Dimensional crossover in the non-linear sigma model

Denjoe O' Connor
Departmento de Física, CINVESTAV, IPN
Apdo. Post. 14-740, México D.F. 07000, México.
denjoe.oconnor@fis.cinvestav.mx

C. R. Stephens
Instituto de Ciencias Nucleares, U.N.A.M.
Circuito Exterior, Apdo. Post. 70-543, México D.F. 04510, México.
stephens@nuclecu.unam.mx

J. A. Santiago
Instituto de Ciencias Nucleares, U.N.A.M.
Circuito Exterior, Apdo. Post. 70-543, México D.F. 04510, México.
santiago@nuclecu.unam.mx

Recibido el 24 de septiembre de 2001; aceptado el 11 de febrero de 2002

We consider dimensional crossover for an $O(N)$ model on a d-dimensional layered geometry of thickness L, in the σ-model limit, using "environmentally friendly" renormalization. We show how to derive critical temperature shifts, giving explicit results to one loop. We also obtain expressions for the effective critical exponents $\delta_{\rm eff}$ and $\beta_{\rm eff}$ that interpolate between their characteristic fixed point values associated with a d and $(d-1)$-dimensional system in the limits $T \to T_c(L)$, with $L(T - T_c(L))^{\nu} \to \infty$, and $T \to T_c(L)$ for L fixed respectively, where $T_c(L)$ is the L-dependent critical temperature of the system.

Keywords: Renormalization group; dimensional crossover; critical phenomena; non-linear σ-model.

Se considera entrecruzamiento dimensional para un modelo de tipo $O(N)$ sobre una película delgada de d dimensiones, en el límite del modelo σ, usando renormalización "ambientalmente amigable. Se muestra como calcular cambios de la temperatura crítica, debido al efecto de tamaño finito, dando resultados explícitos a un lazo. Además, se obtienen expresiones para los exponentes críticos efectivos $\delta_{\rm eff}$ y $\beta_{\rm eff}$ que interpolan entre los valores característicos de punto fijo asociados con un sistema de d y $(d-1)$ dimensiones en los límites $T \to T_c(L)$, con $L(T - T_c(L))^{\nu} \to \infty$, y $T \to T_c(L)$ para L fijo respectivamente, donde $T_c(L)$ es la temperatura crítica del sistema confinado.

Descriptores: Grupo de renormalización; entrecruzamiento dimensional; fenómenos críticos; modelo σ no lineal.

PACS: 64.60.Ak; 05.70.Fh; 11.10.Gh

1. Introduction

Crossover behavior — the interpolation between qualitatively different effective degrees of freedom of a system as a function of scale — is both ubiquitous and extremely important. Calculation of scaling functions associated with crossover behavior is, generally speaking, much more difficult than the calculation of more standard universal quantities, such as critical exponents, the latter being calculable in an approximation scheme suitable for the asymptotic region around one critical point.

An important, non-trivial and experimentally accessible example is seen in the context of confined systems and their analysis via finite size scaling. As far as the fluctuations in a system are concerned there is, in principle, a very marked difference between an "environment" consisting of infinite three dimensional space and a three dimensional box of "size" L. A general formalism for studying such crossover systems using renormalization group (RG) methods is that of "environmentally friendly" renormalization [1]. To access the sensitivity to environment implicit in such a system it is necessary to implement a renormalization programme which is explicitly dependent on the relevant environmental parameters, such as finite size L. By so doing one can access several fixed points of one globally defined RG, and what is more, one may achieve this perturbatively using one uniform approximation scheme.

The main gist of the approach is based on the simple intuition that, viewed as a coarse graining procedure, a "good" coarse graining will be one that when effected to a length scale comparable to any length scale set by the environment will reflect the influence of the latter by changing continuously as a function of scale the type of effective degree of freedom being coarse grained. However, and this is a point to be emphasized, although the intuition is grounded in a coarse graining procedure the actual mechanics are totally different to that of a Kadanoff/Wilson type coarse graining. Instead the formalism is based on the notion of a RG as describing the invariance under reparametrization of a system, an idea which goes back to the original formulation of the field theoretic RG back in the '50's. Just as there are good and bad coarse grainings so there are good and bad reparametrizations. An "environmentally friendly" reparametrization is one that tracks the qualitatively changing nature of the effective degrees of freedom of a crossover system. As has been emphasized previously a necessary condition for an enviromentally friendly RG to satisfy is that the number of fixed points of the RG, defined globally on the space of parameters, be diffeomorphic to the number of points of scale invariance of the system. Unfortunately, favourite forms of field theoretic renormalization, such as minimal subtraction, manifestly do not satisfy this criterion and therefore are of only limited use in describing crossover behavior.

Previously [1], we have considered crossover behavior for an $O(N)$ model in the context of a Landau-Ginzburg-Wilson representation of the underlying lattice model based on a $\lambda\varphi^4$ theory. Given that for a dimensional crossover ε expansion methods cannot work a fixed dimension expansion was used to access the crossover, the essential characteristics of the perturbation theory being based on an expansion around the gaussian fixed point. As is well known, when an $O(N)$ symmetry is spontaneously broken massless Goldstone modes give singularities at large distances for any value of the temperature. The thermodynamics of these spin waves is described in the long distance limit by a Landau-Ginzburg-Wilson Hamiltonian which is that of the field theoretic non-linear σ-model [2]. The appropriate expansion parameter in this case is the temperature, T, and hence perturbation theory corresponds to a low temperature expansion.

Crossover behavior in the context of the non-linear σ-model has been considered previously. In particular, Amit and Goldschmidt extended their original treatment of bicritical systems above the critical temperature using Generalized Minimal Subtraction [3] to below the critical temperature [4] thus describing the crossover between a system exhibiting an $O(N)$ symmetry to that of an $O(M)$ symmetry. In this case ε methods were perfectly feasible due to the fact that the upper critical dimension of the two fixed points was the same. In the context of finite size scaling Brézin and Zinn-Justin described crossover of the non-linear σ-model in the context of a box or a cylinder [5] by treating the lowest infra-red modes of the system non-perturbatively while treating other modes in a perturbative ε expansion. However, this method does not work in the context of a dimensional crossover such as in a thin film where the reduced dimension system also exhibits a non-trivial fixed point.

More recently, the quantum version of the non-linear σ-model [6] has generated a great deal of attention as it can describe the long-wavelength, low-temperature behavior of a two-dimensional quantum Heisenberg ferromagnet which in turn has been proposed as a model of high-temperature superconductors. Given the close analogy between a $(d+1)$-dimensional layered system with periodic boundary conditions and a d-dimensional quantum system it is certainly of interest from this point of view to investigate further dimensional crossover in the context of the non-linear σ-model.

In this paper we will consider dimensional crossover of a ferromagnet with $O(N)$ symmetry in the broken phase using as starting point the non-linear σ-model and utilizing the techniques of environmentally friendly renormalization to access the full universal crossover behavior. One of the benefits of doing so is a better understanding of how environmentally friendly renormalization functions in the context of a low temperature expansion as opposed to an expansion around the critical point.

In Sec. 1 we briefly outline some important features of the non-linear σ-model. In Sec. 2 we consider some formal renormalization results leaving explicit one loop answers for the case of dimensional crossover in a film geometry to Sec. 3. In Sec. 4 we calculate the shift in critical temperature due to finite size effects while in Sec. 5 we derive one-loop expressions for some relevant effective critical exponents in the broken phase. Finally, in Sec. 6 we draw some conclusions.

2. The non-linear σ-model

We begin with the Landau-Ginzburg-Wilson Hamiltonian for a Heisenberg model with $O(N)$ symmetry in the σ-model limit on a d-dimensional ($d < 4$) film geometry of thickness L

$$\mathcal{H}[\varphi_B]=\frac{1}{T_B}\int_0^L\!\!\int d^dx\left(\frac{1}{2}\nabla_\mu\varphi_B^i\nabla_\mu\varphi_B^i-H_B^i(x)\varphi_B^i\right), \quad (1)$$

where $i\in[1,N]$, $\mu\in[1,d]$, T_B is proportional to the temperature of the system and ${\varphi^i}_B$ is subject to the constraint

$$\varphi_B^i\varphi_B^i=1. \quad (2)$$

We will restrict our attention here to the case of periodic boundary conditions.

The partition function Z is obtained by performing the path integral over the order parameter fields, φ_B^i, with the Hamiltonian (1) subject to the constraint (2). Choosing the direction of symmetry breaking to be along the Nth direction we define $\varphi^N=\sigma$ and $\varphi^i=\pi^i,\ \ (i\neq N)$. The constraint implies that $\sigma(x)=\pm(1-\pi^2)^{\frac{1}{2}}$. Thus the partition function becomes

$$Z[H,J]=\int\left[\frac{d\pi_B}{(1-{\pi_B^i}^2)^{\frac{1}{2}}}\right]\mathrm{e}^{-\frac{1}{T_B}\int d^dx\ \ \frac{1}{2}(\nabla\pi_B^i)^2+\frac{1}{2}(\nabla(1-{\pi_B^i}^2)^{\frac{1}{2}})^2-J_B^i\pi_B^i-H_B(1-{\pi_B^i}^2)^{\frac{1}{2}}}\ . \quad (3)$$

Clearly this theory is highly non-polynomial. The non-trivial measure term, which ensures the $O(N)$ invariance of the theory, can of course be exponentiated and expanded in powers of π^2. These terms are necessary to cancel corresponding $O(N)$ non-invariant terms that arise in perturbation theory. We will assume that such terms have been cancelled in the rest of this paper and not consider them further. Rotations are implemented linearly in the $(N-1)$-dimensional π^i-subspace and non-linearly in the $\pi^i - \sigma$ directions. A rotation by an infinitesimal ω^i induces the changes

$$\delta\pi^i(x) = (1 - {\pi^i}^2(x))^{\frac{1}{2}}\omega^i, \tag{4}$$

$$\delta(1 - {\pi^i}^2(x))^{\frac{1}{2}} = -\omega^i\pi^i(x). \tag{5}$$

As long as $|\pi^i| < 1$ the symmetry will remain broken. As $T \to 0$, $\sigma(x) \to 1$.

From the way in which T appears in Eq. (3) we can see that an expansion in terms of temperature is equivalent to an expansion in the number of loops, the only subtlety being that the measure term is then linear in T and therefore contributes to an higher order in T than the other two terms. The free propagator for the π field in the absence of a magnetic field is

$$G_{\pi\pi}(k) = \frac{T_B}{k^2}. \tag{6}$$

The magnetic field coupled to the σ field acts as an IR cutoff. This can be seen by expanding the term $H_B(1 - {\pi_B^i}^2)^{\frac{1}{2}}$ in powers of π. The resulting two-point vertex function is

$$\Gamma_\pi^{(2)}(k) = \frac{k^2 + H_B}{T_B}. \tag{7}$$

From the form of the Hamiltonian, in terms of an expansion in π, there are interactions of arbitrary order. However, interactions with more than four powers of π contribute at higher than one loop order, *i.e.*, more powers of the "small" coupling T. Consequently to first order in T, i.e. one loop, one need only consider the four-point interaction

$$\frac{1}{8T_B}\sum_{k_1k_2k}(k^2 + H_B)\pi_B^i(k_1)\pi_B^i(k - k_1) \times\pi_B^j(-k_2)\pi_B^j(k_2 - k). \tag{8}$$

In this paper we will restrict our attention to $O(T)$ results and therefore will not consider higher order interactions any further.

3. Renormalization

In spite of the fact that the theory is non-polynomial, as is well known [2], it is renormalizable using only two renormalization constants Z_T and Z_π associated with the temperature and the field respectively. The relation between the bare and renormalized parameters is

$$T_B = Z_T T, \qquad \pi_B^i = Z_\pi^{-\frac{1}{2}}\pi^i. \tag{9}$$

To preserve the rotational invariance of the renormalized constraint the field σ must renormalize in the same way as π. Invariance of the term $H_B\sigma_B/T_B$ thereby yields the renormalization of H:

$$H_B = Z_T Z_\pi^{-\frac{1}{2}} H. \tag{10}$$

The bare and renormalized vertex functions are related via

$$\Gamma_\pi^{(N)}(k_i, T, H, L, \kappa) = Z_\pi^{\frac{N}{2}}\Gamma_{\pi B}^{(N)}(k_i, T_B, H_B, L, \Lambda), \tag{11}$$

where κ is an arbitrary renormalization scale and Λ an ultraviolet cutoff. The RG equation, which is a consequence of the κ invariance of the bare theory, follows immediately on differentiating Eq. (11) with respect to κ:

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta_t\frac{\partial}{\partial t} + \beta_H\frac{\partial}{\partial H} - \frac{N}{2}\gamma_\pi\right) \times\Gamma_\pi^{(N)}(k_i, T, H, L, \kappa) = 0, \tag{12}$$

where we have introduced a dimensionless temperature $t = T\kappa^{d-2}$ and $\gamma_\pi = (d\ln Z_\pi)/(d\ln\kappa)$ is the anomalous dimension of the field. The two β-functions are

$$\beta_t = (d-2)t - t\frac{d\ln Z_T}{d\ln\kappa}, \tag{13}$$

$$\beta_H = \frac{d\ln Z_\pi}{d\ln\kappa} - \frac{d\ln Z_T}{d\ln\kappa}. \tag{14}$$

The two renormalization constants must be fixed by normalization conditions. The essence of environmentally friendly renormalization is that in order to obtain a perturbatively well defined description of the crossover the renormalization procedure must depend explicitly on L. The normalization conditions we will use are

$$T\Gamma_\pi^{(2)}(k = 0, t(\kappa, L\kappa), H(\kappa, L\kappa) = \kappa^2, L, \kappa) = \kappa^2, \tag{15}$$

$$\frac{\partial}{\partial k^2}T\Gamma_\pi^{(2)}(k, t(\kappa, L\kappa), H(\kappa, L\kappa){=}\kappa^2, L, \kappa)|_{k=0} = 1. \tag{16}$$

Note that $T\Gamma_\pi^{(2)}$ is just the inverse susceptibility associated with the π field.

4. Explicit results

We now proceed to examine the crossover perturbatively. To one loop,

$$\Gamma^{(2)}_{\pi B}(k=0) = \frac{H_B}{T_B} + \frac{(N-1)}{2}\frac{H_B}{L} \times \sum_{n=-\infty}^{\infty}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{p^2+H_B+\frac{4\pi^2n^2}{L^2}}. \quad (17)$$

Using the normalization conditions expressed in Eqs. (15) and (16) one finds

$$Z_\pi = 1-\frac{(N-1)}{2L\kappa}t\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}y}{(2\pi)^{d-1}}\frac{1}{y^2+1+\frac{4\pi^2n^2}{L^2\kappa^2}}, \quad (18)$$

and

$$Z_T = 1-\frac{(N-2)}{2L\kappa}t\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}y}{(2\pi)^{d-1}}\frac{1}{y^2+1+\frac{4\pi^2n^2}{L^2\kappa^2}}. \quad (19)$$

The β-function β_t is thus given by

$$\beta_t(t,L\kappa) = (d-2)t - \frac{2(N-2)}{L\kappa}t^2 \times \sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}y}{(2\pi)^{d-1}}\frac{1}{(y^2+1+\frac{4\pi^2n^2}{L^2\kappa^2})^2}. \quad (20)$$

There are three different fixed points associated with Eq. (20), a d-dimensional ultraviolet fixed point in the limit $L\kappa\to\infty$, $\kappa\to\infty$; a $(d-1)$-dimensional ultraviolet fixed point in the limit $L\kappa\to 0$, $\kappa\to\infty$; and finally, a zero temperature infrared fixed point in the limit $\kappa\to 0$. However, the approach to $t=0$ depends on whether we consider $\kappa\to 0$ for fixed L or $L\kappa\to\infty$, $\kappa\to 0$.

Because the coupling t was made dimensionless with a factor κ^{d-2}, in the limit $L\kappa\to 0$, $\kappa\to\infty$ one finds that $t\to O(L\kappa)$. However, as emphasized in Ref. 1, this does not imply that in the dimensionally reduced limit that fluctuations are unimportant, as loops enter with a factor $(L\kappa)^{-1}$, which diverges. The issue is made more transparent in this limit by passing to a more appropriate coupling $t' = t/L\kappa$. In the dimensionally reduced limit $t'\to O(d'-2)$, where $d' = d-1$, while in the bulk limit $t'\to O(1/L\kappa)$. Thus t' looks more natural in the dimensionally reduced limit and t in the bulk limit. Of course, these simple changes of variables cannot affect the results for physical quantities, they just help make more transparent what is going on. A coupling that is natural across the entire crossover, the floating coupling, can be introduced [1]. It is defined via $h = a_2 t$ in the present case, where a_2 is the coefficient of t^2 in Eq. (20).

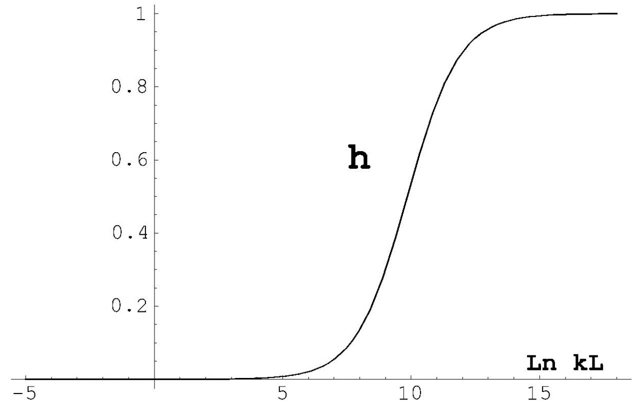

FIGURE 1. Graph of separatrix solution of (21) as function of $\ln\kappa L$.

In terms of the floating coupling one finds

$$\kappa\frac{dh}{d\kappa} = \varepsilon(L\kappa)h - h^2, \quad (21)$$

where

$$\varepsilon(L\kappa) = d-3+4\frac{\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}y}{(2\pi)^{d-1}}\frac{\frac{4\pi^2n^2}{L^2\kappa^2}}{(y^2+1+\frac{4\pi^2n^2}{L^2\kappa^2})^3}}{\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}y}{(2\pi)^{d-1}}\frac{1}{(y^2+1+\frac{4\pi^2n^2}{L^2\kappa^2})^2}}. \quad (22)$$

The quantity $d_{\rm eff} = 2+\varepsilon(L\kappa)$ can be interpreted as a measure of the effective dimension of the system interpolating between d and $d-1$ in the limits $L\kappa\to\infty$ and $L\kappa\to 0$, respectively, where in both cases we are considering $\kappa\to\infty$, *i.e.* the behavior near the critical point.

The corresponding fixed points for h are: $d-2$, $d-3$ and 0. In Fig. 1 we see a plot of h as a function of $\ln\kappa L$ for $d=3$. In this case, as the theory is asymptotically free in two dimensions, the coupling goes to zero in the dimensionally reduced limit, *i.e.*, the $(d-1)$-dimensional ultraviolet fixed point and the trivial infrared fixed points coincide. This dimensional crossover in the coupling is controllable in the low temperature expansion. It is in fact the solution to Eq. (21) that we use as a “small” parameter in the perturbative expansion of all other quantities. It is the fact that h captures the crossover between the different fixed points that gives us a uniform expansion parameter and therefore perturbative control of the crossover. Of course when $d-2$ is not small one really needs to work to higher order and attempt some resummation method. It should be clear however that there is no impediment to continuing this calculation to arbitrary order in the loop expansion.

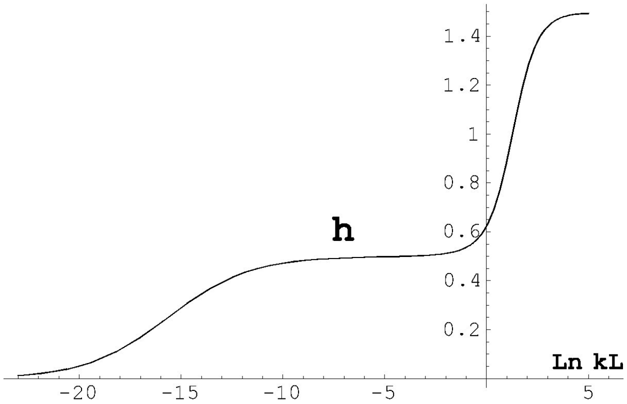

FIGURE 2. Graph of separatrix solution of (21) as function of $\ln \kappa L$.

In Fig. 2 we see the corresponding result for $d = 3.5$. Here, the "double" crossover between the different fixed points is manifest. Asymptotically in the ultraviolet there is a fixed point at $h = 1.5$ that corresponds to the 3.5-dimensional critical point. In the intermediate region h asymptotes to the critical point of the 2.5-dimensional dimensionally reduced theory while, finally, in the infrared $h \to 0$ corresponding to behavior controlled by the zero-temperature fixed point that controls the coexistence curve.

Turning now to the anomalous dimension of the field, γ_π, we find

$$\gamma_\pi = \frac{2(N-1)}{L\kappa} t \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}y}{(2\pi)^{d-1}} \frac{1}{(y^2+1+\frac{4\pi^2 n^2}{L^2\kappa^2})^2}, \quad (23)$$

or in terms of the floating coupling

$$\gamma_\pi = \left(\frac{N-1}{N-2}\right) h. \quad (24)$$

The anomalous dimension also exhibits a dimensional crossover, as can be seen in Fig. 3, for the case $d = 3$, $N = 3$, 4, 5, interpolating between the values

$$\left(\frac{N-1}{N-2}\right)(d-2) \quad \text{and} \quad \left(\frac{N-1}{N-2}\right)(d-3)$$

in the limits $L\kappa \to \infty$ and $L\kappa \to 0$ respectively, where once again we are considering the behavior near the critical point. Note that in contrast to the case of an expansion around the critical point using a φ^4 Landau-Ginzburg-Wilson Hamiltonian γ_π isn't simply the critical exponent η. This is due to the fact that the canonical dimension of the fields π and σ here is zero. The bulk, $L \to \infty$, value of γ_π is $\gamma_\pi(\infty) = (d-2+\eta)$ where the critical exponent $\eta = (d-2)/(N-2)$. In the limit $L\kappa \to 0$, $\kappa \to \infty$ we see that $\gamma_\pi \to (d'-2+\eta')$ where $d' = d-1$ and $\eta' = (d'-2)/(N-2)$ is the critical exponent of the dimensionally reduced system.

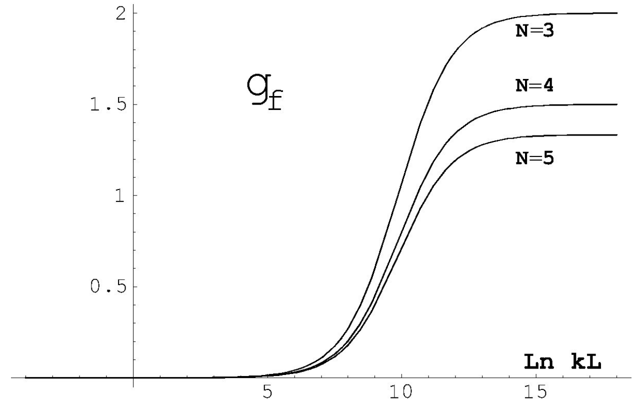

FIGURE 3. Graph of $g_f = \gamma_\pi$ on the separatrix solution of (21) as a function of $\ln \kappa L$.

In the large-N limit, $N \to \infty$, note that $\gamma_\pi \to h = d^e_{\text{eff}}$, where d^e_{eff} is the effective dimension that was found in considerations of dimensional crossover in the large-N limit of a $\lambda\varphi^4$ theory [7].

5. Critical temperature shift

The β function equation is easily integrated to find

$$t(\frac{\kappa}{\kappa_i}, L\kappa) = \frac{1}{t_i^{-1}\left(\frac{\kappa_i}{\kappa}\right)^{d-2} + \int_{\kappa_i}^{\kappa} f(L\kappa')\left(\frac{\kappa'}{\kappa}\right)^{d-2}\frac{d\kappa'}{\kappa'}}, \quad (25)$$

where κ_i and t_i are the initial arbitrary renormalization scale and temperature . The function f is

$$f(L\kappa) = \frac{2(N-2)}{(4\pi)^{\frac{d}{2}}} \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{ds}{s^{\frac{d-2}{2}}} e^{-s} e^{-\frac{n^2L^2\kappa^2}{4s}}. \quad (26)$$

In the limit $L \to \infty$ Eq. (25) becomes

$$t(\frac{\kappa}{\kappa_i}, \infty) = \frac{1}{t_i^{-1}\left(\frac{\kappa_i}{\kappa}\right)^{d-2} + \int_{\kappa_i}^{\kappa} f(\infty)\left(\frac{\kappa'}{\kappa}\right)^{d-2}\frac{d\kappa'}{\kappa'}}, \quad (27)$$

where $f(\infty)$ simply picks out the $n = 0$ term in the sum in $f(L\kappa)$. Choosing the initial temperature and initial scale in Eqs. (25) and (27) to be the same one obtains:

$$\left(\frac{1}{T(L)} - \frac{1}{T(\infty)}\right) = \int_{\kappa_i}^{\kappa} (f(L\kappa') - f(\infty))\kappa'^{d-2}\frac{d\kappa'}{\kappa'}, \quad (28)$$

where a factor of κ^{2-d} has been absorbed into the dimensionless temperature. The interpretation of Eq. (28) is that given a particular renormalization scale κ in two systems of size L and of infinite size then the corresponding temperatures in the two systems are related as above. Given that the limit $\kappa \to \infty$ corresponds to the approach to the critical point we can take this limit in Eq. (28) to find

$$\left(\frac{1}{T_c(L)} - \frac{1}{T_c(\infty)}\right) = \frac{1}{2}(f(L\kappa_i) - f(\infty)). \quad (29)$$

Taking the limit $\kappa_i \to 0$ corresponds to choosing the initial dimensionless temperature to be zero. The shift then becomes

$$\left(\frac{1}{T_c(L)} - \frac{1}{T_c(\infty)}\right) = \frac{b_d}{L^{d-2}}, \quad (30)$$

where the dimension dependent constant b_d is

$$b_d = \frac{(N-2)}{2\pi^{\frac{d}{2}}}\Gamma(\frac{d-2}{2})\zeta(d-2). \quad (31)$$

The result in Eq. (30) is fully in agreement with the expectations of finite size scaling [8] with the exponent $\nu = 1/(d-2)$. In the limit $d \to 3$ $b_d \to \infty$ as there is a divergence in the ζ function at $d = 2$. This corresponds to the fact that the the shift is ill defined due to the non-existence of a critical point in two dimensions as discussed by Barber and Fisher [9].

6. Effective exponents

A useful set of universal scaling functions are defined by effective critical exponents that interpolate between those characteristic of the end points of the crossover of interest. Here, given that the non-linear σ-model is restricted to the broken phase we concentrate on the two effective exponents $\beta_{\rm eff}$ and $\delta_{\rm eff}$ defined as

$$\delta_{\rm eff}^{-1} = \left.\frac{d\ln\sigma}{d\ln H}\right|_{t_c(L)}, \quad \beta_{\rm eff} = \left.\frac{d\ln\sigma}{d\ln(t_c(L)-t)}\right|_{H=0}, \quad (32)$$

where $\delta_{\rm eff}$ is defined along the critical isotherm of the finite size system and $\beta_{\rm eff}$ on the coexistence curve of the finite size system. To derive $\beta_{\rm eff}$ we solve the RG equation for the magnetization, $\bar{\varphi}$ with initial condition $\bar{\varphi}(\kappa = 0) = 1$. We then substitute the anomalous dimension expressed in Eq. (24) and consider the limit $L\kappa \to \infty$ to find

$$\beta_{\rm eff} = \beta - (t_c(L)-t)\frac{d}{dt}\left(\frac{1}{2}\int_0^\infty \Delta\gamma_\pi(x,t)\frac{dx}{x}\right), \quad (33)$$

where $\beta = \nu(d-2+\eta)/2$ is the bulk exponent and

$$\Delta\gamma_\pi = \gamma_\pi - (d-2+\eta).$$

In the limit

$$(t_c(L)-t) \to 0, \qquad L^{d-2}(t_c(L)-t) \to \infty$$

one finds that $\beta_{\rm eff} \to \nu(d-2+\eta)/2$, while in the limit

$$(t_c(L)-t) \to 0, \qquad L^{d-2}(t_c(L)-t) \to 0$$

one obtains $\beta_{\rm eff} \to \nu'(d'-2+\eta')/2$. At one-loop order, using Eq. (24), $\beta_{\rm eff}$ interpolates between the above asymptotic values, where now

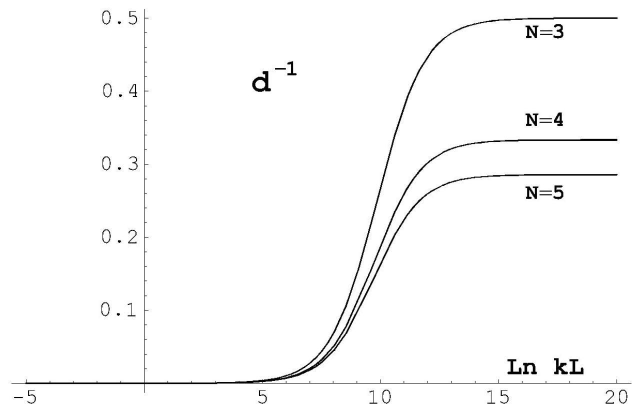

FIGURE 4 Graph of $1/d = 1/\delta_{\rm eff}$ on the separatrix solution of (21) as function of $\ln\kappa L$.

$$\nu = 1/(d-2), \quad \eta = (d-2)/(N-2), \quad \nu' = 1/(d'-2),$$

$$\eta' = (d'-2)/(N-2), \quad \text{and} \quad d' = d-1.$$

Similarly, one finds for $\delta_{\rm eff}$ that

$$\delta_{\rm eff}^{-1} = \frac{\gamma_\pi}{(2d-\gamma_\pi-2\gamma_t)}, \quad (34)$$

where $\gamma_t \equiv (\beta_t/t)$. In the limit

$$(t_c(L)-t) \to 0, \qquad L^{1/\nu}(t_c(L)-t) \to \infty$$

one has $\gamma_\pi \to (d-2+\eta)$ and $\gamma_t \to 0$. Hence, we see that

$$\delta_{\rm eff} \to (d+2-\eta)/(d-2+\eta).$$

In the limit

$$(t_c(L)-t) \to 0, \quad L^{1/\nu}(t_c(L)-t) \to 0$$

one finds that $\gamma_\pi \to (d'-2+\eta')$ and $\gamma_t \to 1$. Thus, in this limit

$$\delta_{\rm eff} \to (d'+2-\eta')/(d'-2+\eta'),$$

where d' and η' are as above. At the one-loop level γ_π is as given by Eq. (24) and $\gamma_t = \varepsilon(L\kappa) - h$. In Fig. 4 we see a graph of $\delta_{\rm eff}^{-1}$ as a function of $\ln L\kappa$ for $d = 3$ and $N = 3$, 4, 5 where the full dimensional crossover is evident.

7. Conclusions

In this paper we have used environmentally friendly renormalization to consider dimensional crossover in the context of a non-linear σ-model on a d-dimensional film geometry with periodic boundary conditions. Using an explicitly L dependent renormalization we derived one loop formulas for the anomalous dimension of the Goldstone field and for the β-function, describing the flow of the temperature as a function of RG scale. We found that there were three fixed points exhibited in the one differential equation. A d-dimensional critical point, a $(d-1)$-dimensional critical point and a zero temperature infra-red fixed point. The β function when integrated described the global flow between all three of these fixed points. Critical temperature shifts are a particularly interesting consequence of finite size behavior. Here, we showed how such shifts could explicitly be calculated finding an expression to one-loop in agreement with finite size scaling arguments.

Finally, we derived one-loop expressions for the two effective critical exponents $\beta_{\rm eff}$ and $\delta_{\rm eff}$ showing how they interpolated between the asymptotic expressions associated with the corresponding d and $(d-1)$-dimensional critical exponents. Evidently there is much more that could be done, such as deriving the full equation of state etc. Of particular interest will be to adapt the results to that of a quantum non-linear σ-model and apply them to the case of a high-temperature superconductor. The same mathematical model, though with a quite different physical interpretation, will also describe a d-dimensional relativistic, quantum field theoretic non-linear σ-model. We hope to return to these interesting issues in a future publication.

Acknowledgements

This work was supported by Conacyt grants 30422-E and 32399-E.

1. D. O'Connor and C. R. Stephens, *Nucl. Phys.* **B260** (1991) 297; *Phys. Rev. Lett.* **72** (1994) 506; *Int. J. Mod. Phys.* **A9** (1994) 2805; *Physics Reports* **363** 4-6 (2002) 425.
2. A. A. Migdal, *Ah. Eksp. Teor. Fiz.* **69** (1975) 810; A. M. Polyakov, *Phys. Lett.* **B59** (1975) 79; E. Brézin and J. Zinn-Justin, *Phys. Rev. Lett.* **36** (1976) 691.
3. D. J. Amit and Y. Y. Goldschmidt, *Ann. Phys.* **114** (1978) 356.
4. D. J. Amit, Y. Y. Goldschmidt and L. Peliti, *Ann. Phys.* **116**, (1978) 1.
5. E. Brézin and J. Zinn-Justin, *Nucl. Phys.* **B257[FS14]** (1985) 867.
6. S. Chakravarty, B. I. Halperin and D.R. Nelson, *Phys. Rev. Lett.* **60** (1988) 1057; *Phys. Rev.* **B39** (1989) 2344.
7. D. O'Connor, C.R. Stephens and A.J. Bray, *J. Stat. Phys.* **87** (1997) 273.
8. M. E. Fisher and M. N. Barber, *Phys. Rev. Lett.* **28** (1972) 1516.
9. M. Barber and M. E. Fisher, *Ann. Phys.* **77** (1973) 1.