The depolarization field in polarizable objects of general shape

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The polarization of particles or biological cells is commonly investigated by measuring the impedance of suspensions or by a variety of single particle methods, that exploit different force effects. For biological cells the most striking frequency-dependent changes in polarizability result from structural (Maxwell-Wagner) polarization phenomena. Explicit solutions of the Laplace equation are available only for objects with finite surfaces of the second degree. Thus, dielectric models consider the structural properties of cells by assuming spherical or ellipsoidal geometries, since only in very few cases is the effective local field $E_i(r)$ in the presence of a dielectric object known. This concerns dielectric bodies of special shape, which are exposed to a special electric field $E_0(r)$. In the present paper an approximation procedure is presented for the general case, allowing to calculate the depolarization field $E_i(r)$, which is generated in the presence of an arbitrarily shaped dielectric object, introduced into a field space $\vec{E}_0(\vec{r})$. Contrary to recent numerical methods (finite element technique), which require extensive computer resources due to the unavailability of analytical solutions, the here presented approach results in closed analytical expressions. The applicability of the method is demonstrated for a non-ellipsoidal cylindrical dielectric by measuring its dipole moment in a microwave field. The accordance with the calculated results is found to be one order of magnitude better than it would be in the commonly practiced procedure, where the cylinder is substituted by a spheroid of the same axis relation.

Keywords: Dielectrics; polarization; depolarization field; bioelectronics.

La polarización de partículas o celdas biológicas se estudia comúnmente a través de la medición de la impedancia de suspensiones, o bien por una variedad de métodos de partículas sencillas, que se basan en efectos diferentes de fuerza. Para celdas biológicas, los cambios en la polarización más notables dependiendo de la frecuencia, resultan de fenómenos estructurados de polarización (Efecto Maxwell-Wagner). Modelos dieléctricos consideran las propiedades estructurales de celdas suponiendo geometrías esféricas o elipsoidales, puesto que solo en muy pocos casos se conoce el campo efectivo local $E_i(r)$ en presencia de un objeto dieléctrico. Esto se refiere a cuerpos dieléctricos de forma especial, que están expuestos a un campo eléctrico especial $E_0(r)$. En el presente trabajo se muestra un procedimiento de aproximación para el caso general que permite el cálculo del campo local $E_i(r)$ generado en presencia de un objeto dieléctrico de forma arbitraria, introducido en el espacio de campo $\vec{E}_0(\vec{r})$. La aplicabilidad del método es demostrada para un cilindro dieléctrico no-elipsoidal a través de la medición de su momento dipolar en un campo de microondas. La correspondencia con los resultados calculados se encuentra un orden de magnitud mejor que en el procedimiento común de aproximar el cilindro por un esferoide con las mismas relaciones axiales.

Descriptores: Dieléctricos; campo local, polarización, bioelectrónica.

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1. Introduction

Only in very few cases can the electric field $\vec{E}(\vec{r})$ in the presence of matter with dielectric constant κ be calculated. In such cases an object of special shape is usually brought into a special electric field $\vec{E}_0(\vec{r})$. With an intelligent guess or ansatz is then shown, that the boundary conditions of the fields on the surface of the dielectric material are fulfilled [1-5].

Dielectric models applied in diverse areas of research consider the structural properties of matter by assuming spherical or general ellipsoidal geometries. This is widely used, e.g., in the case of the interaction of a.c. electric fields with colloidal particles and biological cells. A variety of methods, based on impedance measurements and on different force effects are employed for the dielectric characterization of single objects [6-10]. The general Laplace solution for the polarization of single-shell ellipsoids, a standard model of biological cells [11, 12], was also derived for the meteorological problem of dust particles covered by a water layer [13]. In the Laplace model, a homogeneous ellipsoid always exhibits a constant internal local field. Integrating over this field leads to the induced dipole moment and thus to analytical expressions related to force actions on the particles.

Already in such important cases as a cube or a cylinder is it difficult to calculate the depolarization factors without assuming a spheroid as substituting body shape. But even then is the best shape of it to be chosen, and the design of the next approximation step, not straightforward. Driven by the growing interest and impact of physical contributions to life sciences [14-16], complex geometries, such as rods and cylinders, need to be considered. Characterized by the unavailability of analytical solutions for the field distribution within such dielectric bodies, finite element numerical techniques have been developed recently [17-20], with the compromise of claiming considerable computer resources, though.

This paper deals with an approximation procedure for the calculation of the depolarization field $\vec{E}_i(\vec{r})$ in a material body of general shape with a dielectric constant κ, which is brought into any given field $\vec{E}_0(\vec{r})$, yielding an analytical solution.

2. Depolarization field calculation

The problem, which is treated here, can be formulated as follows: The internal field $\vec{E}_i(\vec{r})$ generates a polarization

This polarization induces on the surface element $\Delta\vec{F}$ of the dielectric body a polarization charge

$$\Delta q = \sigma_{pol} \cdot \Delta F = \vec{P} \cdot \Delta\vec{F},$$

which by virtue of the Coulomb law, together with the unperturbed field $\vec{E}_0(\vec{r})$, generates finally the depolarization field, such that

$$\vec{E}_i = (\vec{r}_1) = \vec{E}_0(\vec{r}_1) - \oiint \frac{\vec{r}_{12}}{4\pi\varepsilon_0 r_{12}^3}(\vec{P}(\vec{r}_2) \cdot \Delta\vec{F}_2). \quad (1)$$

The integration is carried out over the surface of the dielectric body; $\Delta\vec{F}$ points outward, and $\vec{r}_{12}$ combines the origin at $\vec{r}_1$ with the integration element at $\vec{r}_2$. The relation between $\vec{E}_0(\vec{r})$ and $\vec{E}_i(\vec{r})$ is supposed by us to be lineal:

$$\begin{aligned}\vec{P}(\vec{r}) &= \varepsilon_0(\kappa-1)\vec{E}_i(\vec{r}) \\ &= \varepsilon_0(\kappa-1)\alpha(\vec{r})\vec{E}_0(\vec{r}) \end{aligned} \quad (2)$$

In general, $\alpha(\vec{r})$ is a tensor, as the directions of $\vec{E}_i$ and $\vec{E}_0$ are not necessarily parallel. It further depends on the coordinates inside the sample due to the locally different action of the polarization charges. In order to calculate $\vec{P}(\vec{r})$ or $\vec{E}i(\vec{r})$ from Eqs. (1, 2), we make the assumption, that α does not depend on $\vec{r}$, which of course is exactly fulfilled only in homogeneous ellipsoids. Here it is an approximation, which allows us to get to viable solutions which will be tested at the end by an experimental comparison.

The polarization, established inside the dielectric, is due to the displacement of electrical charges enforced by the field $\vec{E}_0(\vec{r})$. Surface charges are built up and counteract the complete displacement corresponding to the field $\vec{E}_0(\vec{r})$. We will suppose here, that the whole set of charges experiences the same displacement, which means, that α = constant. We further suppose, that the polarization vector $\vec{P}$ points more or less into the direction of $\vec{E}_0(\vec{r})$, *i.e.*, we will consider the projection of the field, generated by the polarization charges, on the direction of $\vec{E}_0(\vec{r})$:

$$\begin{aligned}\vec{P}(\vec{r}_1) &= \varepsilon_0(\kappa-1)\alpha_1\vec{E}_0(\vec{r}_1) \\ &= \varepsilon_0(\kappa-1)\left\{\vec{E}_0(\vec{r}_1) - \oiint \frac{(\kappa-1)\alpha_1}{4\pi} \cdot \frac{\vec{E}_0(\vec{r}_2)\cdot d\vec{F}_2}{\vec{r}_{12}^3} \cdot \frac{\vec{E}_0(\vec{r}_1)\cdot\vec{r}_{12}}{\vec{E}_0^2(\vec{r}_1)} \cdot \vec{E}_0(\vec{r}_1)\right\}, \\ \alpha_1 &= \left\{1 + \frac{\kappa-1}{4\pi}\oiint \frac{\left(\vec{E}_0(\vec{r}_2)\cdot d\vec{F}_2\right)}{\vec{r}_{12}^3} \cdot \frac{\left(\vec{E}_0(\vec{r}_1)\cdot r_{12}\right)}{\vec{E}_0^2(\vec{r}_1)}\right\}^{-1}. \end{aligned} \quad (3)$$

This value α_1 allows to consider a first approximation of the polarization $\vec{P}_1$, which on the surface of the dielectric generates charges, and thus an additional field inside the dielectric. The problem would be completely solved, if the total field at any place fulfills already the condition

$$\vec{E}_i = \vec{P}_1/\varepsilon_0(\kappa-1),$$

but in general, the polarization $\vec{P}_1$ of the first approximation step will not be sufficient to describe the real situation, and a field $\vec{E}_1(\vec{r})$ keeps acting on the dielectric with the effect of an additional polarization $\vec{P}_2(\vec{r})$,

$$\begin{aligned}\vec{E}_1(\vec{r}) = \vec{E}_0(\vec{r}) - \oiint \frac{\vec{r}_{12}}{4\pi\varepsilon_0\, r_{12}^3}\left(\vec{P}_2(\vec{r}_2)\cdot d\vec{F}_2\right) \\ -\vec{P}_1(\vec{r}_1)/\varepsilon_0(\kappa-1). \end{aligned} \quad (4)$$

$\vec{P}_2(\vec{r})$can be calculated with $\vec{E}_1(\vec{r})$ in the same way, as $\vec{P}_1(\vec{r})$ was calculated with $\vec{E}_0(\vec{r})$.

The number of approximation steps needed to achieve the best result depends on the complexity of the shape of the dielectric body, as well as the allowed error of the result.

3. The depolarization field in a dielectric cylinder

Exact solutions are known for the sphere, the infinitesimal thin wire, and the infinitesimal extended disk. When our approach is applied here, already the first approximation step gives the exact solution, as it should be, when

$$\alpha(\vec{r}) = \alpha_1 = constant.$$

The polarization of a prolate spheroid (Fig. 1) results with Eq. 2 in

$$\alpha=\left\{1-(\kappa-1)q^2\left(1+\frac{\sqrt{q^2+1}}{2}\cdot \ell n\frac{\sqrt{q^2+1}-1}{\sqrt{q^2+1}+1}\right)\right\}^{-1}, \quad (5)$$

where $q = b^2/(a^2 + b^2)$.

For an oblate ellipsoid one gets

$$\alpha = \left\{1-(\kappa-1)(q^2+1)(q\, arc\, tan\frac{1}{q}-1)\right\}^{-1}$$

and consequently with q $\rightarrow$ ∞ (or a = b) we have for the sphere-shaped dielectric

$$\alpha = 3/(\kappa+2),$$

and thus the known result

$$\vec{P} = 3\varepsilon_0\frac{\kappa-1}{\kappa+2}\cdot\vec{E}_0.$$

Not so straightforward is the situation in the case of a cylinder in a homogeneous electric field $\vec{E}_0(\vec{r})$ (see Fig. 2). We get

$$\oiint \frac{\vec{E}_0(\vec{r}_2)\cdot d\vec{F}_2}{\vec{r}_{12}^3}\,\frac{\vec{E}_0(\vec{r}_1)\cdot\vec{r}_{12}}{\vec{E}_0^2(\vec{r}_1)} = 2\int_0^R \frac{2\pi a\, da\cdot L}{(a^2+L^2)^{3/2}}$$
$$= \int_L^{(R^2+L^2)^{1/2}} \frac{r\, dr}{r^3}\cdot 4\pi L$$
$$= 4\pi\left[1-(1+R^2/L^2)^{-1/2}\right],$$

$$P_1 = \frac{\varepsilon_0(\kappa-1)\vec{E}_0}{1+(\kappa-1)\left(1-(1+R^2/L^2)^{-1/2}\right)}. \quad (6)$$

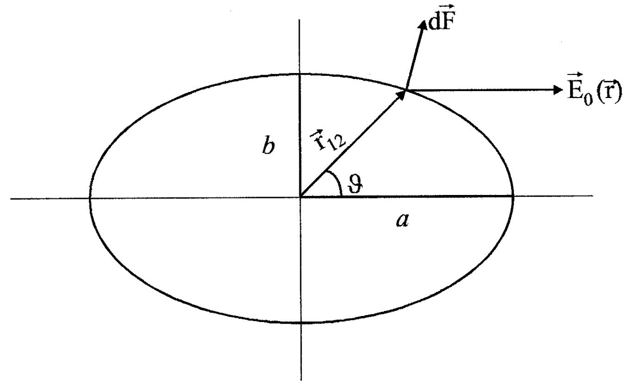

FIGURE 1. Prolate ellipsoid.

Such a homogeneous polarization is only the first approximation. Due to the choice of the origin at $z = 0$, the by P generated field will be too weak in the transversal plane at $z = 0$, but along the z -axis at the limiting faces of the cylinder it is too strong. A field $\vec{E}_1(\vec{r})$ remains as given in Eq. (4), which delivers the depolarization at the cylinder top and bottom faces in a second approximation step.

The integrations, involved in this step, are quite tedious and will not be carried out exactly here. We proceed instead as follows: $\vec{E}_1(\vec{r})$ is largest at the center of the plane cylinder faces, thus we put the origin $\vec{r}_1$ at the center of one of this faces S (see Fig. 2) and concentrate all the charge $\sigma_1 = \left|\vec{P}_1\right|$ at this center. (The real $\vec{E}_1$ might be slightly larger, but this effect is compensated by a stronger inclination against the surface). The charge at the opposite side face acts on $\vec{r}_1$ after the Coulomb law, and provided $L/R \gg 1$, like a point charge $\pi R^2\sigma_1/4\pi\varepsilon_0\cdot 4L^2$, but for L/R $\gg$ 1 like an extended charged disk with $\sigma_1/2\varepsilon_0$. A suitable interpolation of this field for the complete range L/R is about $(\sigma_1/2\varepsilon_0)(1+8L^2/R^2)^{-1}$. The normal component of the field belonging to the side face, which contains $\vec{r}_1$ is $\sigma_1/2\varepsilon_0$, and consequently results the normal component $\vec{E}_{1\perp S}$ at both side faces S to

$$\vec{E}_{1\perp S} = E_0 - \frac{\sigma_1}{2\varepsilon_0} - \frac{\sigma_1}{2\varepsilon_0}(1+8L^2/R^2)^{-1} - \frac{\sigma_1}{\varepsilon_0(\kappa-1)}.$$

The normal component of $\vec{E}_1$ at the cylinder cover area C (see Fig. 2) is approximately

$$\vec{E}_{1\perp C} = \frac{\pi R^2\sigma_1\cdot R}{4\pi\varepsilon_0\left[R^2+(L-z)^2\right]^{3/2}} - \frac{\pi R^2\sigma_1\cdot R}{4\pi\varepsilon_0\left[R^2+(L+z)^2\right]^{3/2}}.$$

The surface integral of α_2 contains then the following contributions: 2π from the side faces containing $\vec{r}_1$, $2\pi\,[1-(1+R^2/4L^2)^{1/2}]$ from the opposite side face, and from the cover area C

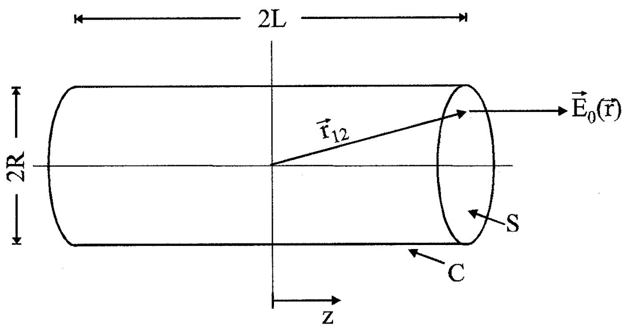

FIGURE 2. Dielectric cylinder of length 2L and diameter $2R$.

$$\frac{R^3}{4}\left[\frac{\varepsilon_0}{\sigma_1}-\frac{1}{2}-\frac{1}{2}\left(1+8L^2/R^2\right)^{-1}-\frac{1}{\kappa-1}\right]^{-1}\times\int_{-L}^{L}\frac{2\pi R\,(L-z)\,dz}{[R^2+(L-z)^2]^{\,3/2}}\times\left\{\left[R^2+(L-z)^2\right]^{-3/2}-\left[R^2+(L+z)^2\right]^{-3/2}\right\}.$$

The first part of the integral results in

$$\int_{R}^{(R^2+4L^2)^{1/2}}(1/r^5)dr,$$

the second part is approximated after Fig. 3 with $L/R \gg 1$ to

$$\int_{R}^{2L}\frac{r\,dr}{r^3(2L+R-r)^3}=\frac{1}{8R^2L^2}-O\left(\frac{L}{R}\right),$$

and with $L/R \ll 1$ the contribution of the cover area tends to zero of order L^2/R^2.

As a total, the surface integral approaches the value 4π for $L/R \ll 1$, and

$$9\pi/4-(11\pi/32)R^2/L^2$$

for $L/R \gg 1$, which can be approximated by

$$4\pi-7\pi/4-(1+11R^2/56L^2)^{-1}.$$

Thus

$$\alpha_2=\left\{1+(\kappa-1)\left[1-\frac{7}{16}(1+11R^2/56L^2)^{-1}\right]\right\}^{-1}.$$

The surface charge density $\sigma_2=\varepsilon_0(\kappa-1)\alpha_2\vec{E}_{1\perp}$ when added to σ_1 has the effect of reducing the charge density of the side faces, but increasing it on the cover area close to the side faces.

The dipole moment of the cylinder in a second approximation yields then

$$\vec{p}=\pi R^2\varepsilon_0(\kappa-1)\left\{1+(\kappa-1)\left[1-(1+R^2/L^2)^{-1/2}\right]\right\}^{-1}\vec{E}_0\times\left\{2L+\frac{L-2L\,(1+R^2/L^2)^{-1/2}-L(1+8L^2/R^2)^{-1}+(R^2+2L^2)\cdot(R^2+4L^2)^{-1/2}-R}{(\kappa-1)^{-1}+1-7/16(1+11R^2/56L^2)}\right\}\quad(7)$$

The first part of the equation describes a homogeneous polarization under consideration of only the charge density situated on the side faces S, and the remaining part considers, as a second approximation, a slight depolarization contribution at both ends of the cylinder.

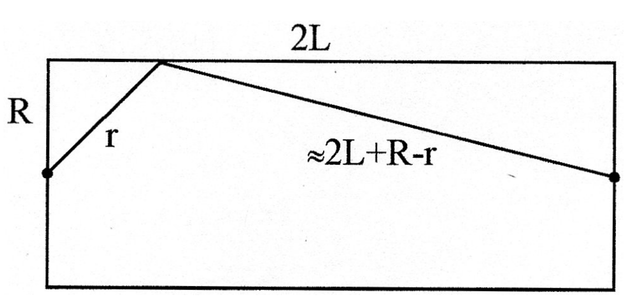

FIGURE 3. Long cylinder.

4. Comparison with experimental results

The dipole moment of a dielectric cylinder with diameter $2R = 1.225$ mm and a length of $2L = 4.460$ mm, made of corundum with κ = 7.44$\pm1\%$, can be calculated, following Eq.(7), to

$$|\vec{p}|\,/\varepsilon_0\left|\vec{E}_0\right|=29.67\text{ mm}^3-2.80\text{ mm}^3=26.87\text{ mm}^3.$$

The larger part of the total corresponds to the first approximation step. The correction due to depolarization effects of the cylinder top and bottom faces, *i.e.* the second approximation step, results in about 10% of the first one, which is quite remarkable.

This result has been experimentally verified, using a microwave cavity resonator in the X-band. The resonance frequency of the empty cavity is $\upsilon_0 = 9351$ MHz. The introduction of the corundum cylinder into the cavity space of maximum electric field causes a shift of the resonance frequency by $\Delta\upsilon = 19.4$ MHz $\pm 1\%$. In the first order of perturbation theory, the dipole moment $\vec{p}$ of the introduced sample is given by

$$|\vec{p}|/\varepsilon_0 \left|\vec{E}_0\right| = V_c \cdot \Delta\upsilon/2\upsilon_0,$$

where $V_c = 2.579 \cdot 10^4$ mm^3 is the volume of the H_{105} -resonator cavity, applied in this experiment. Thus, a reduced dipole moment $|\vec{p}|/\varepsilon_0 \left|\vec{E}_0\right| = 26.75$ mm$^3 \pm 1\%$ has been measured, which agrees quite well with the calculated value of 26.87 mm^3.

Would the cylinder be approximated by a prolate spheroid of the same volume with an axis relation $a/b = R/L$, one gets with a homogeneous polarization along the terms of Eq. (5) $|\vec{p}|/\varepsilon_0 \left|\vec{E}_0\right| = 24.1$ mm^3. This value deviates strongly from the measured result. Thus, the substitution of a cylinder by a spheroid is not a good approximation.

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