Debye potentials adapted to cylindrical coordinates

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Recibido el 19 de marzo de 2002; aceptado el 3 de junio de 2002

Using the method of adjoint operators, the solution to the source-free Maxwell equations is expressed in terms of two real Debye potentials adapted to circular, parabolic or elliptic cylindrical coordinates. Analogous expressions are obtained for the solutions of the Einstein vacuum field equations linearized about the Minkowski space-time.

Keywords: Electromagnetic field; linearized Einstein theory.

Usando el método de operadores adjuntos, se expresa la solución de las ecuaciones de Maxwell sin fuentes en términos de dos potenciales de Debye reales adaptados a las coordenadas cilíndricas circulares, parabólicas o elípticas. Se obtienen expresiones análogas para las soluciones de las ecuaciones de campo de Einstein para el vacío linealizadas alrededor del espacio-tiempo de Minkowski.

Descriptores: Campo electromagnético; teoría de Einstein linealizada.

PACS: 03.50.De; 04.30.-w

1. Introduction

It is well known that the electromagnetic field in a source-free region can be expressed as

$$\mathbf{E} = \frac{1}{c}\partial_t(\mathbf{r}\times\nabla\psi_{\rm M}) - \nabla\times(\mathbf{r}\times\nabla\psi_{\rm E}),$$
$$\mathbf{B} = -\frac{1}{c}\partial_t(\mathbf{r}\times\nabla\psi_{\rm E}) - \nabla\times(\mathbf{r}\times\nabla\psi_{\rm M}), \qquad (1)$$

where $\psi_{\rm E}$ and $\psi_{\rm M}$ are two real solutions of the scalar wave equation, called Debye potentials [1–4]. It can be shown that Eqs. (1) actually represent the most general solution to the source-free Maxwell equations (see, *e.g.*, Ref. 5). Expressions in Eqs. (1) can be derived by solving the Maxwell equations by separation of variables in spherical coordinates [6], but a simpler derivation is provided by Wald's method of adjoint operators [7], which also yields the electromagnetic potentials. Equations (1) are adapted to the spherical coordinates and are useful in the multipole expansion of the electromagnetic field [1–3].

The Einstein vacuum field equations linearized about the Minkowski space-time can be written in a form analogous to that of the source-free Maxwell equations, with the curvature perturbation in place of the electromagnetic field, and the solution to these equations can be expressed in the form [5]

$$E_{ij} = \frac{1}{c}\partial_t U_{ij}(\psi_{\rm M}) - V_{ij}(\psi_{\rm E}),$$
$$B_{ij} = -\frac{1}{c}\partial_t U_{ij}(\psi_{\rm E}) - V_{ij}(\psi_{\rm M}), \qquad (2)$$

where

$$U_{jk}(\psi) \equiv iL_j X_k\psi + iL_k X_j\psi, \quad V_{jk}(\psi) \equiv \varepsilon_{jlm}\partial_l U_{mk}(\psi),$$

ε_{ijk} is the Levi-Cività symbol,

$$\mathbf{L} \equiv -i\mathbf{r}\times\nabla, \qquad \mathbf{X} \equiv i\nabla\times\mathbf{L} - \nabla,$$

and E_{ij} and B_{ij} are the components of the curvature perturbations (see Eqs. (18) below). Equations (2) can also be obtained by separation of variables in spherical coordinates [8] and by means of the method of adjoint operators, which gives, in the first place, the corresponding metric perturbations [9].

The solution to the source-free Maxwell equations and to the Einstein vacuum field equations linearized about the flat space-time can be written in forms adapted to Cartesian or (circular, parabolic or elliptic) cylindrical coordinates. Using the method of separation of variables one obtains the expressions

$$\mathbf{E} = \frac{1}{c}\partial_t(\mathbf{e}_z\times\nabla\psi_{\rm M}) - \nabla\times(\mathbf{e}_z\times\nabla\psi_{\rm E}),$$
$$\mathbf{B} = -\frac{1}{c}\partial_t(\mathbf{e}_z\times\nabla\psi_{\rm E}) - \nabla\times(\mathbf{e}_z\times\nabla\psi_{\rm M}), \qquad (3)$$

where $\mathbf{e}_z$ is a unit vector along the z-axis (see, *e.g.*, Refs. 10 and 11) and

$$E_{ij} = \frac{1}{c}\partial_t W_{ij}(\psi_{\rm M}) - Z_{ij}(\psi_{\rm E}),$$
$$B_{ij} = -\frac{1}{c}\partial_t W_{ij}(\psi_{\rm E}) - Z_{ij}(\psi_{\rm M}), \qquad (4)$$

where [8]

$$W_{ij}(\psi) \equiv iM_i N_j\psi + iM_j N_i\psi, \quad Z_{ij}(\psi) \equiv \varepsilon_{imn}\partial_m W_{nj}(\psi),$$

and

$$\mathbf{M} \equiv -i\mathbf{e}_z\times\nabla, \qquad \mathbf{N} \equiv i\nabla\times\mathbf{M}.$$

In all cases, the potentials $\psi_{\rm E}$ and $\psi_{\rm M}$ satisfy the scalar wave equation. Expressions in Eqs. (3) are useful, for instance, in the study of the propagation of electromagnetic waves in waveguides (taking the z-axis along the axis of the waveguide).

The aim of this paper is to give a short and elementary derivation of Eqs. (3) and (4), using the method of adjoint operators, and to obtain the corresponding vector potential and metric perturbation, respectively. In Sec. 2 we obtain the solution to the Maxwell equations and in Sec. 3 the case of the linearized Einstein vacuum field equations is considered. Throughout this paper the summation convention is applied. Lower case Greek indices run from 0 to 3 and lower case Latin indices run from 1 to 3.

2. Solution to the source-free Maxwell equations

If the Cartesian components of the electromagnetic field tensor, $F_{\alpha\beta}$, are expressed in terms of the four-potential, $A_\alpha = (-\phi, \mathbf{A})$, in the usual manner,

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \tag{5}$$

where $\partial_\alpha \equiv \partial/\partial x^\alpha$, then the source-free Maxwell equations are given by $\partial^\alpha F_{\alpha\beta} = 0$ or, equivalently, by $[\mathcal{E}(A_\gamma)]_\beta = 0$, where $\mathcal{E}$ is the differential operator

$$\begin{aligned}[\mathcal{E}(A_\gamma)]_\beta &\equiv \partial^\alpha(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &= (\delta^\gamma_\beta \partial^\alpha \partial_\alpha - \partial^\gamma \partial_\beta) A_\gamma \end{aligned} \tag{6}$$

and the tensor indices are raised or lowered by means of the Minkowski metric $(\eta_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1) = (\eta^{\alpha\beta})$.

Equation (5) is (locally) equivalent to

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0,$$

which implies that

$$\partial^\alpha \partial_\alpha F_{\beta\gamma} = \partial_\beta \partial^\alpha F_{\alpha\gamma} - \partial_\gamma \partial^\alpha F_{\alpha\beta} = (\delta^\rho_\gamma \partial_\beta - \delta^\rho_\beta \partial_\gamma)[\mathcal{E}(A_\sigma)]_\rho.$$

(This shows that when the source-free Maxwell equations are satisfied, each Cartesian component of the electromagnetic field satisfies the wave equation.) Setting

$$\mathcal{T}(A_\gamma) \equiv \partial_1 A_2 - \partial_2 A_1 = F_{12}$$

and $\mathcal{O}(f) \equiv \partial^\alpha \partial_\alpha f$, we have the operator identity

$$\mathcal{O}\mathcal{T}(A_\gamma) = (\delta^\rho_2 \partial_1 - \delta^\rho_1 \partial_2)[\mathcal{E}(A_\gamma)]_\rho = \mathcal{S}\mathcal{E}(A_\gamma),$$

where $\mathcal{S}$ is the differential operator

$$\mathcal{S}(b_\rho) \equiv (\delta^\rho_2 \partial_1 - \delta^\rho_1 \partial_2) b_\rho. \tag{7}$$

If the adjoint, $\mathcal{A}^\dagger$, of a linear differential operator, $\mathcal{A}$, that maps m-index tensor fields into n-index tensor fields, is the linear differential operator that maps n-index tensor fields into m-index tensor field defined by

$$[\mathcal{A}(t_{\alpha\beta\cdots})]_{\rho\sigma\cdots} s^{\rho\sigma\cdots} - t_{\alpha\beta\cdots}[\mathcal{A}^\dagger(s^{\rho\sigma\cdots})]^{\alpha\beta\cdots} = \partial_\alpha v^\alpha,$$

where v^α is some vector field (for details see, *e.g.*, Refs. 12, 7, 9), then

$$(\mathcal{A}+\mathcal{B})^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger, \qquad (\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger \mathcal{A}^\dagger$$

and the operator $\mathcal{E}$, defined by Eq. (6), is self-adjoint ($\mathcal{E}^\dagger = \mathcal{E}$); thus, from the identity $\mathcal{O}\mathcal{T} = \mathcal{S}\mathcal{E}$ it follows that $\mathcal{T}^\dagger \mathcal{O}^\dagger = \mathcal{E}\mathcal{S}^\dagger$. Hence, if ψ is a function such that $\mathcal{O}^\dagger(\psi) = 0$, then $\mathcal{E}(\mathcal{S}^\dagger(\psi)) = 0$, *i.e.*, $A^\rho = [\mathcal{S}^\dagger(\psi)]^\rho$ satisfies the source-free Maxwell equations. Using the fact that $\partial_\alpha^\dagger = -\partial_\alpha$ one finds that

$$\mathcal{O}^\dagger = (\partial^\alpha \partial_\alpha)^\dagger = \partial_\alpha^\dagger \partial^{\alpha\,\dagger} = (-\partial_\alpha)(-\partial^\alpha) = \partial_\alpha \partial^\alpha = \partial^\alpha \partial_\alpha$$

and, from Eq. (7),

$$\mathcal{S}^\dagger = (\delta^\rho_2 \partial_1 - \delta^\rho_1 \partial_2)^\dagger = -\delta^\rho_2 \partial_1 + \delta^\rho_1 \partial_2,$$

thus

$$[\mathcal{S}^\dagger(\psi)]^\rho = (\delta^\rho_1 \partial_2 - \delta^\rho_2 \partial_1)\psi.$$

Therefore, if $\psi_{\rm M}$ satisfies the wave equation, $\partial^\alpha \partial_\alpha \psi_{\rm M} = 0$,

$$A^\rho = \delta^\rho_1 \partial_2 \psi_{\rm M} - \delta^\rho_2 \partial_1 \psi_{\rm M} \tag{8}$$

is the four-potential of a solution of the source-free Maxwell equations. Explicitly, Eq. (8) gives

$$A_1 = \partial_y \psi_{\rm M}, \quad A_2 = -\partial_x \psi_{\rm M}, \quad A_3 = 0, \quad \phi = 0. \tag{9}$$

The z-component of the electric field corresponding to the potentials in Eqs. (9) is

$$E_z = -\partial_z \phi - (1/c)\,\partial_t A_3 = 0;$$

hence, Eqs. (9) is not the most general solution to the source-free Maxwell equations.

In order to obtain an expression for the most general solution of the source-free Maxwell equations we now take

$$\mathcal{T}(A_\gamma) \equiv \partial_3 A_0 - \partial_0 A_3 = F_{30}$$

and

$$\mathcal{O}(f) \equiv \partial^\alpha \partial_\alpha f,$$

as before. Then we have

$$\mathcal{O}\mathcal{T}(A_\gamma) = (\delta^\rho_0 \partial_3 - \delta^\rho_3 \partial_0)[\mathcal{E}(A_\gamma)]_\rho = \mathcal{S}\mathcal{E}(A_\gamma),$$

where now $\mathcal{S}$ is the differential operator

$$\mathcal{S}(b_\rho) \equiv (\delta^\rho_0 \partial_3 - \delta^\rho_3 \partial_0) b_\rho.$$

Proceeding as above, one finds that if $\partial^\alpha \partial_\alpha \psi_{\rm E} = 0$, then

$$A^\rho = \delta^\rho_3 \partial_0 \psi_{\rm E} - \delta^\rho_0 \partial_3 \psi_{\rm E} \tag{10}$$

is another solution of the source-free Maxwell equations. Thus, by virtue of the linearity of the Maxwell equations, the superposition of the four-potentials (8) and (10), given by

$$\phi = -\mathbf{e}_z \cdot \nabla\psi_{\rm E}, \qquad \mathbf{A} = \mathbf{e}_z \frac{1}{c}\partial_t\psi_{\rm E} - \mathbf{e}_z \times \nabla\psi_{\rm M}, \quad (11)$$

is a solution of the Maxwell equations. The electromagnetic field generated by the potentials in Eqs. (11) is precisely that given by Eqs. (3). If $\psi_{\rm E}$ and $\psi_{\rm M}$ are separable solutions to the wave equation in (circular, parabolic, or elliptic) cylindrical coordinates, the fields given by Eqs. (3) are separable solutions to the Maxwell equations in that coordinate system [10,11].

The Debye potentials $\psi_{\rm E}$ and $\psi_{\rm M}$ in Eqs. (1) are independent in the sense that the electromagnetic field generated by a potential $\psi_{\rm E}$ cannot be generated by a potential $\psi_{\rm M}$. In fact, the field generated by $\psi_{\rm M}$ satisfies the condition $\mathbf{r} \cdot \mathbf{E} = 0$, while the field generated by $\psi_{\rm E}$ satisfies $\mathbf{r} \cdot \mathbf{B} = 0$ and there is no non-trivial well-behaved electromagnetic field such that $\mathbf{r} \cdot \mathbf{E}$ and $\mathbf{r} \cdot \mathbf{B}$ vanish. By contrast, as is well-known, it is possible to have electromagnetic fields with $E_z = B_z = 0$. Using Eqs. (3) one finds that the conditions $E_z = 0$, $B_z = 0$ amount to

$$(\partial_x^2 + \partial_y^2)\psi_{\rm E} = 0, \qquad (\partial_x^2 + \partial_y^2)\psi_{\rm M} = 0,$$

respectively. The first of these equations is locally equivalent to the existence of a function χ such that

$$\partial_x\psi_{\rm E} = \partial_y\chi, \qquad \partial_y\psi_{\rm E} = -\partial_x\chi. \quad (12)$$

On the other hand, since $\psi_{\rm E}$ obeys the wave equation, it follows that

$$((1/c^2)\partial_t^2 - \partial_z^2)\psi_{\rm E} = 0,$$

which implies that $\psi_{\rm E}$ is of the form

$$\psi_{\rm E} = f(x,y,u) + g(x,y,v),$$

where $u \equiv z - ct$, $v \equiv z + ct$. Taking, for instance, $\psi_{\rm E} = f(x,y,u)$, which (if the field is not static) corresponds to waves propagating along the positive z-axis, the terms containing $\psi_{\rm E}$ in Eq. (11) can be rewritten as

$$\begin{aligned} \phi &= -\partial_z\psi_{\rm E} = -\partial_u\psi_{\rm E} = \frac{1}{c}\partial_t\psi_{\rm E}, \\ \mathbf{A} &= \mathbf{e}_z\frac{1}{c}\partial_t\psi_{\rm E} = -\mathbf{e}_z\partial_u\psi_{\rm E} = -\mathbf{e}_z\partial_z\psi_{\rm E} \\ &= -\nabla\psi_{\rm E} - \mathbf{e}_z \times \nabla\chi, \end{aligned}$$

therefore, by means of the gauge transformation

$$\mathbf{A} \mapsto \mathbf{A} + \nabla\psi_{\rm E}, \qquad \phi \mapsto \phi - \partial_0\psi_{\rm E},$$

one obtains the potentials generated by $\psi_{\rm M} = \chi$ (see Eqs. (11)). (Note that, owing to Eqs. (12), χ also obeys the wave equation).

3. Solution to the linearized Einstein vacuum field equations

In the linearized Einstein theory it is assumed that the space-time metric, $g_{\alpha\beta}$, can be written as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $|h_{\alpha\beta}| \ll 1$, then, the curvature tensor of $g_{\alpha\beta}$, to first order in $h_{\alpha\beta}$ is

$$\begin{aligned} K_{\alpha\beta\gamma\delta} = \frac{1}{2}(\partial_\alpha\partial_\delta h_{\beta\gamma} - \partial_\beta\partial_\delta h_{\alpha\gamma} \\ +\partial_\beta\partial_\gamma h_{\alpha\delta} - \partial_\alpha\partial_\gamma h_{\beta\delta}). \quad (13) \end{aligned}$$

Therefore, the Einstein vacuum field equations linearized about the Minkowski space-time,

$$K_{\alpha\beta} - \frac{1}{2}K_\gamma{}^\gamma\eta_{\alpha\beta} = 0,$$

where $K_{\alpha\beta} \equiv K^\gamma{}_{\alpha\gamma\beta}$, amount to $[\mathcal{E}(h_{\alpha\beta})]_{\gamma\delta} = 0$ with

$$\begin{aligned} [\mathcal{E}(h_{\rho\sigma})]_{\alpha\beta} \equiv \frac{1}{2}(\partial_\alpha\partial^\gamma h_{\gamma\beta} + \partial_\beta\partial^\gamma h_{\gamma\alpha} - \partial^\gamma\partial_\gamma h_{\alpha\beta} \\ -\partial_\alpha\partial_\beta h^\gamma{}_\gamma + \eta_{\alpha\beta}\partial^\gamma\partial_\gamma h^\delta{}_\delta - \eta_{\alpha\beta}\partial^\gamma\partial^\delta h_{\gamma\delta}). \quad (14) \end{aligned}$$

Equivalently, $[\mathcal{E}(h_{\rho\sigma})]_{\alpha\beta} = K_{\alpha\beta} - \frac{1}{2}K_\gamma{}^\gamma\eta_{\alpha\beta}$, which implies that $[\mathcal{E}(h_{\rho\sigma})]_\alpha{}^\alpha = -K_\alpha{}^\alpha$; therefore,

$$K_{\alpha\beta} = (\delta^\mu_\alpha\delta^\nu_\beta - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu})[\mathcal{E}(h_{\rho\sigma})]_{\mu\nu}. \quad (15)$$

Equation (13) implies that $K_{\alpha\beta\gamma\delta} = K_{\beta\gamma\delta\alpha}$ and

$$\partial_\alpha K_{\beta\gamma\delta\epsilon} + \partial_\beta K_{\gamma\alpha\delta\epsilon} + \partial_\gamma K_{\alpha\beta\delta\epsilon} = 0$$

hence [9]

$$\begin{aligned} \partial^\alpha\partial_\alpha K_{\beta\gamma\delta\epsilon} = \partial_\beta\partial_\delta K_{\gamma\epsilon} - \partial_\beta\partial_\epsilon K_{\gamma\delta} \\ -\partial_\gamma\partial_\delta K_{\beta\epsilon} + \partial_\gamma\partial_\epsilon K_{\beta\delta}. \quad (16) \end{aligned}$$

Thus,

$$\begin{aligned} \partial^\alpha\partial_\alpha K_{1230} &= \partial_3(\partial_1 K_{20} - \partial_2 K_{10}) - \partial_0(\partial_1 K_{23} - \partial_2 K_{13}) \\ &= \varepsilon_{ij3}\partial_i\delta^\alpha_j(\delta^\beta_0\partial_3 - \delta^\beta_3\partial_0)K_{\alpha\beta}. \end{aligned}$$

Letting $\mathcal{T}(h_{\alpha\beta}) \equiv K_{1230}$, $\mathcal{O} = \partial^\alpha\partial_\alpha$ and making use of Eq. (15) we find the identity $\mathcal{OT} = \mathcal{SE}$, with

$$\mathcal{S}(b_{\alpha\beta}) = \varepsilon_{ij3}\partial_i\delta^{(\alpha}_j(\delta^{\beta)}_0\partial_3 - \delta^{\beta)}_3\partial_0)b_{\alpha\beta},$$

where the parentheses denote symmetrization on the indices enclosed. Since the operator $\mathcal{E}$ defined by Eq. (14) is self-adjoint, it follows that if $\psi_{\rm M}$ is a function such that $\mathcal{O}^\dagger(\psi_{\rm M}) = 0$ then $h^{\alpha\beta} = [\mathcal{S}^\dagger(\psi_{\rm M})]^{\alpha\beta}$ satisfies the Einstein vacuum field equations linearized about the Minkowski space-time. The adjoints of $\mathcal{O}$ and $\mathcal{S}$ are given by $\mathcal{O}^\dagger = \partial^\alpha\partial_\alpha$ and

$$[\mathcal{S}^\dagger(\psi_{\rm M})]^{\alpha\beta} = \varepsilon_{ij3}\partial_i\delta_j^{(\alpha}(\delta_0^{\beta)}\partial_3 - \delta_3^{\beta)}\partial_0)\psi_{\rm M};$$

therefore

$$h^{\alpha\beta} = \varepsilon_{ij3}\delta_j^{(\alpha}(\delta_0^{\beta)}\partial_3 - \delta_3^{\beta)}\partial_0)\partial_i\psi_{\rm M} \qquad (17)$$

satisfies the linearized Einstein vacuum field equations if $\psi_{\rm M}$ is a solution of the wave equation.

When $K_{\alpha\beta} = 0$, the curvature perturbation $K_{\alpha\beta\gamma\delta}$ has only ten independent components that can be represented by the two traceless, symmetric tensors E_{ij} and B_{ij} defined by [5,9]

$$E_{ij} \equiv K_{0i0j}, \qquad B_{ij} \equiv -\frac{1}{2}K_{0i}{}^{\rho\sigma}\varepsilon_{\rho\sigma 0j}, \qquad (18)$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\varepsilon_{0123} = 1$. Since the metric perturbations (17) are such that

$$h_{00} = h_{33} = h_{03} = 0,$$

from Eqs. (13) and (18) one finds that $E_{33} = K_{0303} = 0$; which means that Eq. (17) represents a "transverse electric" field and that the potential $\psi_{\rm M}$ alone cannot produce the general solution to the linearized Einstein equations.

Hence, making use of Eqs. (16) and (15), we consider now the identity

$$\begin{aligned}\partial^\alpha\partial_\alpha K_{0303} &= \partial_0\partial_0 K_{33} - 2\partial_0\partial_3 K_{03} + \partial_3\partial_3 K_{00}\\ &= [\partial_0\partial_0(\delta_3^\alpha\delta_3^\beta - \tfrac{1}{2}\eta^{\alpha\beta}) - 2\partial_0\partial_3\delta_0^{(\alpha}\delta_3^{\beta)}\\ &\quad + \partial_3\partial_3(\delta_0^\alpha\delta_0^\beta + \tfrac{1}{2}\eta^{\alpha\beta})][\mathcal{E}(h_{\rho\sigma})]_{\alpha\beta},\end{aligned}$$

which is of the form $\mathcal{OT} = \mathcal{SE}$, with

$$\mathcal{T}(h_{\alpha\beta}) = K_{0303}, \qquad \mathcal{O} = \partial^\alpha\partial_\alpha,$$

and

$$\begin{aligned}\mathcal{S}(b_{\alpha\beta}) = [&\partial_0\partial_0(\delta_3^\alpha\delta_3^\beta - \tfrac{1}{2}\eta^{\alpha\beta}) - 2\partial_0\partial_3\delta_0^{(\alpha}\delta_3^{\beta)}\\ &+\partial_3\partial_3(\delta_0^\alpha\delta_0^\beta + \tfrac{1}{2}\eta^{\alpha\beta})](b_{\alpha\beta}).\end{aligned}$$

Then one finds that

$$\begin{aligned}[\mathcal{S}^\dagger(\psi)]^{\alpha\beta} = &(\delta_3^\alpha\delta_3^\beta - \tfrac{1}{2}\eta^{\alpha\beta})\partial_0\partial_0\psi - 2\delta_0^{(\alpha}\delta_3^{\beta)}\partial_0\partial_3\psi\\ &+(\delta_0^\alpha\delta_0^\beta + \tfrac{1}{2}\eta^{\alpha\beta})\partial_3\partial_3\psi\end{aligned}$$

and therefore

$$\begin{aligned}h_{\alpha\beta} = &(\eta_{3\alpha}\eta_{3\beta} - \tfrac{1}{2}\eta_{\alpha\beta})\partial_0\partial_0\psi_{\rm E}\\ &-2\eta_{0(\alpha}\eta_{\beta)3}\partial_0\partial_3\psi_{\rm E} + (\eta_{0\alpha}\eta_{0\beta} + \tfrac{1}{2}\eta_{\alpha\beta})\partial_3\partial_3\psi_{\rm E} \qquad (19)\end{aligned}$$

satisfies the linearized Einstein vacuum field equations if $\psi_{\rm E}$ satisfies the wave equation. The curvature perturbations generated by (19) satisfy $B_{33} = 0$ and therefore this field is "transverse magnetic". Any linear combination of the metric perturbations (17) and (19) is also a solution to the linearized Einstein equations; hence

$$\begin{aligned}h_{00} &= -2\left(\partial_z^2 + \frac{1}{c^2}\partial_t^2\right)\psi_{\rm E},\\ h_{0i} &= -4\delta_{3i}\frac{1}{c}\partial_t\partial_z\psi_{\rm E} + 2\varepsilon_{3ki}\partial_z\partial_k\psi_{\rm M}, \qquad (20)\\ h_{ij} &= -2\delta_{ij}\left(\partial_z^2 - \frac{1}{c^2}\partial_t^2\right)\psi_{\rm E} - 4\delta_{3i}\delta_{3j}\frac{1}{c^2}\partial_t^2\psi_{\rm E}\\ &\quad +4\varepsilon_{3k(i}\delta_{j)3}\frac{1}{c}\partial_t\partial_k\psi_{\rm M}\end{aligned}$$

(obtained by multiplying Eqs. (17) and (19) by -4 and adding the results), satisfies the linearized Einstein equations for any two real solutions of the wave equation. By means of a straightforward but somewhat lengthy computation one finds that the curvature perturbations corresponding to Eq. (20) are given by Eqs. (4). Since the most general solution to the equations for the curvature perturbations is of the form in Eq. (4) (see Ref. 8), it follows that the most general solution to the Einstein vacuum field equations linearized about the Minkowski space-time is given by Eqs. (20), up to the gauge transformations

$$h_{\alpha\beta} \mapsto h_{\alpha\beta} + \partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha,$$

where ξ_α is an arbitrary vector field, which leave the curvature perturbations $K_{\alpha\beta\gamma\delta}$ invariant.

As in the case of expressions in Eqs. (3), the potentials $\psi_{\rm E}$ and $\psi_{\rm M}$ appearing in Eqs. (4) and (20) are not independent; the perturbations with $E_{33} = B_{33} = 0$ can be expressed in terms of either $\psi_{\rm E}$ or $\psi_{\rm M}$ alone. Indeed, E_{33} and B_{33} vanish if

$$(\partial_x^2 + \partial_y^2)\psi_{\rm E} = 0 \qquad \text{and} \qquad (\partial_x^2 + \partial_y^2)\psi_{\rm M} = 0.$$

Taking, as in Sec. 2, $\psi_{\rm E} = f(x,y,u)$, $\psi_{\rm M} = 0$, where $u = z - ct$ and $(\partial_x^2 + \partial_y^2)f = 0$, the only nonvanishing components of the metric perturbation (20) are given by

$$\begin{aligned}h_{00} &= -4\partial_0^2\psi_{\rm E}, \qquad h_{03} = -4\partial_0\partial_3\psi_{\rm E} = 4\partial_0^2\psi_{\rm E},\\ h_{33} &= -4\partial_3^2\psi_{\rm E} = -4\partial_0^2\psi_{\rm E}. \qquad (21)\end{aligned}$$

Then, under the gauge transformation

$$h_{\alpha\beta} \mapsto h_{\alpha\beta} + \partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha,$$

with $\xi_0 = 2\partial_0\psi_{\rm E}$, $\xi_1 = 0 = \xi_2$, $\xi_3 = 2\partial_3\psi_{\rm E}$, one obtains the metric perturbation generated by $\psi_{\rm E} = 0$ and $\psi_{\rm M} = \chi$, where χ is defined by Eqs. (12).

The perturbed metric determined by Eqs. (21) is

$$\begin{aligned}g_{\alpha\beta}dx^\alpha dx^\beta &= (\eta_{\alpha\beta} + h_{\alpha\beta})dx^\alpha dx^\beta\\ &= dx^2 + dy^2 + dz^2 - c^2dt^2 + F(x,y,u)(dz - cdt)^2,\end{aligned}$$

with $F \equiv -4\partial_u^2 f$. This metric is not only a solution to the Einstein vacuum field equations linearized about the

Minkowski metric, but also an exact solution to the Einstein vacuum field equations for any function $F(x, y, u)$ such that $(\partial_x^2 + \partial_y^2)F = 0$ (see, *e.g.*, Ref. 13).

4. Concluding remarks

The only drawback of the method of adjoint operators in its present form is that it is not known in advance how many potentials are necessary to express the most general solution of a given system of linear partial differential equations. In the two cases considered in this paper we know that two real potentials are sufficient since the solution of the Maxwell equations or of the equations for the curvature perturbations obtained by separation of variables can be expressed in terms of two real potentials. The method of adjoint operators gives not only the electromagnetic field tensor and the curvature perturbations but also the vector potential and the metric perturbations in an extremely simple way.

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