10^4$  times larger than the dissipative terms. These results suggest that the horizontal components of  $\mathbf{g}$  cannot be omitted in Eq. (9) for a region  $\mathcal{D}(L)$  larger than  $200 \times 200$  km<sup>2</sup> and should be considered for  $\mathcal{D}(L)$  between  $\mathcal{D}(L)$   $10 \times 10$  and  $200 \times 200$  km<sup>2</sup>.

TABLE I. Magnitudes in  $\text{ms}^{-2}$  of terms in the  $u^1$ -equation for flows with horizontal scale  $L$  (m),  $U = 10 \text{ ms}^{-1}$ ,  $H = 10^4 \text{ m}$ ,  $f = 2\Omega \sin \phi$ ,  $\phi = 45^\circ$ ,  $g = 10 \text{ ms}^{-2}$  [5], and  $x^1 = L/2$ ,  $x^2 = x^3 = 0$ ,  $r = \sqrt{(x^1)^2 + a^2}$ ,  $a = 6378 \text{ km}$ .

|        | $\frac{du}{dt} =$ | $-\frac{1}{\rho} \frac{\partial p}{\partial x^1}$ | $+fv$     | $-fw$           | $+\frac{\partial}{\partial x^3} K_z \frac{\partial u}{\partial x^3}$ | $+\frac{\partial}{\partial x^1} K_H \frac{\partial u}{\partial x^1}$ | $-\frac{ga^2 x^1}{r^3}$ |
|--------|-------------------|---------------------------------------------------|-----------|-----------------|----------------------------------------------------------------------|----------------------------------------------------------------------|-------------------------|
| $L$    | $\frac{U^2}{L}$   | $\frac{\Delta P}{\rho L}$                         | $fU$      | $\frac{fHU}{L}$ | $\frac{KU}{H^2}$                                                     | $\frac{KU}{L^2}$                                                     |                         |
| $10^6$ | $10^{-4}$         | $10^{-3}$                                         | $10^{-3}$ | $10^{-5}$       | $10^{-6}$                                                            | $10^{-10}$                                                           | $10^0$                  |
| $10^5$ | $10^{-3}$         | $10^{-2}$                                         | $10^{-3}$ | $10^{-4}$       | $10^{-6}$                                                            | $10^{-8}$                                                            | $10^{-1}$               |
| $10^4$ | $10^{-2}$         | $10^{-1}$                                         | $10^{-3}$ | $10^{-3}$       | $10^{-6}$                                                            | $10^{-6}$                                                            | $10^{-2}$               |

Let us consider the calculation of the pressure field of an isothermic atmosphere with zero velocity with respect to the earth. Using the equation of state  $p = \mathcal{R}T\rho$ , where  $T$  and  $\mathcal{R}$  are temperature and the gas constant, and the pressure at the earth's surface,  $p(x^i = 0) \equiv p_0$ , the exact Eqs. (8) yield

$$p(x^i) = p_0 e^{-ba(1-a/r)}$$

where  $b \equiv g/\mathcal{R}T$  and  $r = [(x^1)^2 + (x^2)^2 + (x^3 + a)^2]^{1/2}$ . This is the expected pressure field on the entire terrestrial sphere, with spherical surfaces having a constant pressure. In contrast, the approximate Eqs. (9) yield

$$p^0(x^i) = p_0 \exp(-bx^3).$$

According to this equation, the tangent plane  $x^1x^2$  is a constant-pressure surface with  $p = p_0$ . A simple way of estimating the validity region of Eqs. (9) is given by the relative error of  $p^0(x^i)$  with respect to the exact pressure  $p(x^i) = p_0$  on the terrestrial sphere,  $\Delta p^0 = |p^0/p - 1| \times 100$ . Figure 6 of Ref. 18 shows the graph of  $\xi = \sqrt{2}L$  v.s.  $\Delta p^0$ , where  $\xi/2$  is the length of the diagonal of a given domain  $\mathcal{D}(L) = 2L \times 2L$  (see Ref. 18, Fig. 2) and  $x^3 = -a + \sqrt{a^2 - \xi^2}$ . From this figure we obtain  $\Delta p^0 = 1, 5, 10\%$  for  $L \simeq 20, 50$  and  $70 \text{ km}$ , respectively. The corresponding domains are  $\mathcal{D}(L) \simeq 40 \times 40, 100 \times 100$  and  $140 \times 140 \text{ km}^2$ . If we consider that the largest accepted error of  $p^0(x^i)$  is  $5\%$ , the validity domain of Eqs. (9) is bounded by  $100 \times 100 \text{ km}^2$ ,

$$\mathcal{D}(L) \subset 100 \times 100 \text{ km}^2.$$

Another criterion to estimate the validity domain of (9) is given by the observed horizontal pressure fluctuation on the terrestrial sphere. It is known that this fluctuation is  $\delta p/\rho \sim 10^3 \text{ m}^2\text{s}^{-2}$  for a horizontal scale  $L_s = 10^3 \text{ km}$  on the sphere (see, e.g., [37, Table 2.1]). This yields the estimate  $\rho^{-1}\partial p/\partial x_s \sim 10^{-3} \text{ ms}^{-2}$  used in the standard scale analysis of the momentum equations in curvilinear spherical coordinates  $x_s = a \cos \phi_c(\lambda - \lambda_c)$ ,  $y_s = a(\phi - \phi_c)$ ,  $z_s = r - a$  [2,37]. Let us calculate  $\rho^{-1}\partial p^0/\partial s$  where  $s$  is the length of a circular arc that starts at the origin  $x^i = 0$ . We have  $\xi = a \sin(s/a)$ ,  $x^3 = -a[1 - \cos(s/a)]$  and  $\partial p^0/\partial s = bp^0 \sin(s/a)$ . The density obtained from the equation  $\rho = p^0/\mathcal{R}T$ , yields  $\rho^{-1}\partial p^0/\partial s = g \sin(s/a)$ . If we impose the condition that  $\rho^{-1}\partial p^0/\partial s$  cannot be larger than the observed pressure fluctuation, we get

$g \sin(s/a) \leq 10^{-3} \text{ ms}^{-2}$ , and hence  $s_{\max} = 10^{-4}a = 637.8 \text{ m}$  and  $L_{\max} \simeq a \sin(s_{\max}/a) = 637.8 \text{ m}$ .

In the next sections we analyze some computational mesoscale models that use the approximate Eqs. (9). These models have been used on domains  $\mathcal{D}(L)$  larger than  $100 \times 100 \text{ km}^2$  for the analysis of data provided by meteorological networks but no correction of Eqs. (9) have been reported. This suggests that:

- the number of data is not sufficient to appreciate the error in Eq. (9) generated by the omission of the horizontal components of the exact gravity acceleration  $\mathbf{g}$ , and
- the results given by computational models that use equations such as (9) should be reanalyzed by solving the exact momentum equations.

## 2.2. The RAMS model

Since the primary reference of RAMS [9] is Pilke's book [1], we begin with the analysis of momentum equations reported in Ref. 1. A cartesian coordinate system  $xyz$  with the  $z$  axis normal to the earth at a point with latitude  $\phi$  is used. In this case  $x = x^1$ ,  $y = x^2$ ,  $z = x^3$ ,  $\mathbf{i} = \hat{\mathbf{x}}^1$ ,  $\mathbf{j} = \hat{\mathbf{x}}^2$ ,  $\mathbf{k} = \hat{\mathbf{x}}^3$ , the direction of  $\mathbf{i}$ ,  $\mathbf{j}$  is not defined explicitly in [1]. The gravity and centripetal accelerations are approximated by  $\mathbf{g}, -g\mathbf{k} = \mathbf{G} - \vec{\Omega} \times (\vec{\Omega} \times \mathbf{R})$ , with  $g = 9.8 \text{ ms}^{-2}$ . Thus, the equation of motion is

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\rho} \nabla p - g\mathbf{k} - 2\vec{\Omega} \times \mathbf{u}$$

[1, Eq. (2-33)] or in scalar form

$$\frac{\partial u^i}{\partial t} = -u^j \frac{\partial u^i}{\partial x^j} - \frac{1}{\rho} \frac{\partial p}{\partial x^i} - g\delta_{i3} - 2\varepsilon_{ijk}\Omega_j u^k \quad (10)$$

[1, Eq. (2-45)]. As we saw above, these equations are independent of the direction of the  $x, y$  axes and are valid on a domain  $\mathcal{D}(L)$  bounded by  $100 \times 100 \text{ km}^2$ , but there is no comment about this validity domain in [1].

The Eqs. (10) are written explicitly in chapter 3 of [1] with the Coriolis force

$$-2\vec{\Omega} \times \mathbf{u} = 2\Omega [\mathbf{i}(v \sin \phi - w \cos \phi) - \mathbf{j}u \sin \phi + \mathbf{k}u \cos \phi]$$

[1, p. 31,34]. The direction of the  $x, y$  axes is not defined but this form corresponds to  $-2\vec{\Omega} \times \mathbf{V}$  with

$$\vec{\Omega} = \Omega (\mathbf{j} \cos \phi + \mathbf{k} \sin \phi) \quad \text{and} \quad \mathbf{u} = i\mathbf{u} + \mathbf{j}v + \mathbf{k}w$$

and therefore the  $x$  ( $y$ ) axis is tangent to the parallel circle (meridian) at a point with latitude  $\phi$  and is positive eastward (northward). The pressure gradient is written in terms of the potential temperature  $\theta$  [1, Eq. (2-21)] and the Exner function  $\pi$  [1, Sec. 4.4],

$$\theta \equiv T(p_0/p)^{R/C_p} \quad \pi \equiv C_p(p/p_0)^{R/C_p},$$

which yield

$$\frac{1}{\rho} \frac{\partial p}{\partial x^i} = \theta \frac{\partial \pi}{\partial x^i}. \quad (11)$$

In order to simplify the treatment of the lower boundary conditions, mesoscale models replace the vertical coordinate  $z$  by a terrain-following coordinate  $\sigma$  [1,4-7]. Since the coordinate system  $xy\sigma$  is nonorthogonal, one can use the tensor formalism to get the dynamic equations in such coordinates. The procedure is given in [1] but we shall give some details to show how factor  $g$  appears in the horizontal momentum equations via the pressure gradient.

Consider the transformation equations

$$x^1 = \tilde{x}^1 \quad x^2 = \tilde{x}^2 \quad x^3 = x^3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \quad (12)$$

or equivalently  $\tilde{x}^3 = \tilde{x}^3(x^1, x^2, x^3)$ . The position vector  $\mathbf{R}$  in terms of  $\tilde{x}^i$  and  $x^i$  yields the covariant vectors  $\tau_i$  and contravariant vectors  $\eta^i$ ,

$$\tau_i \equiv \frac{\partial \mathbf{R}}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \hat{\mathbf{x}}^j \quad \eta^i \equiv \nabla \tilde{x}^i = \hat{\mathbf{x}}^j \frac{\partial \tilde{x}^i}{\partial x^j}. \quad (13)$$

The contravariant form of exact equations (8) is obtained from the expression of Eq. (7) in terms of the  $\tau_i$ 's. From (13) we get  $\hat{\mathbf{x}}^j = \tau_i \partial \tilde{x}^i / \partial x^j$  and inserting into  $\mathbf{g} = g^j \hat{\mathbf{x}}^j$  yields  $\mathbf{g} = \tilde{g}^i \tau_i$  with  $\tilde{g}^i = g^j \partial \tilde{x}^i / \partial x^j$ . In particular, for (12) the horizontal components of  $\mathbf{g}$  remain unchanged,

$$\tilde{g}^i = g^i = -ga^2 r^{-3} x^i \quad (i=1,2) \quad \tilde{g}^3 = \frac{\partial \tilde{x}^3}{\partial x^j} g^j. \quad (14)$$

For the pressure gradient we have

$$\nabla p = \eta^j \partial p / \partial \tilde{x}^j = \tau_i \tilde{G}^{ij} \partial p / \partial \tilde{x}^j$$

where  $\tilde{G}^{ij}$  is the inverse matrix of the metric tensor  $\tilde{G}_{ij} = (\partial x^k / \partial \tilde{x}^i) (\partial x^k / \partial \tilde{x}^j)$ . In terms of the Exner function, we have

$$\frac{1}{\rho} \nabla p = \tau_i \tilde{G}^{ij} \theta \frac{\partial \pi}{\partial \tilde{x}^j}. \quad (15)$$

Thus, the exact momentum equations (8) with  $F^i = 0$  are

$$\frac{\partial \tilde{u}^i}{\partial t} = -\tilde{u}^j \tilde{u}_{,j}^i - \theta \tilde{G}^{ij} \frac{\partial \pi}{\partial \tilde{x}^j} - \tilde{g}^i - 2\tilde{\varepsilon}^{ijl} \tilde{\Omega}_j \tilde{u}_l, \quad (16)$$

where  $\tilde{u}^i = u^j \partial \tilde{x}^i / \partial x^j$  are the contravariant components of  $\mathbf{u}$ ,  $\tilde{u}_{,j}^i$  is the covariant derivative of  $\tilde{u}^i$ , and  $\tilde{\Omega}_j$ ,  $\tilde{u}_l$  are the covariant components of  $\vec{\Omega}$ ,  $\mathbf{u}$ .

The factor  $g$  appears in the pressure gradient (15) as follows. Using the components of  $\tilde{G}^{ij}$  [1, Eq. (6-29)] we have

$$\tilde{G}^{ij} \frac{\partial \pi}{\partial \tilde{x}^j} = \frac{\partial \pi}{\partial \tilde{x}^i} + \frac{\partial \tilde{x}^3}{\partial \tilde{x}^i} \frac{\partial \pi}{\partial \tilde{x}^3} \quad \text{for } i=1,2 \quad (17)$$

where the chain rule yields

$$\frac{\partial \pi}{\partial \tilde{x}^3} = \frac{\partial x^3}{\partial \tilde{x}^3} \frac{\partial \pi}{\partial x^3}. \quad (18)$$

Following Pielke, the hydrostatic approximation  $\rho^{-1} \partial p / \partial x^3 = -g$  and Eq. (11) are used to get

$$\partial \pi / \partial x^3 = -g / \theta$$

and Eq. (18) becomes  $\partial \pi / \partial \tilde{x}^3 = -g \theta^{-1} \partial x^3 / \partial \tilde{x}^3$ . For the terrain following-coordinate

$$\tilde{x}^3 \equiv \sigma = s \frac{z - z_G(x, y)}{s - z_G(x, y)}, \quad (19)$$

where  $s$  is a constant and  $z_G(x, y)$  is the terrain elevation on  $(x, y)$ , we have  $\partial \pi / \partial \tilde{x}^3 = -s^{-1} (s - z_G) g \theta^{-1}$  [1, Eq. (6-49)] and (17) is

$$\tilde{G}^{ij} \frac{\partial \pi}{\partial \tilde{x}^j} = \frac{\partial \pi}{\partial \tilde{x}^i} - \frac{g}{\theta} \frac{\sigma - s}{s} \frac{\partial z_G}{\partial \tilde{x}^i} \quad (i=1,2).$$

Thus the exact horizontal momentum Eqs. (16) are

$$\begin{aligned} \frac{\partial \tilde{u}^i}{\partial t} = & -\tilde{u}^j \tilde{u}_{,j}^i - \theta \frac{\partial \pi}{\partial \tilde{x}^i} + g \frac{\sigma - s}{s} \frac{\partial z_G}{\partial \tilde{x}^i} \\ & + g^i - 2\tilde{\varepsilon}^{ijl} \tilde{\Omega}_j \tilde{u}_l \quad (i=1,2) \end{aligned}$$

where  $g^i$  are the horizontal components of  $\mathbf{g}$  (14). If the approximation  $\mathbf{g} \sim -g\mathbf{k}$  is used,  $g^1$  and  $g^2$  are replaced by 0 to obtain

$$\begin{aligned} \frac{\partial \tilde{u}^i}{\partial t} = & -\tilde{u}^j \tilde{u}_{,j}^i - \theta \frac{\partial \pi}{\partial \tilde{x}^i} + g \frac{\sigma - s}{s} \frac{\partial z_G}{\partial \tilde{x}^i} \\ & - 2\tilde{\varepsilon}^{ijl} \tilde{\Omega}_j \tilde{u}_l \quad (i=1,2). \end{aligned} \quad (20)$$

These are basically the equations (6-56,57) reported in [1] where the factor  $g$  appears without the use of the exact  $\mathbf{g}$ . Additional references [8,10] of RAMS use the approximation  $\mathbf{g} \sim -g\mathbf{k}$  to get horizontal momentum equations similar to Eq. (20) where the factor  $g$  appears in  $\nabla p$  via the chain rule and the hydrostatic approximation as above.

In agreement with the use of the approximation  $\mathbf{g} \sim -g\mathbf{k}$  in Refs. 1 and 8, the documentation [10, p. 6] of RAMS reports the momentum equations

$$\begin{aligned} \frac{du}{dt} = & -\theta \frac{\partial \pi'}{\partial x} + f v + \nabla \cdot K_m \nabla u \\ \frac{dv}{dt} = & -\theta \frac{\partial \pi'}{\partial y} - f u + \nabla \cdot K_m \nabla v \\ \frac{dw}{dt} = & -\theta \frac{\partial \pi'}{\partial z} - \frac{g \theta_v}{\theta_0} + \nabla \cdot K_m \nabla w, \end{aligned} \quad (21)$$

where only the vertical momentum equation has the factor  $g$ ,  $f = 2\Omega \sin \phi_c$ , and we use the notation

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

$$\nabla \cdot K_m \nabla u \equiv \frac{\partial}{\partial x} \left( K_m \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_m \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_m \frac{\partial u}{\partial z} \right).$$

Since the approximation  $\mathbf{g} \sim -g\mathbf{k}$  is used, the horizontal momentum equations (21) are valid on a domain  $\mathcal{D}(L) \subset 100 \times 100 \text{ km}^2$ , but the documentation [10] does not comment on this aspect.

### 2.3. The HOTMAC model

The HOTMAC model uses the momentum equations in tangent-plane  $x_i$  coordinates

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x^j} = -\frac{1}{\rho} \frac{\partial P}{\partial x^i} - g\delta_{i3} - 2\varepsilon_{ijk} \Omega_j u_k + v \frac{\partial^2 u_i}{\partial x^j \partial x^j}$$

where the approximation  $\mathbf{g} \sim -g\mathbf{k}$  is used [11]. These equations are written in terms of the terrain-following vertical coordinate

$$z^* = \bar{H} \frac{z - z_g(x, y)}{H - z_g(x, y)} \quad (22)$$

with  $H = \bar{H} + z_g(x, y)$  [12, Eqs. (1,7)], where  $\bar{H}$  is the initial value of  $H$  and  $z_g(x, y)$  is the ground elevation. The horizontal momentum equations in coordinates  $x_1 x_2 z^*$  contain the factor  $g$ , which appears via the pressure gradient as above. In fact, the horizontal pressure gradient in terms of the Exner function  $\Pi = C_p (P/P_0)^\varepsilon$ , the virtual potential temperature  $\Theta_v \equiv (P_0/P)^\varepsilon T_v = C_p \Pi^{-1} T_v$  and  $z^*$  [12],

$$\left( \frac{\partial P}{\partial x^i} \right)_z = \rho \Theta_v \left[ \left( \frac{\partial \Pi}{\partial \tilde{x}^i} \right)_{z^*} + \frac{\partial z^*}{\partial x^i} \frac{\partial \Pi}{\partial z^*} \right] \quad \text{for } i = 1, 2.$$

The hydrostatic equation

$$\langle \rho \rangle^{-1} \partial P / \partial z = -[1 - \beta(\Theta_v - \langle \Theta_v \rangle)] g,$$

with  $\rho / \langle \rho \rangle = 1 - \beta(\Theta_v - \langle \Theta_v \rangle)$ , and the chain rule yields

$$\frac{\partial \Pi}{\partial z^*} = \frac{\partial z}{\partial z^*} \frac{\partial \Pi}{\partial z} = \frac{H - z_g - g}{\bar{H}} \frac{1}{\Theta_v},$$

where  $\langle \cdot \rangle$  indicates an average over a horizontal plane. This and the Reynold's decomposition  $u_i = \bar{U}_i + u'_i$  ( $\bar{U}_i$  denotes ensemble average) yield the horizontal momentum equations [12]

$$\begin{aligned} \frac{DU}{Dt} &= f(V - V_g) + g \frac{\bar{H} - z^*}{\bar{H}} \left( 1 - \frac{\langle \Theta_v \rangle}{\Theta_v} \right) \frac{\partial z_g}{\partial x} + \frac{\partial}{\partial x} \left( K_x \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_{xy} \frac{\partial U}{\partial y} \right) + \frac{\bar{H}}{H - z_g} \frac{\partial}{\partial z^*} (-\overline{u'w'}) + C_1 \frac{DV}{Dt} \\ &= -f(U - U_g) + g \frac{\bar{H} - z^*}{\bar{H}} \left( 1 - \frac{\langle \Theta_v \rangle}{\Theta_v} \right) \frac{\partial z_g}{\partial y} + \frac{\partial}{\partial x} \left( K_{xy} \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_y \frac{\partial V}{\partial y} \right) + \frac{\bar{H}}{H - z_g} \frac{\partial}{\partial z^*} (-\overline{v'w'}) + C_2 \end{aligned} \quad (23)$$

where

$$C_i = g \left[ \langle \Theta_v \rangle \langle \Theta_v(H) \rangle^{-1} - (\langle \Theta_v \rangle \Theta_v^{-1} - 1) z^* (\bar{H})^{-1} - 1 \right] \frac{\partial H}{\partial x^i}$$

and the geostrophic wind components  $U_g, V_g$  are given in [12, Eq. (8)]. Model variables written in upper case represent ensemble averages. In agreement with the use of  $\mathbf{g} \sim -g\mathbf{k}$ , the horizontal components of  $\mathbf{g}$  are omitted and therefore the validity domain of Eqs. (23) is  $\mathcal{D}(L) \subset 100 \times 100 \text{ km}^2$ . This region is congruent with the earliest purpose of studying the airflow on a region of  $25 \times 25 \text{ km}^2$  [12]. In other work Yamada and Bunker [13] neglect the terms  $C_i$  to study atmospheric transport and diffusion of airborne over a region of  $30 \times 30 \text{ km}^2$ . In a subsequent work [14], the terms  $C_1, C_2$  were replaced by  $G_u (U_{\text{obs}} - U)$ ,  $G_v (V_{\text{obs}} - V)$ , respectively, to simulate airflow with a four-dimensional data assimilation technique, where  $G_u, G_v$  are nudging coefficients and  $U_{\text{obs}}, V_{\text{obs}}$  are observations of  $U, V$  at each 6 hr observation interval. However, the original small-domain conception of Eqs. (23) was missed since the resulting momentum equations [14, Eqs. 3,4], which do not

consider the horizontal components  $g^1, g^2$  (14) of  $\mathbf{g}$  or their linear approximations  $-gx/a$  and  $-gy/a$ , were used to assimilate data from an observation network on a domain  $\mathcal{D}(L) \sim 1600 \times 1300 \text{ km}^2$ .

### 2.4. The ARPS model

The ARPS model considers dynamic equations in tangent-plane coordinates  $xyz$ , and each meteorological variable is decomposed as follows

$$\psi(x, y, z, t) = \psi_0(z) + \bar{\psi}(x, y, z, t)$$

where  $\psi_0$  corresponds to an atmospheric base state which is horizontally homogeneous and  $\bar{\psi}$  is a perturbation of  $\psi_0$  [16, p. 117]. This decomposition shows the local conception of ARPS, since a base state that depends only of  $z$  is correct in a vicinity of the origin  $x = y = z = 0$ . The ARPS documentation [16] does not point out this local character, instead it asserts that the model is suitable for use on

scales ranging from a few meters to hundreds of kilometers [16, p. 113], but the validity domain is small, as is shown below.

Chapter 7 of Ref. 16 describes the map projection options to define the computational domain and says that “the map factors are not included in the dynamic equations of ARPS version 4.0, but will be in a future release”. “If one chooses the Lambert conformal projection and uses a relatively small domain (less than 1000 km), then the effect of the map factor is negligible” (the role of map projections is discussed in Sec. 3). Accordingly, the model can be applied on a domain  $\mathcal{D}(L)$  as large as  $1000 \times 1000 \text{ km}^2$ . This assertion is incorrect since

- (i) on a domain larger than  $700 \times 700 \text{ km}^2$ , the troposphere is below the tangent  $xy$  plane, and
- (ii) the approximation  $\mathbf{g} \sim -g\mathbf{k}$  is used in momentum equations.

In fact, the model solves dynamic equations in terrain following coordinates  $\xi = x, \eta = y, \zeta = \zeta(x, y, z)$  with  $\zeta(x, y, z)$  similar to Eq. (19) [16, p. 114, 199] which yield the horizontal momentum equations [16, p. 119]

$$\begin{aligned} \frac{dv^*}{dt} &= - \left[ \frac{\partial}{\partial \eta} J_3(p' - \alpha \text{Div}^*) + \frac{\partial}{\partial \zeta} J_2(p' - \alpha \text{Div}^*) \right] \\ &\quad - \rho^* f u + \sqrt{G} D_v \\ \frac{du^*}{dt} &= - \left[ \frac{\partial}{\partial \xi} J_3(p' - \alpha \text{Div}^*) + \frac{\partial}{\partial \zeta} J_1(p' - \alpha \text{Div}^*) \right] \\ &\quad + \rho^* [f v - \tilde{f} w] + \sqrt{G} D_u \end{aligned} \quad (24)$$

where the horizontal components  $g^1, g^2$  of  $\mathbf{g}$  are omitted;  $J_1 = -\partial z / \partial \xi, J_2 = -\partial z / \partial \eta, \sqrt{G} = |\partial z / \partial \zeta|$ , the notation  $u^* = \rho^* u$  with  $\rho^* = \bar{\rho} \sqrt{G}$  is used and  $\alpha \text{Div}^*$  is an artificial divergence damping term to attenuate acoustic waves. In fact, Eqs. (24) are obtained from the momentum equation for a Boussinesq fluid

$$\frac{\partial \vec{V}}{\partial t} + \nabla \cdot (\vec{V} \vec{V}) = -\nabla \frac{p'}{\rho_0} + B\mathbf{k}$$

with  $B = -g\rho'/\bar{\rho}$  and  $\mathbf{g} \sim -g\mathbf{k}$  [16, p. 350]. As we saw in Sec. 2.1, the last approximation is valid on a domain  $\mathcal{D}(L)$  bounded by  $100 \times 100 \text{ km}^2$ .

### 3. Map projections

Map projections have been used in numerical weather prediction models to consider the earth's sphericity [4,7,22,23]. Map projections generate curvilinear coordinates  $x_p y_p z_p$  which are defined below and will be called *projection coordinates*.

The spherical coordinates of a point with position vector  $\mathbf{R} = \hat{\mathbf{Y}}^i Y^i$  are defined by

$$Y^1 = r \cos \phi \cos \lambda \quad Y^2 = r \cos \phi \sin \lambda \quad Y^3 = r \sin \phi \quad (25)$$

where  $r = \|\mathbf{R}\|$ ,  $\phi$  is the latitude and  $\lambda$  the longitude. The projection of a point  $(\lambda, \phi, r = a)$  on the earth is a point  $(x_p, y_p)$  in a cartesian system  $x_p y_p$  given by a pair of equations

$$x_p = x_p(\lambda, \phi) \quad y_p = y_p(\lambda, \phi). \quad (26)$$

The center  $(\lambda_c, \phi_c, r = a)$  of the domain  $\mathcal{D}$  is projected on the origin of the system  $x_p y_p$ ,  $x_p(\lambda_c, \phi_c) = y_p(\lambda_c, \phi_c) = 0$ . If a point has spherical coordinates  $(\lambda, \phi, r)$ , its projection coordinates  $(x_p, y_p, z_p)$  are defined by Eq. (26) and

$$z_p = r - a. \quad (27)$$

Since conformal map projections are used in meteorology,  $x_p, y_p, z_p$  define an orthogonal curvilinear system.

#### 3.1. Equations in projection coordinates

In order to obtain dynamic equations in projection coordinates, we first consider some expressions in spherical coordinates. Let  $s^1 = \lambda, s^2 = \phi, s^3 = r$ . The substitution of Eqs. (25) into  $\mathbf{R} = \hat{\mathbf{Y}}^i Y^i$  yields orthonormal vectors  $\hat{\mathbf{s}}^i = \mathbf{s}^i / h_{si}$  with  $\mathbf{s}^i = \partial \mathbf{R} / \partial s^i$  and  $h_{si} = \|\mathbf{s}^i\|$ . The relation between  $\hat{\mathbf{Y}}^j$  and  $\hat{\mathbf{s}}^i$  is

$$\hat{\mathbf{s}}^i = \mathbb{R}_{ij} \hat{\mathbf{Y}}^j \quad (28)$$

where  $\mathbb{R}$  is given by

$$\mathbb{R}(\lambda, \phi) = \begin{pmatrix} -\sin \lambda & \cos \lambda & 0 \\ -\sin \phi \cos \lambda & -\sin \phi \sin \lambda & \cos \phi \\ \cos \phi \cos \lambda & \cos \phi \sin \lambda & \sin \phi \end{pmatrix} \quad (29)$$

and satisfies  $\mathbb{R}_{ik} \mathbb{R}_{ij} = \delta_{ij}$ . Hence  $\mathbf{V}_Y$  (3) is transformed as follows:

$$\mathbf{V}_Y = \dot{\mathbf{Y}}^j \hat{\mathbf{Y}}^j = \dot{s}^i \frac{\partial Y^j}{\partial s^i} \hat{\mathbf{Y}}^j = \dot{s}^i h_{si} \hat{\mathbf{s}}^i = u_s^i \hat{\mathbf{s}}^i$$

where  $u_s^i \equiv \dot{s}^i h_{si}$ . From the identity  $\mathbf{V}_Y = V_Y^k \hat{\mathbf{Y}}^k = u_s^i \hat{\mathbf{s}}^i$  and Eq. (28) we get  $V_Y^k = \mathbb{R}_{ik} u_s^i$ ; this and Eq. (28) yield  $\mathbf{A}_Y = \hat{\mathbf{Y}}^k \dot{V}_k = \hat{\mathbf{s}}^j \mathbb{R}_{jk} (\mathbb{R}_{ik} \dot{u}_s^i + \mathbb{R}_{ik} u_s^i)$  and therefore

$$\mathbf{A}_Y = \hat{\mathbf{s}}^j a_s^j, \quad a_s^j \equiv \dot{u}_s^j + \mathbb{R}_{jk} \mathbb{R}_{ik} u_s^i. \quad (30)$$

Let us now consider calculations in projection coordinates with  $x_p^1 = x_p, x_p^2 = y_p, x_p^3 = z_p$ . The orthogonality of coordinates  $x_p^i$  means the orthogonality of the unitary vectors

$$\hat{\mathbf{x}}_p^i = \mathbf{x}_p^i / h_{pi} \quad \text{with } \mathbf{x}_p^i = \partial \mathbf{R} / \partial x_p^i, \quad h_{pi} = \|\mathbf{x}_p^i\|. \quad (31)$$

The contravariant vectors in spherical and projection coordinates,  $\eta_s^j = \nabla s^j$  and  $\eta_p^i = \nabla x_p^i$ , can be written as follows

$$\eta_s^j = \nabla s^j = \frac{\hat{\mathbf{s}}^j}{h_{sj}} \quad \eta_p^i = \nabla x_p^i = \frac{\hat{\mathbf{x}}_p^i}{h_{pi}}. \quad (32)$$

These identities and the chain rule yield

$$\begin{aligned}\frac{\hat{\mathbf{x}}_p^i}{h_{pi}} &= \nabla x_p^i = \hat{\mathbf{Y}}^k \frac{\partial x_p^i}{\partial Y^k} = \hat{\mathbf{Y}}^k \frac{\partial s^j}{\partial Y^k} \frac{\partial x_p^i}{\partial s^j} \\ &= \nabla s^j \frac{\partial x_p^i}{\partial s^j} = \frac{\hat{\mathbf{s}}^j}{h_{sj}} \frac{\partial x_p^i}{\partial s^j}\end{aligned}$$

or

$$\hat{\mathbf{x}}_p^i = h_{pi} \sum_j \hat{\mathbf{s}}^j h_{sj}^{-1} \partial x_p^i / \partial s^j,$$

and hence

$$h_{pi} = \left[ \sum_j \left( h_{sj}^{-1} \frac{\partial x_p^i}{\partial s^j} \right)^2 \right]^{-1/2}.$$

For the Eqs. (26) and (27) we have  $\partial x_p^i / \partial s^3 = \delta_{i3}$  and therefore  $h_{p3} = 1$ ,

$$h_{pi} = \left[ \sum_{j=1,2} \left( \frac{1}{h_{sj}} \frac{\partial x_p^i}{\partial s^j} \right)^2 \right]^{-1/2} \quad (i = 1, 2), \quad (33)$$

and

$$\hat{\mathbf{x}}_p^i = T_{ij} \hat{\mathbf{s}}^j \quad \text{with } T_{ij} = \frac{h_{pi}}{h_{sj}} \frac{\partial x_p^i}{\partial s^j}, \quad (34)$$

without summation in repeated indices in the last equation, and  $T_{3i} = T_{i3} = \delta_{3i}$ . We assume that  $\mathbb{T} = \{T_{ij}\}$  defines a rotation of  $\hat{\mathbf{s}}^1$  and  $\hat{\mathbf{s}}^2$ , then  $\hat{\mathbf{x}}_p^1 \times \hat{\mathbf{x}}_p^2 = \hat{\mathbf{x}}_p^3$ ,  $\det(T_{ij}) = 1$ , and the normalization and orthogonality of  $\hat{\mathbf{x}}_p^i$  and  $\hat{\mathbf{s}}^j$  yield  $T_{11}T_{22} - T_{12}T_{21} = 1$ ,

$$T_{11}^2 + T_{12}^2 = T_{21}^2 + T_{22}^2 = 1 \quad T_{11}T_{21} + T_{12}T_{22} = 0. \quad (35)$$

Hence  $T_{11} = T_{22} \equiv T_1$ ,  $T_{12} = -T_{22} \equiv T_2$  and

$$\mathbb{T} = \begin{pmatrix} T_1 & T_2 & 0 \\ -T_2 & T_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (36)$$

If  $b_p^i$  and  $b_s^j$  denote the physical components of a vector  $\mathbf{b}$  in coordinates  $x_p^i$  and  $s^j$ , respectively, that is,  $\mathbf{b} = \hat{\mathbf{x}}_p^i b_p^i = \hat{\mathbf{s}}^j b_s^j$ , Eq. (34) yields

$$b_p^i = T_{ij} b_s^j. \quad (37)$$

In particular, the physical components of  $\mathbf{V}_Y$  in projection coordinates are

$$u_p^1 = T_1 u_s^1 + T_2 u_s^2 \quad u_p^2 = -T_2 u_s^1 + T_1 u_s^2 \quad u_p^3 = u_s^3. \quad (38)$$

Another expression of  $u_p^i$  is obtained by combining  $\mathbf{V}_Y = \hat{\mathbf{Y}}^k \dot{Y}^k$  with Eq. (31) and the chain rule, namely

$$u_p^i = h_{pi} \dot{x}_p^i. \quad (39)$$

If  $a_p^i$  denote the physical components of  $\mathbf{A}_Y$  in projection coordinates, Eq. (37) yields  $a_p^i = T_{ij} a_s^j$ . From Eq. (37) we get

$u_s^j = T_{mj} u_p^m$  and inserting it into Eq. (30) for  $a_s^j$  yields

$$\begin{aligned}a_p^i &= T_{ij} \left( \frac{d}{dt} T_{mj} u_p^m + R_{jk} \dot{R}_{lk} T_{ml} u_p^l \right) \\ &= \frac{du_p^i}{dt} + Q_{im} u_p^m\end{aligned} \quad (40)$$

where we use  $T_{ij} T_{mj} = \delta_{im}$  and  $\mathbb{Q} \equiv \mathbb{T} \dot{\mathbb{T}}^t + \mathbb{T} \mathbb{R} \dot{\mathbb{R}}^t \mathbb{T}^t$ . In order to compare the expressions in projection coordinates with those reported in the literature, the following notation will be used:

- (i)  $u_s = u_s^1, v_s = u_s^2, w_s = u_s^3$ ,
- (ii)  $\hat{\lambda} = \hat{s}^j, \hat{\phi} = \hat{s}^2, \hat{\mathbf{r}} = \hat{\mathbf{s}}^3$ ,
- (iii)  $u_p = u_p^1, v_p = u_p^2, w_p = u_p^3$ ,
- (iv)  $\hat{\mathbf{x}}_p = \hat{\mathbf{x}}_p^1, \hat{\mathbf{y}}_p = \hat{\mathbf{x}}_p^2, \hat{\mathbf{z}}_p = \hat{\mathbf{x}}_p^3$ , and
- (v)  $h_x = h_{p1}, h_y = h_{p2}$ .

From (35) and (36) we get the antisymmetric matrix  $\mathbb{T} \dot{\mathbb{T}}^t$ ,  $\mathbb{R} \dot{\mathbb{R}}^t$  is obtained from (29) and  $\dot{s}^i = u_s^i / h_{si}$ , and using (38) one gets  $Q_{im} u_p^m$  or in matrix form

$$\mathbb{Q} \begin{pmatrix} u_p \\ v_p \\ w_p \end{pmatrix} = \begin{pmatrix} -(r^{-1} u_s \tan \phi + \xi) v_p + r^{-1} u_p v_p \\ (r^{-1} u_s \tan \phi + \xi) u_p + r^{-1} u_p v_p \\ -r^{-1} (u_p^2 + v_p^2) \end{pmatrix}$$

with  $\xi = T_1 \dot{T}_2 - T_2 \dot{T}_1$ . To compute the Coriolis force in projection coordinates we define  $e = 2\Omega \cos \phi$  and  $f = 2\Omega \sin \phi$ . In spherical coordinates we have  $2\vec{\Omega} = e\hat{\phi} + f\hat{\mathbf{r}}$  and using Eq. (37) we get  $2\vec{\Omega} = \hat{\mathbf{x}}_p e T_2 + \hat{\mathbf{y}}_p e T_1 + \hat{\mathbf{z}}_p f$  and

$$\begin{aligned}2\vec{\Omega} \times \mathbf{V}_Y &= \hat{\mathbf{x}}_p (e w_p T_2 - f v_p) + \hat{\mathbf{y}}_p (-e w_p T_1 + f u_p) \\ &\quad + \hat{\mathbf{z}}_p e (v_p T_2 - u_p T_1).\end{aligned}$$

Using (32) and the chain rule, the pressure gradient takes the form

$$\nabla p = \left( \hat{\mathbf{x}}_p \frac{1}{h_x} \frac{\partial}{\partial x_p} + \hat{\mathbf{y}}_p \frac{1}{h_y} \frac{\partial}{\partial y_p} + \hat{\mathbf{z}}_p \frac{\partial}{\partial z_p} \right) p.$$

Substituting the above results into the exact momentum equation (4) with  $\mathbf{F} = \mathbf{0}$  and the approximation  $ga^2/r^2 \sim g$  yields

$$\begin{aligned}\frac{du_p}{dt} + \frac{1}{h_x \rho} \frac{\partial p}{\partial x_p} &= v_p \left( f + \frac{u_s \tan \phi}{r} + \xi \right) \\ &\quad - e w_p T_1 - \frac{u_p w_p}{r} \\ \frac{dv_p}{dt} + \frac{1}{h_y \rho} \frac{\partial p}{\partial y_p} &= -u_p \left( f + \frac{u_s \tan \phi}{r} + \xi \right) \\ &\quad + e w_p T_2 - \frac{v_p w_p}{r} \\ \frac{dw_p}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial z_p} + g &= e (u_p T_1 - v_p T_2) + \frac{u_p^2 + v_p^2}{r}\end{aligned} \quad (41)$$

with  $u_s = u_p T_1 - v_p T_2$  [Eq. (37)],  $\xi = T_1 \dot{T}_2 - T_2 \dot{T}_1$  and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u_p}{h_x} \frac{\partial}{\partial x_p} + \frac{v_p}{h_y} \frac{\partial}{\partial y_p} + w_p \frac{\partial}{\partial z_p}.$$

### 3.2. Map projections and topography

Terrain elevation data are defined with respect to an ellipsoidal earth model [38,39], but some models consider that the data are defined with respect to a sphere without considering the method to define such a sphere [10,16,28]. This omission is unfortunate because an incorrect choice of the sphere radius can generate significant errors in the terrain elevation data. For instance, the data base GTOPO30 [39] has data defined with respect to the ellipsoid WGS84 (World Geodetic System 1984) with axes  $a = 6378$ ,  $b = 6357$  km [38]. The relative difference  $(a - b)/b$  is small but the difference  $a - b = 21$  km is equal to the average height of the troposphere. A method to define properly a spherical earth model with the parameters of an ellipsoid is given in Ref. 24. Thus we can consider that the terrain data are defined with respect to a sphere  $\mathcal{S}_a$  with radius  $a$ . If  $h_s(\lambda, \phi)$  denotes the terrain elevation at point  $(\lambda, \phi, r = a)$  of  $\mathcal{S}_a$ , the set of points  $\{\lambda, \phi, h_s\}$  defines the true earth surface which is called *geoid*. In practice the geoid is known on a discrete set of points  $\{\lambda_k, \phi_k, h_{sk}\}_{k=1}^N$  known as the *digital elevation model*.

Some mesoscale models use the following procedure to define the topography. If a terrain datum has spherical coordinates  $(\lambda, \phi, r = h_s + a)$ , a point  $(x_p, y_p)$  is computed with a map projection (26) and it is assumed that the terrain elevation at such a point is  $h_s$ . Of course,  $(x_p, y_p)$  belongs to the  $x_p y_p$  plane in the abstract  $x_p y_p z_p$  space, but of the terrain height at the domain center  $(\lambda_c, \phi_c, r = a)$  is *defined* as the datum  $h_s(\lambda_c, \phi_c)$ , the  $x_p y_p$  plane coincides with the  $xy$  plane tangent to the sphere  $\mathcal{S}_a$  at  $(\lambda_c, \phi_c)$ . Additionally, if the scale of the  $x, y, z$  and  $x_p, y_p, z_p$  axes is the same, then every point  $(x_p, y_p, z_p \equiv h_p)$  defines a point in the  $xyz$  coordinate system (see details in Ref. 24). To clarify the geometrical meaning of  $(x_p, y_p, h_p)$ , let us remember that if a point  $P$  has spherical coordinates  $(\lambda, \phi, r = h_s + a)$ , its *unique* and *correct* cartesian coordinates  $x, y, z$  are obtained from Eqs. (5) and (25). Thus, if  $(x_p, y_p, h_p)$  has coordinates  $(\lambda, \phi, r = h_s + a)$ , the corresponding cartesian coordinates are given by

$$\begin{pmatrix} x \\ y \\ z + a \end{pmatrix} = \mathbb{R}(\lambda_c, \phi_c) \begin{pmatrix} (h_s + a) \cos \phi \cos \lambda \\ (h_s + a) \cos \phi \sin \lambda \\ (h_s + a) \sin \phi \end{pmatrix} \quad (42)$$

where  $\mathbb{R}(\lambda_c, \phi_c)$  is given by Eq. (29). This shows that  $x, y, z$  and  $x_p, y_p, z_p$  are different. In particular, if  $P$  is a point in the geoid with coordinates  $(\lambda_k, \phi_k, r_k = h_{sk} + a)$  and the corresponding projection coordinates  $x_{pk}, y_{pk}, h_{pk}$  are seen as the cartesian coordinates of a point  $P'$  in physical space, it is clear the  $P'$  does not belong to the geoid in general. The documentation of HOTMAC [15] and ARPS [16] does not report the use of Eq. (42) to obtain the correct coordinates

$x_k, y_k, z_k$ ; instead,  $x_{pk}, y_{pk}, h_{pk}$  are considered to be valid approximations of  $x_k, y_k, z_k$ ,

$$x_{pk} \sim x_k \quad y_{pk} \sim y_k \quad h_k \sim z_k,$$

and the set  $\{x_{pk}, y_{pk}, h_{pk}\}$  is a reliable digital elevation model of the geoid with respect to the  $xyz$  system; but this is not necessarily the case. Since map projections generate a minimum distortion of the sphere  $\mathcal{S}_a$ ,  $x_{pk}, y_{pk}$  are similar to  $x_k, y_k$  over a wide domain  $\mathcal{D}(L)$ . For instance, Figs. 4,5 of Ref. 24 show that the relative error  $|y_{pk} - y_k|/y_k$  is very small for  $|y_k| \leq 1665$  km with several map projections. The problem lies in  $z_{pk}$ ; if  $(x_{pk}, y_{pk})$  is close to the origin  $(x = 0, y = 0, z = 0) = (\lambda_c, \phi_c, r = a)$ , the difference  $|z_{pk} - z_k|$  is small but it increases rapidly with  $x_{pk}$  or  $y_{pk}$ . For example, the correct cartesian coordinates of a point on  $\mathcal{S}_a$  with projection coordinates  $x_{pk} = z_{pk} = 0$ ,  $y_{pk} = 650$  (km) are  $x_k = 0$ ,  $y_k = 650 \pm 8$ ,  $z_k = -33$  (km). The data  $h_k$  reported in GTOPO30 for Mexico have an uncertainty of  $\pm 30$  m [39]; in this case the approximation  $z_k \sim h_k$  is valid on a domain  $\mathcal{D}(L)$  bounded by Ref. 24

$$\mathcal{D}_h \sim 60 \times 60 \text{ km}^2$$

which is very small for some mesoscale applications, as we shall see below.

The ARPS documentation [16] describes three map projections that the user can employ to define the computational domain and the topography. The latter is defined with projection coordinates  $x_{pk}, y_{pk}, h_{pk}$  when a terrain datum has spherical coordinates  $(\lambda_k, \phi_k, h_k)$ . If all calculations were done in projection coordinates, the use of the “terrain data”  $\{x_{pk}, y_{pk}, h_{pk}\}$  would be correct. However, the model ARPS combine the data  $\{x_{pk}, y_{pk}, h_{pk}\}$ , with dynamic equations in coordinates  $xyz$ . In fact, “the map factors are not included in the dynamic equations of ARPS version 4.0”, but “If one chooses the Lambert conformal projection and uses a relatively small model domain (less than 1000 km), then the effect of the map factors is negligible” [16, Sec. 7.1]. This assertion can lead to an incorrect use of the model because

- (i) the momentum equations use the approximation  $\mathbf{g} \sim -g\mathbf{k}$  which is valid on  $\mathcal{D}(L) \subset 100 \times 100 \text{ km}^2$ ,
- (ii) the troposphere is below the  $xy$  plane when  $\mathcal{D}(L)$  is larger than  $700 \times 700 \text{ km}^2$ , and
- (iii) the data  $\{x_{pk}, y_{pk}, h_{pk}\}$  are a reliable estimation of the geoid on a domain  $\mathcal{D}(L) \subseteq 60 \times 60 \text{ km}^2$ .

According to Ref. 15, the model HOTMAC uses the Universal Transverse Mercator (UTM) projection [38] to define the horizontal coordinates and the topography. This means that the horizontal coordinates are projection coordinates  $x_p, y_p$  defined by the UTM projection. However, in Sec. 2.3 we saw that the model equations are written and solved in coordinates  $xyz^*$  (22). Thus, the validity domain of HOTMAC is  $\mathcal{D}_h \sim 60 \times 60 \text{ km}^2$ , since the corresponding terrain



data  $\{x_{pk}, y_{pk}, h_{pk}\}$  and the approximation  $\mathbf{g} \sim -g\mathbf{k}$  are valid on such a domain. The domain  $\mathcal{D}_h$  is congruent with the original conception of HOTMAC, which was developed for meteorological simulations on domains that are  $25 \times 25$  [12] and  $30 \times 30$  km<sup>2</sup> [13]. However, this conception was missed in Ref. 14, where the approximation  $\mathbf{g} \sim -g\mathbf{k}$  and the UTM system are used to assimilate meteorological data on a domain  $\mathcal{D}(L) \sim 1300 \times 1600$  km<sup>2</sup>. In this domain the separation between the  $xy$  plane and the sphere  $S_a$  reaches up to 33 km. In another application [20] a domain of  $120 \times 150$  km<sup>2</sup> was used to analyze air pollution data in Mexico City; in this case the separation between  $S_a$  and the  $xy$  plane reaches 1.6 km.

The use of a terrain-following vertical coordinate  $\sigma_{zp}$ , which is defined with a terrain elevation  $z_{hp}(x, y)$  obtained via map projections, introduces the error of  $z_{hp}(x, y)$  directly into dynamic equations such as (23) and (24). Of course, the solution of this problem is simple since all that is needed is to eliminate the use of map projections to define the topography (which is a procedure without computational, physical or mathematical advantages) and use a correct coordinate transformation, as is done in Ref. 24.

### 3.3. Map projections in RAMS model

The correct use of map projections consists in transforming terrain data and each meteorological variable into the space of projection coordinates  $x_p y_p z_p$ , and solving the dynamic equations in such a space (equations such as (41)). This procedure is used in models RAMS and MM5 but there are some inconsistencies, as we shall see below.

The horizontal domain  $\mathcal{D}(L)$  and the terrain elevation of model RAMS are defined with a rotated Polar-Stereographic Projection (PSP) [10, p. 6], “where the pole of projection is rotated to an area near the center of the domain, thus minimizing the distortion of the projection in the main area of interest” and “The appropriate map factors are used in all horizontal derivative terms”. Documentation [10] and additional references [1, 8, 9] do not describe the procedure for introducing the “map factors” in horizontal derivatives, but there is an inconsistency in the use of the PSP.

There are two ways to introduce the map factors into dynamic equations. The first one consists in using the map projection equations (26) of the PSP to transform the dynamic equations in spherical coordinates  $\lambda \phi r$  into equations in projection coordinates  $x_p y_p z_p$ , as was done in section 3.1. The second way consists in using the Eqs. (26), (27) and (42) to transform the equations in coordinates  $xyz$  into equations in coordinates  $x_p y_p z_p$ . In each case, the exact acceleration  $\mathbf{g} = -ga^2 r^{-3} \mathbf{R}$  has to be used to get momentum equations in coordinates  $x_p y_p z_p$  that are valid on any domain  $\mathcal{D}(L)$ . However, in Sec. 2.2 we saw that the documentation [10, p. 6] and complementary Refs. 1 and 8 of RAMS use the approximation  $\mathbf{g} \sim -g\mathbf{k}$ . Therefore, independently of the method used to introduce the map factors into model equations, the momentum equations are valid on a domain  $\mathcal{D} \subset 200 \times 200$  km<sup>2</sup>. This domain is very small with respect to that of  $1764 \times 1764$  km<sup>2</sup> used to analyze the meteorologi-

cal data of the project of Investigación sobre Materia Particulada y Deterioro Atmosférico - Aerosol and Visibility Evaluation (IMADA-AVER), which was conducted by the U.S. Department of Energy and the Instituto Mexicano del Petróleo in 1997 [21].

### 3.4. Model MM5 version 2

Versions 2 and 3 of model MM5 [26, 27] have the same choice of three map projections for defining the model domain. The documentation of Program Terrain [28] describes, in detail, the use of these map projections in defining the topography. The model computes the input information in projection coordinates  $x_p y_p$ , and  $z_p$  is replaced by the terrain-following coordinate

$$\sigma = \frac{p_0(z_p) - p_{top}}{p^*(x_p, y_p)} \quad (43)$$

where  $p_{top}$  is a constant,  $p^* = p_s(x_p, y_p) - p_{top}$ ,  $p_s$  is the pressure on the topography, and  $p_0(z_p)$  is the pressure of an atmospheric reference state which obeys the hydrostatic equation  $dp_0/dz_p = -g\rho_0(z_p)$ , and the equation of state  $p_0 = RT_0\rho_0$  with  $T_0 = T_{00} + A \log(p_0/p_{00})$ . From these equations, the expression of  $z_p$  in terms of  $\sigma$  follows, and using the chain rule one gets the relationships

$$\begin{aligned} \left( \frac{\partial}{\partial x_p} \right)_{z_p} &= \left( \frac{\partial}{\partial x_p} \right)_\sigma - \frac{\sigma}{p^*} \frac{\partial p^*}{\partial x_p} \frac{\partial}{\partial \sigma} \\ \left( \frac{\partial}{\partial y_p} \right)_{z_p} &= \left( \frac{\partial}{\partial y_p} \right)_\sigma - \frac{\sigma}{p^*} \frac{\partial p^*}{\partial y_p} \frac{\partial}{\partial \sigma} \\ \frac{\partial}{\partial z_p} &= -\frac{\rho_0 g}{p^*} \frac{\partial}{\partial \sigma} \end{aligned} \quad (44)$$

which has to be used to get the correct momentum equations in coordinates  $x_p y_p \sigma$ , but to simplify the notation we use the left side of Eq. (44).

The momentum equations reported in the documentation of MM5 version 2 [26] are

$$\frac{du_p}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial x_p} = v_p \left( f + u_p \frac{\partial m}{\partial y} \right) - ew_p - \frac{u_p w_p}{a} + D_u \quad (45)$$

$$\frac{dv_p}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial y_p} = -u_p \left( f + u_p \frac{\partial m}{\partial y} \right) - \frac{v_p w_p}{a} + D_v \quad (46)$$

$$\frac{dw_p}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial z_p} + g = eu_p + \frac{u_p^2 + v_p^2}{a} + D_w \quad (47)$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_p \frac{\partial}{\partial x_p} + v_p \frac{\partial}{\partial y_p} + w_p \frac{\partial}{\partial z_p}$$

where the gradient pressure is written in terms of  $x_p y_p z_p$ , instead of using the right side of Eq. (44). In the next section we show that the correct Eqs. (41) can be rewritten with  $m = 1/h_x = 1/h_y$  and

$$r^{-1} u_s \tan \phi + \xi = u_p \partial m / \partial y_p - v_p \partial m / \partial x_p.$$

Thus, a comparison with the correct equations (41) shows that:

- (i) the Eqs. (45)-(47) omit  $m$  in the horizontal pressure gradient and the advection terms of  $d/dt$ ,
- (ii) in Eq. (45) the term  $-v_p \partial m / \partial x_p$  is absent and  $ew_p$  has to be replaced by  $ew_p T_1$ ,
- (iii) in Eq. (46) the terms  $-v_p \partial m / \partial x_p$  and  $ew_p T_2$  are absent,
- (iv) in Eq. (47) the term  $-ev_p T_2$  is absent and  $eu_p$  has to be replaced by  $eu_p T_1$ .

In order to obtain additional information about the theoretical formulation of MM5 version 2, the equations of Ref. 25 cited in the documentation of MM5 version 2 were considered. In this reference the hydrostatic approximation is applied to the vertical momentum equation and the vertical coordinate  $\sigma$  Eq. (43), is used. The momentum equations are

$$\begin{aligned} \frac{\partial p^* u}{\partial t} &= -\frac{\partial p^* u u}{\partial x} - \frac{\partial p^* \sigma u}{\partial x} - \frac{\sigma R_d T p^*}{p} \frac{\partial p^*}{\partial x} - p^* \frac{\partial \phi}{\partial x} \\ &\quad + f p^* (v - v_g) + p^* (F_u + D_u + P_u + C_u) \\ \frac{\partial p^* v}{\partial t} &= -\frac{\partial p^* u v}{\partial x} - \frac{\partial p^* \sigma v}{\partial x} - f p^* (u - u_g) \\ &\quad + p^* (F_v + D_v + P_v + C_v) \end{aligned} \quad (48)$$

with  $A_u = p^* (F_u + D_u + P_u + C_u)$ ,  $A_v = p^* (F_v + D_v + P_v + C_v)$ , where  $F$ ,  $D$ ,  $P$ ,  $C$ ,  $R$  are horizontal eddy diffusion, horizontal fourth order smoothing, vertical eddy flux convergence, cumulus transport term, and radiative heating, respectively. From Eqs. (48) we get the following conclusions: (i) The absence of the factors  $h_i = \|\partial \mathbf{R} / \partial x_p^i\|$  implies  $h_i = 1$  and therefore the horizontal coordinates are the cartesian coordinates  $xy$  of a plane tangent to the sphere  $S_a$ . Thus, the horizontal components of  $\mathbf{g}$  are necessary for modeling on a domain larger than  $100 \times 100 \text{ km}^2$ . However, (ii) the horizontal components of  $\mathbf{g}$  or their linear approximation are absent in (48) and therefore the approximation  $\mathbf{g} \sim -g\mathbf{k}$  is used, although the Eqs. (48) were used in Ref. 25 to simulate convection on a region with each side measuring 600 km.

### 3.5. Model MM5 version 3

The momentum equations of model MM5 version 3 [27] are

$$\begin{aligned} \frac{du_p}{dt} + \frac{m}{\rho} \frac{\partial p}{\partial x_p} &= v_p \left( f + u_p \frac{\partial m}{\partial y_p} - v_p \frac{\partial m}{\partial x_p} \right) \\ &\quad - ew_p \cos \alpha - \frac{u_p w_p}{a} + D_u \\ \frac{dv_p}{dt} + \frac{m}{\rho} \frac{\partial p}{\partial y_p} &= -u_p \left( f + u_p \frac{\partial m}{\partial y_p} - v_p \frac{\partial m}{\partial x_p} \right) \\ &\quad + ew_p \sin \alpha - \frac{v_p w_p}{a} + D_v \\ \frac{dw_p}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial z_p} + g &= e (u_p \cos \alpha - v_p \sin \alpha) \\ &\quad + \frac{u_p^2 + v_p^2}{a} + D_w, \end{aligned} \quad (49)$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + mu_p \frac{\partial}{\partial x_p} + mv_p \frac{\partial}{\partial y_p} + w_p \frac{\partial}{\partial z_p}$$

where  $m$  is the map-scale factor,  $e = 2\Omega \cos \phi$ ,  $f = 2\Omega \sin \phi$ , where  $\phi$  is the latitude,  $\alpha = \lambda - \lambda_c$ ,  $\lambda$  is longitude and  $\lambda_c$  is central longitude. These equations are similar to the correct ones (41) but there are some differences.

The Mexican meteorological service uses model MM5 version 3 [29,30] to carry out the operational analysis on a domain  $\mathcal{D} \sim 3330 \times 3330 \text{ km}^2$  defined by a Lambert projection [29]. The projection is [28]

$$x_p = R_s(\phi) \sin(k\alpha) \quad y_p = -R_s(\phi) \cos(k\alpha) \quad (50)$$

with

$$\begin{aligned} R_s(\phi) &= (am \cos \phi) / k, \\ m &= [\tan B / \tan B_1]^k \cos \phi_1 / \cos \phi, \\ B &= 2^{-1} (90^\circ - \phi), \\ B_1 &= 2^{-1} (90^\circ - \phi_1) \end{aligned}$$

and

$$k = \frac{\log [\cos \phi_1 / \cos \phi_2]}{\log [\tan (\pi/4 - \phi_1/2) / \tan (\pi/4 - \phi_2/2)]}.$$

In this case the Eqs. (33) yield  $h_x = h_y = r/am$ ,  $T_1 = \cos k\alpha$ ,  $T_2 = -\sin k\alpha$ ,  $\xi = -ku_s/r \cos \phi$ . Some algebraic manipulations and  $r \sim a$  yield

$$\frac{u_s \tan \phi}{r} + \xi = u_p \frac{\partial m}{\partial y_p} - v_p \frac{\partial m}{\partial x_p} \quad (51)$$

and the correct Eqs. (41) are

$$\begin{aligned} \frac{du_p}{dt} + \frac{m}{\rho} \frac{\partial p}{\partial x_p} &= v_p \left( f + u_p \frac{\partial m}{\partial y_p} - v_p \frac{\partial m}{\partial x_p} \right) \\ \frac{dv_p}{dt} + \frac{m}{\rho} \frac{\partial p}{\partial y_p} &= -u_p \left( f + u_p \frac{\partial m}{\partial y_p} - v_p \frac{\partial m}{\partial x_p} \right) \\ \frac{dw_p}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial z_p} + g &= e (u_p \cos k\alpha + v_p \sin k\alpha) \frac{dv_p}{dt} \\ &\quad + \frac{m}{\rho} \frac{\partial p}{\partial y_p} + \frac{u_p^2 + v_p^2}{a}, \end{aligned} \quad (52)$$

where the approximation  $r \sim a$  is used. According to (49), with  $D_u = D_v = D_w = 0$ , the MM5 equations are

$$\begin{aligned} \frac{du_p}{dt} + \frac{m}{\rho_p} \frac{\partial p_p}{\partial x_p} &= v_p \left( f + u_p \frac{\partial m}{\partial y} - v_p \frac{\partial m}{\partial x} \right) \\ &\quad - ew_p \cos \alpha - \frac{u_p w_p}{a} \\ \frac{dv_p}{dt} + \frac{m}{\rho_p} \frac{\partial p_p}{\partial y_p} &= -u_p \left( f + u_p \frac{\partial m}{\partial y} - v_p \frac{\partial m}{\partial x} \right) \\ &\quad + ew_p \sin \alpha - \frac{v_p w_p}{a} \\ \frac{dw_p}{dt} + \frac{1}{\rho_p} \frac{\partial p_p}{\partial z_p} + g &= e(u_p \cos \alpha - v_p \sin \alpha) \\ &\quad + \frac{u_p^2 + v_p^2}{a}. \end{aligned} \quad (53)$$

We can see the following differences:

- (i) MM5 equations (53) have  $\cos \alpha$  and  $-\sin \alpha$  instead of the correct coefficients  $\cos k\alpha$  and  $\sin k\alpha$ , respectively. Factor  $k$  appears in  $T_1, T_2$  by means of the Jacobian matrix  $\partial x_p^i / \partial s^j$  of the projection equations (50), whereas the MM5 equations omit such a factor. To get an idea of the magnitude of  $k$  we have  $k = 0.34$  for  $\phi_1 = 15^\circ, \phi_2 = 25^\circ$  and  $k = 3.4$  for  $\phi_1 = 35^\circ, \phi_2 = 55^\circ$ .
- (ii) Although the terms  $ew_p \cos \alpha$  and  $ew_p \sin \alpha$  are small, their incorrectness increases the error when the equations (53) are solved numerically.
- (iii) The incorrect terms  $eu_p \cos \alpha, -ev_p \sin \alpha$  are small with respect to  $g$ , but their magnitude is equal to or larger than that of  $dw_p/dt$ . For example, for large scale synoptic systems  $dw_p/dt \sim 10^{-7} \text{ ms}^{-2}$ ,  $eu_p \cos \alpha \sim 10^{-3} \text{ ms}^{-2}$  for  $|\phi| \leq 30^\circ$  and  $ev_p \sin \alpha \sim 10^{-3} \text{ ms}^{-2}$  for  $|\phi| \geq 45^\circ$  [37, Table 2.2]. This invalidates the use of the  $w_p$ -equation to compute vertical motions which are important for the prognostic of rainfall.

The Mercator projection used by MM5 is [28]

$$x_p = a\alpha \quad y_p = a \ln [(1 + \sin \phi) / \cos \phi]. \quad (54)$$

In this case, the Eqs. (33) yield  $h_x = h_y = r/am$  with  $m = 1/\cos \phi, T_1 = 1, T_2 = \xi = 0$  and using  $r \sim a$ . Equation (51) holds true and therefore the correct Eqs. (41) have the form

$$\begin{aligned} \frac{du_p}{dt} + \frac{m}{\rho_p} \frac{\partial p_p}{\partial x_p} &= v_p \left( f + u_p \frac{\partial m}{\partial y} - v_p \frac{\partial m}{\partial x} \right) \\ &\quad - ew_p - \frac{u_p w_p}{r} \end{aligned}$$

$$\begin{aligned} \frac{dv_p}{dt} + \frac{m}{\rho_p} \frac{\partial p_p}{\partial y_p} &= -u_p \left( f + u_p \frac{\partial m}{\partial y} - v_p \frac{\partial m}{\partial x} \right) \\ &\quad - 0 - \frac{v_p w_p}{r} \\ \frac{dw_p}{dt} + \frac{1}{\rho_p} \frac{\partial p_p}{\partial z_p} + g &= eu_p + \frac{u_p^2 + v_p^2}{r}. \end{aligned} \quad (55)$$

According to (49) the equations of MM5 version 3 are

$$\begin{aligned} \frac{du_p}{dt} + \frac{m}{\rho_p} \frac{\partial p_p}{\partial x_p} &= v_p \left( f + u_p \frac{\partial m}{\partial y} - v_p \frac{\partial m}{\partial x} \right) \\ &\quad - ew_p \cos \alpha - \frac{u_p w_p}{a} \\ \frac{dv_p}{dt} + \frac{m}{\rho_p} \frac{\partial p_p}{\partial y_p} &= -u_p \left( f + u_p \frac{\partial m}{\partial y} - v_p \frac{\partial m}{\partial x} \right) \\ &\quad + ew_p \sin \alpha - \frac{v_p w_p}{a} \end{aligned}$$

$$\begin{aligned} \frac{dw_p}{dt} + \frac{1}{\rho_p} \frac{\partial p_p}{\partial z_p} + g &= e(u_p \cos \alpha - v_p \sin \alpha) \\ &\quad + \frac{u_p^2 + v_p^2}{a}. \end{aligned} \quad (56)$$

There are the following differences:

- (i) The Eqs. (56) have the coefficients  $\cos \alpha$  and  $\sin \alpha$  instead of the correct ones, 1 and 0, respectively. We have  $T_2 = \xi = 0$  since the Jacobian matrix  $\partial x_p^i / \partial s^j$  of (54) is diagonal.
- (ii) As above, the incorrect terms in horizontal momentum equations ( $-ew_p \cos \alpha$  and  $+ew_p \sin \alpha$ ) are small but they increase the numerical errors.
- (iii) The incorrect terms  $eu_p \cos \alpha$  and  $-ev_p \sin \alpha$  in the vertical momentum equation invalidate its use in computing  $w_p$  (and predicting rainfall) since at least one term is larger than  $dw_p/dt$ , as we saw above [37].

The documentation of MM5 version 3 [27, pp. 8-5] comments that Eqs. (49) include terms ( $eu_p$  and  $ev_p$ ) representing the usually neglected components of the Coriolis force, but the examples above show that such terms are incorrect in projection coordinates.

## 4. Conclusions

There has been an important effort to develop and calibrate computational mesoscale models which use the approximation  $\mathbf{g} \sim -g\mathbf{k}$  [1,4-17]. This approximation yields a simple momentum equation useful for theoretical analysis [3], but not for numerical simulations of real atmospheric flows on a domain larger than  $100 \times 100 \text{ km}^2$ . The estimation of this validity domain (Sec. 2.1) ignores important aspects of a real flow such as stratification or time dependence. Since

these factors can generate qualitatively different flows from the exact and approximate momentum equations (8) and (9) because of the nonlinearities of such equations [40], the validity domain of the approximation  $\mathbf{g} \sim -g\mathbf{k}$  may be significantly smaller. On the other hand, Pielke [17] suggests that the numerical modeling of some real mesoscale flows may require a domain with 5000 km on a side. A partial solution to the conflict is the use of the exact value of  $\mathbf{g}$ , which allows the use of the tangent-plane system  $xyz$  on a domain  $\mathcal{D}(L) \subset 700 \times 700 \text{ km}^2$ , since the troposphere is below the  $xy$  plane for  $\mathcal{D}(L) \supset 700 \times 700 \text{ km}^2$ . Of course, the optimal solution is the development of computational models that explicitly consider the earth sphericity. The use of map projections into the dynamic equations is legitimate if the latter are written correctly in projection coordinates  $x_p y_p z_p$ .

The mesoscale models considered in this work have an increasing number of users in Mexican institutions devoted to meteorological research [31-35] and operational meteorological analysis [29,30]. This has been motivated by the results found by some users. For instance, some authors [33] claim that version 2 of MM5 generates the meteorological products that Mexican Institutions should use to prevent disasters caused by severe storms in México City, but the analysis of sections 3.4 and 3.5 shows that versions 2 and 3 of MM5 cannot be used to predict rainfall. These incongruences show that the data available on Mexican territory are not enough to validate mesoscale models.

The use of mesoscale models in México has not considered the solution of operational problems inherent in any meteorological network. For instance, the Mexican meteorological service uses the model MM5 version 3 to predict meteorological conditions. This model defines the initial conditions with data generated by global meteorological models, which in turn use the data from the rawinsondes in Mexico at 12 hr observation intervals. The meteorological service has the open problem of validating data from rawinsonde and surface stations, but this problem is not considered with the use of MM5. Of course, the reliability of MM5 depends heavily on the data from global models which in turn depend on the quality of data provided by the Mexican meteorological service. To this we must add the fact that the estimation of meteorological conditions with the data from a network is an unsolved open problem [41]. This means that the initial conditions used by a model like MM5 are not optimal and therefore the resulting numerical predictions have to be carefully validated [42].

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