Fast algorithm for bilinear transforms in optics

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Recibido el 4 de mayo de 2001; aceptado el 15 de enero de 2002

The fast algorithm for calculating the bilinear transform in the optical system is proposed. This algorithm is based on the coherent-mode representation of the cross-spectral density function of the illumination. The algorithm is computationally efficient when the illumination is partially coherent. Numerical examples are studied and compared with theoretical results.

Keywords: Bilinear transform; fast algorithm; cross-spectral density, coherent-mode representation.

Se propone un algoritmo rápido para calcular la transformación bilineal en un sistema óptico. Este algoritmo está basado en la representación en modos coherentes de la función de densidad espectral cruzada de la iluminación. El algoritmo es eficiente computacionalmente cuando la iluminación es parcialmente coherente. Se estudian ejemplos numéricos y se comparan con resultados teóricos.

Descriptores: Transformación bilineal; algoritmo rápido; densidad espectral cruzada; representación en modos coherentes

PACS: 42.25.Kb; 42.30.Va; 42.30.Kq; 42.15.Eq.

1. Introduction

In consequence of the quadratic relation between the optical field and intensity, an inherent nonlinearity exists in almost all optical systems. As well known, the output $g(y)$ of any non-linear system can be expressed as a functional of the input signal $f(x)$, which is represented by the Volterra series [1]

$$g(y) = q_0(y) + \sum_{n=1}^{\infty} \int \cdots \int \mathrm{d}x_1 \ldots \mathrm{d}x_n f(x_1) \ldots \times f(x_n)\, q_n(y; x_1, \ldots, x_n), \quad (1)$$

where $q_n(...)$ denotes the nth-order Volterra kernel of the system. Saleh [2] showed that many optical systems and processes can be represented either exactly or approximately by the third term of this series, *i.e.*,

$$g(y) = \iint_{-\infty}^{\infty} f(x_1)\, f^*(x_2)\, q_2(y; x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2. \quad (2)$$

He called the transform described by Eq. (2) a bilinear transform (BLT) and gave a comprehensive analysis of the properties of its kernel for various optical systems.

In spite of all its mathematical attractiveness the BLT approach has so far limited application for numerical simulation of optical systems in view of the complexity of the required calculations and, as a consequence, the enormous time needed for its computer realization. In this connection there is a strong need for a computationally efficient method for calculating the BLT. Recently we proposed such a method for calculating the BLT in partially coherent optical imaging system [3]. This method is based on the coherent-mode representation of the cross-spectral density function of the illumination and allows to reduce the needed computational effort by a factor up to two orders in comparison with the direct calculation. In this paper, we describe the generalization of the proposed method for calculating the BLT in an arbitrary optical system and illustrate its efficiency by two examples of calculating the intensity distribution when optical system meets either the condition of image formation or the condition of Fourier spectrum formation. After the analogy of the FFT algorithm we refer to the proposed method as FBLT algorithm.

2. BLT in optics and its computer realization

Let us consider an elementary optical system with a single thin converging lens shown in Fig. 1. We will assume that an object with the complex amplitude transmittance $t(\mathbf{x})$ in a point $\mathbf{x} = (x, y)$ is illuminated by a stochastic quasi-monochromatic scalar wave field $V(\mathbf{x})$ (to keep the notation as simple as possible, here and further on, we suppress the explicit dependence of some of considered quantities on temporal frequency ν), which can be completely characterized by the cross-spectral density function [4]

$$W(\mathbf{x}_1, \mathbf{x}_2) = \langle V(\mathbf{x}_1) V^*(\mathbf{x}_2) \rangle, \tag{3}$$

where the angular brackets represent the statistical average taken over the ensemble and the asterisk denotes the complex conjugate. Then, as it is well known [see, *e.g.*, Ref. 5], the intensity distribution in the output plane of the system, within the paraxial approximation, is given by

$$I(\mathbf{u}) = \iint_{-\infty}^{\infty} t(\mathbf{x}_1) t^*(\mathbf{x}_2) W(\mathbf{x}_1, \mathbf{x}_2) \times H(\mathbf{u}; \mathbf{x}_1) H^*(\mathbf{u}; \mathbf{x}_2) \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2, \tag{4}$$

where

$$H(\mathbf{u}; \mathbf{x}) = \frac{1}{\lambda^2 z_1 z_2} \exp\left(i\frac{\pi}{\lambda z_1}\mathbf{x}^2\right) \exp\left(i\frac{\pi}{\lambda z_2}\mathbf{u}^2\right) \times \int_{-\infty}^{\infty} P(\mathbf{p}) \exp\left[i\frac{\pi}{\lambda}\left(\frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f}\right)\mathbf{p}^2\right] \times \exp\left[-i\frac{2\pi}{\lambda z_2}\mathbf{p}\cdot\left(\mathbf{u} + \frac{z_2}{z_1}\mathbf{x}\right)\right] \mathrm{d}\mathbf{p}, \tag{5}$$

λ is the mean wavelength, f is the focal distance of the lens, and $P(\mathbf{p})$ is the aperture function of the lens. Comparing Eq. (4) with Eq. (2), one can find that this equation describes the BLT of the object function $t(\mathbf{x})$ with the Volterra kernel

$$q_2(\mathbf{u}; \mathbf{x}_1, \mathbf{x}_2) = W(\mathbf{x}_1, \mathbf{x}_2) H(\mathbf{u}; \mathbf{x}_1) H^*(\mathbf{u}; \mathbf{x}_2). \tag{6}$$

If the geometry in Fig. 1 satisfies the lens law,

$$1/z_1 + 1/z_2 = 1/f$$

(the image formation condition), the corresponding BLT kernel of the system takes the same form as Eq. (6), but with

$$H(\mathbf{u}; \mathbf{x}) = \frac{1}{\lambda^2 z_1 z_2} \exp\left(i\frac{\pi}{\lambda z_1}\mathbf{x}^2\right) \exp\left(i\frac{\pi}{\lambda z_2}\mathbf{u}^2\right) \times \int_{-\infty}^{\infty} P(\mathbf{p}) \exp\left[-i\frac{2\pi}{\lambda z_2}\mathbf{p}\cdot\left(\mathbf{u} + \frac{z_2}{z_1}\mathbf{x}\right)\right] \mathrm{d}\mathbf{p}, \tag{7}$$

which is known as the amplitude spread function of optical system.

If $z_2 = f$ (the Fourier transform condition), the corresponding BLT kernel again has the form of Eq. (6) with

$$H(\mathbf{u}; \mathbf{x}) = \frac{1}{i\lambda f} \exp\left[i\frac{\pi}{\lambda f}\left(1 - \frac{z_1}{f}\right)\mathbf{x}^2\right] P\left(\mathbf{u} + \frac{z_1}{f}\mathbf{x}\right) \exp\left(-i\frac{2\pi}{\lambda f}\mathbf{u}\cdot\mathbf{x}\right). \tag{8}$$

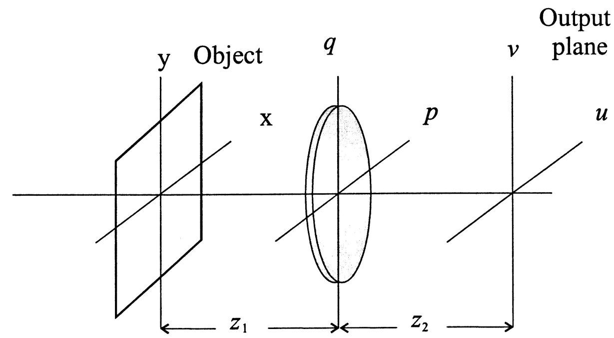

FIGURE 1. Single lens optical system.

It is obvious that knowledge of the functions $t(\mathbf{x})$, $W(\mathbf{x}_1, \mathbf{x}_2)$ and $H(\mathbf{u}; \mathbf{x})$ allows to calculate BLT (4) with the kernel given by Eqs. (6-8). Let us evaluate the computational complexity of such a calculation. The dominant portion of calculating the intensity distribution from Eq. (4) is the multiplication of four 4-D functions $t(\mathbf{x}_1) t^*(\mathbf{x}_2)$, $W(\mathbf{x}_1, \mathbf{x}_2)$, $H(\mathbf{u}; \mathbf{x}_1)$ and $H^*(\mathbf{u}; \mathbf{x}_2)$. To realize the numerical multiplication of these functions, it is necessary to multiply their samples for all possible combinations of sampling points taken one by one in each of three planes $\mathbf{u}$, $\mathbf{x}_1$ and $\mathbf{x}_2$. Hence, assuming that the illumination field, object and amplitude spread function have each been adequately represented by $N \times N$ sampling points, one finds that the total number of operations required to compute $I(\mathbf{u})$ is proportional to

$$C = \left(N^2\right)^3 = N^6. \tag{9}$$

The magnitude of this number can easily result in an unacceptably long computation time. Thus, for example, when $N = 100$ and the computational speed is 10^6 operations per second, the computer run time needed for calculation of $I(\mathbf{u})$ is about 300 h. Clearly, an alternative approach to the calculation of intensity distribution is desired as a way to reduce the computational effort.

3. FBLT algorithm

According to Wolf's theory of partial coherence in the space-frequency domain [4], the cross-spectral density function $W(\mathbf{x}_1, \mathbf{x}_2)$ of a wide class of sources may be represented in the form of the Mercer expansion, *i.e.*,

$$W(\mathbf{x}_1, \mathbf{x}_2) = \sum_{n=0}^{\infty} \lambda_n \varphi_n(\mathbf{x}_1) \varphi_n^*(\mathbf{x}_2), \tag{10}$$

where λ_n are the eigenvalues and $\varphi_n(\mathbf{x})$ are the orthonormal eigenfunctions of the homogeneous Fredholm integral equation

$$\int_{-\infty}^{\infty} W(\mathbf{x}_1, \mathbf{x}_2) \varphi_n(\mathbf{x}_2) \mathrm{d}\mathbf{x}_2 = \lambda_n \varphi_n(\mathbf{x}_1). \tag{11}$$

The expansion (11) represents the cross-spectral density function of the illumination field as a superposition of spatially coherent mutually uncorrelated elementary modes.

Substituting for $W(\mathbf{x}_1,\mathbf{x}_2)$ from Eq. (10) into Eq. (4), after a straightforward calculation we obtain

$$I(\mathbf{u}) = \sum_{n=0}^{\infty} I_n(\mathbf{u}), \tag{12}$$

where

$$I_n(\mathbf{u}) = \lambda_n \left| \int_{-\infty}^{\infty} t(\mathbf{x})\, \varphi_n(\mathbf{x})\, H(\mathbf{u};\mathbf{x})\, \mathrm{d}\mathbf{x} \right|^2 \tag{13}$$

represents the intensity distribution formed by the n-th coherent mode of illumination field with the weight λ_n. The eigenvalues λ_n may be arranged in a converging sequence

$$\lambda_0 \geq \lambda_1 \geq ... \geq \lambda_n \geq ... \geq 0, \tag{14}$$

and hence, it is possible to truncate the summation in Eq. (12) to a finite number M of expansion terms which ensures the admissible value of the relative error of approximation,

$$\triangle = \sum_{n=M}^{\infty} \int_{-\infty}^{\infty} I_n(\mathbf{u})\, \mathrm{d}\mathbf{u} / \int_{-\infty}^{\infty} I(\mathbf{u})\, \mathrm{d}\mathbf{u}. \tag{15}$$

It is evident that this error decreases with increase of the number M. In Ref. 6 the concept of the effective number $\mathfrak{N}$ of uncorrelated modes needed to represent the illumination field is introduced, and its upper bound is defined by the following inequality:

$$\mathfrak{N} \leq \left[\int_{-\infty}^{\infty} W(\mathbf{x},\mathbf{x})\, \mathrm{d}\mathbf{x} \right]^2 / \iint_{-\infty}^{\infty} |W(\mathbf{x}_1,\mathbf{x}_2)|^2\, \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2. \tag{16}$$

It is also noted there that this number may be used to establish an optimal point for truncating the orthogonal representation of the intensity distribution. As it follows from our examples given in the next section, when the upper bound of the effective number $\mathfrak{N}$ of uncorrelated modes of the illumination field is used to truncate the summation in Eq. (12), the relative error of intensity approximation does not exceed a few percent. Such an error is quite admissible when resolving many practical problems of actual optical design.

Now, let us evaluate the computational complexity of intensity calculation in accordance with the proposed method. The dominant portion of the intensity calculation from Eqs. (12) and (13) is the consecutive multiplication of 4-D function $H(\mathbf{u};\mathbf{x})$ by 2-D function $(\lambda_n)^{1/2}\, t(\mathbf{x})\, \varphi_n(\mathbf{x})$, followed by the calculation of a square absolute value of the product for every n-th expansion term. To realize the numerical calculation of every expansion term in Eq. (12), it is necessary to multiply the samples of this functions for all possible combinations of sampling points taken one by one in each of the planes $\mathbf{u}$ and $\mathbf{x}$, and then to multiply the obtained product by its conjugate value. Hence, again using $N \times N$ sampling points and truncating the summation in Eq. (12) to the effective number $\mathfrak{N}$ of uncorrelated modes, one finds that the number of operations needed to compute $I(\mathbf{u})$ by the proposed algorithm is proportional to

$$C = \mathfrak{N}\left[\left(N^2\right)^2 + N^2 \right] = \mathfrak{N} N^2 \left(N^2 + 1\right) \tag{17}$$

or, for rather large N,

$$C \approx \mathfrak{N} N^4. \tag{18}$$

As shown in Ref. 7, the value of $\mathfrak{N}$ increases with decrease in the degree of coherence of the illumination field. For a completely coherent illumination, $\mathfrak{N} = 1$, and the computational effort C decreases to N^4. For a partially coherent illumination, C increases linearly with $\mathfrak{N}$, *i.e.*, the computational effort is lager the more incoherent the illumination. For sufficiently large values of $\mathfrak{N}$, the illumination may be generally considered to be completely incoherent. In this case, Eq. (4) reduces to [5]:

$$I(\mathbf{u}) = I_0 \iint_{-\infty}^{\infty} |H(\mathbf{u};\mathbf{x})|^2\, |t(\mathbf{x})|^2\, \mathrm{d}\mathbf{x}, \tag{19}$$

where I_0 is a constant. By analogy with the foregoing, it is straightforward matter to show that this time the number of operations needed to compute $I(\mathbf{u})$ reduces again to N^4.

Comparison of the computational efficiency of the direct calculation and the proposed algorithm for different values of $\mathfrak{N}$ is illustrated by a schematic picture in Fig 2. It is evident from this figure that the FBLT algorithm can be efficiently employed to calculate the intensity distribution when $\mathfrak{N} \leq N$. For the same values of N and the computational speed that are in the example of the previous section, the computer run time needed for calculation of $I(\mathbf{u})$ from Eqs. (12) and (13) takes from 2 min to 3 h, depending on the degree of coherence of the illumination.

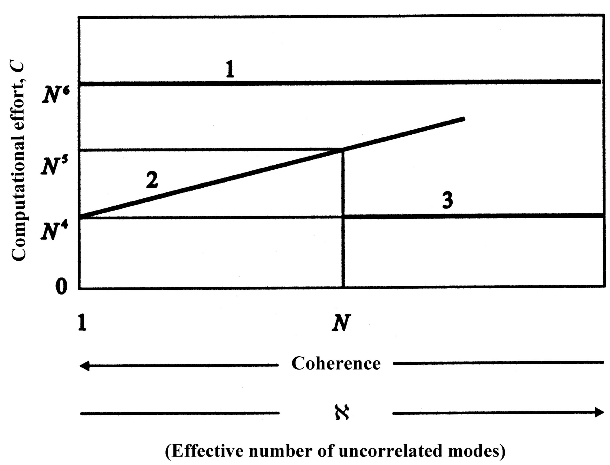

FIGURE 2. Estimation of the computational effort C as a function of coherence (effective number $\mathfrak{N}$ of uncorrelated modes of illumination): 1 - the direct method in accordance with Eq. (4); 2 - the FBLT algorithm; 3 - the direct method in accordance with Eq. (19).

4. Examples of FBLT calculations

To illustrate the application of the proposed algorithm, let us consider two examples of calculating the intensity distribution (4) in the output plane of the optical system shown in Fig. 1 for two specific cases, *i.e.*, formation of the image of an object and formation of the Fourier spectrum of an object.

As an object we choose the 1-D Dirac comb function, *i.e.*,

$$t(x) = \sum_k \delta(x - kx_0). \qquad (20)$$

This object was studied for the following two reasons. First, both the ideal image and the exact Fourier spectrum of such an object have the same form of the Dirac comb function. Secondly, the choice of this object allows the result of integrating in Eq. (4) to be obtained in an explicit analytic form, a fact that gives us a chance to evaluate the accuracy of the proposed algorithm.

Taking into account the 1-D character of our object, and for the sake of simplicity, as an illumination field, we consider the secondary 1-D Gaussian Schell-model source [7] that is characterized by a cross-spectral density function of the form

$$W(x_1, x_2) = I_0 \exp\left(-\frac{x_1^2 + x_2^2}{4\sigma_I^2}\right)\exp\left[-\frac{(x_1 - x_2)^2}{2\sigma_\mu^2}\right], \qquad (21)$$

where I_0, σ_I^2 and σ_μ^2 are positive constants. This type of source was chosen because it exhibits the essential features of many sources encountered in practice and yet it can be analyzed mathematically with relative easy. For this source

$$\lambda_n = I_0 \left(\frac{\pi}{a+b+c}\right)^{1/2}\left(\frac{b}{a+b+c}\right)^n, \qquad (22)$$

and

$$\varphi_n(x) = \left(\frac{2c}{\pi}\right)^{1/4}\frac{1}{(2^n n!)^{1/2}} H_n\left(x\sqrt{2c}\right)\exp\left(-cx^2\right), \qquad (23)$$

where

$$a = 1/4\sigma_I^2,\ b = 1/2\sigma_\mu^2,\ c = \left(a^2 + 2ab\right)^{1/2}, \qquad (24)$$

and $H_n(...)$ is the Hermite polynomial of order n. As shown in Ref. 6, for this source the effective number of uncorrelated modes is determined by the inequality

$$\mathfrak{N} \leq \left(1 + 4/\beta^2\right)^{1/2}, \qquad (25)$$

where $\beta = \sigma_\mu/\sigma_I$ is a measure of the global coherence of the source.

At last, we consider that the lens in Fig. 1 is free of aberrations and has a circular aperture of radius R. The amplitude spread function of such an optical system under certain conditions [5] is given by

$$H(\rho) = \alpha \exp\left(i\frac{\pi}{\lambda z_2}\rho^2\right)\frac{J_1(\pi R\rho/\lambda f)}{\pi R\rho/\lambda f}, \qquad (26)$$

where $\rho = \left(u^2 + v^2\right)^{1/2}$, $J_1(...)$ is the first-order Bessel function, and α, here and further on, is a dimensionless coefficient.

At first we suppose that the optical system forms the image of an object without magnification ($z_1 = z_2 = 2f$). Then, substituting for $t(x)$, $W(x_1, x_2)$ and $H(u; x)$ from Eqs. (20), (21) and (26) into the 1-D version of Eq. (4) and making use of the sifting property of the Dirac function, it is straightforward matter to obtain the following expression for the theoretical image intensity distribution:

$$I(u) = \alpha I_0 \sum_{m,l} A_{ml} \frac{J_1[\pi R(u + mx_0)/\lambda f]}{\pi R(u + mx_0)/\lambda f} \times \frac{J_1[\pi R(u + lx_0)/\lambda f]}{\pi R(u + lx_0)/\lambda f}, \qquad (27)$$

where

$$A_{ml} = \exp\left[-\frac{x_0^2}{4\sigma_I^2}\left(m^2 + l^2\right)\right]\exp\left[-\frac{x_0^2}{2\sigma_\mu^2}(m - l)^2\right]. \qquad (28)$$

By analogy, but this time using the FBLT algorithm with due regard for the truncation of summation in Eq.(12), we obtain the following approximation of the image intensity distribution (27):

$$\hat{I}(u) = \alpha I_0 \sum_{n=0}^{M-1} B_n \left[\sum_k C_{nk}\frac{J_1[\pi R(u + kx_0)/\lambda f]}{\pi R(u + kx_0)/\lambda f}\right]^2, \qquad (29)$$

where

$$B_n = \frac{1}{2^n n!}\left(\frac{b}{a+b+c}\right)^n, \qquad (30)$$

and

$$C_{nk} = H_n\left(kx_0\sqrt{2c}\right)\exp\left(-ck^2x_0^2\right). \qquad (31)$$

Now we suppose that the optical system realizes the Fourier transform of an object ($z_1 = z_2 = f$). For the sake of simplicity we neglect the vignetting effect, *i.e.*, the limitation of the effective object size by the finite lens aperture. In this case, making use of Eqs. (8), (20) and (21), by analogy with the foregoing, one can find that the intensity distribution (4) takes the form

$$I(u) = \alpha I_0 \left[A_0 + 2 \sum_{m \neq l} A_{ml} \cos \left[\frac{2\pi}{\lambda f} u x_0 (m - l) \right] \right], \quad (32)$$

where

$$A_0 = \sum_k \exp \left(-\frac{x_0^2}{2\sigma_I^2} k^2 \right), \quad (33)$$

and A_{ml} are the same as in Eq. (28).

Using the FBLT algorithm, we obtain the following approximation of the intensity distribution (32):

$$\hat{I}(u) = \alpha I_0 \sum_{n=0}^{M-1} B_n \left[C_{n0} + 2 \sum_{m \neq l} C_{nml} \times \cos \left[\frac{2\pi}{\lambda f} u x_0 (m - l) \right] \right], \quad (34)$$

where

$$C_{n0} = \sum_k H_n^2 \left(k x_0 \sqrt{2c} \right) \exp \left(-2 x_0^2 c k^2 \right), \quad (35)$$

$$C_{nml} = H_n \left(m x_0 \sqrt{2c} \right) H_n \left(l x_0 \sqrt{2c} \right) \times \exp \left[-c x_0^2 \left(m^2 + l^2 \right) \right], \quad (36)$$

and B_n is the same as in Eq. (30).

To evaluate the quality of our approximation, we realized numerical calculations of the intensity distribution $I(u)$ in accordance with Eqs. (27), (29), (32) and (33). When calculating we put $x_0 = 2.44\lambda f/R$, which is twice greater than the Rayleigh limit of resolution for our optical system, and $\sigma_I = 2\sigma_\mu = 10x_0$, which corresponds to the case of the true partial coherence $(\beta = 0.5)$. We truncated the summation over indexes k, m, l to nine central Dirac impulses in the object and varied the number M of the terms in the modal expansion.

The results of calculations are shown in Fig. 3 and Fig. 4. As can be seen in these figures, with the increase of the number M the approximate intensity distributions come closer to the theoretical curves. When the number M is equal to the effective number $\mathfrak{N}$ of uncorrelated modes of illumination (in our example $\mathfrak{N} = 4$), the relative error of the FBLT algorithm makes up approximately 1% and, when $M = 2\mathfrak{N}$, it becomes negligible.

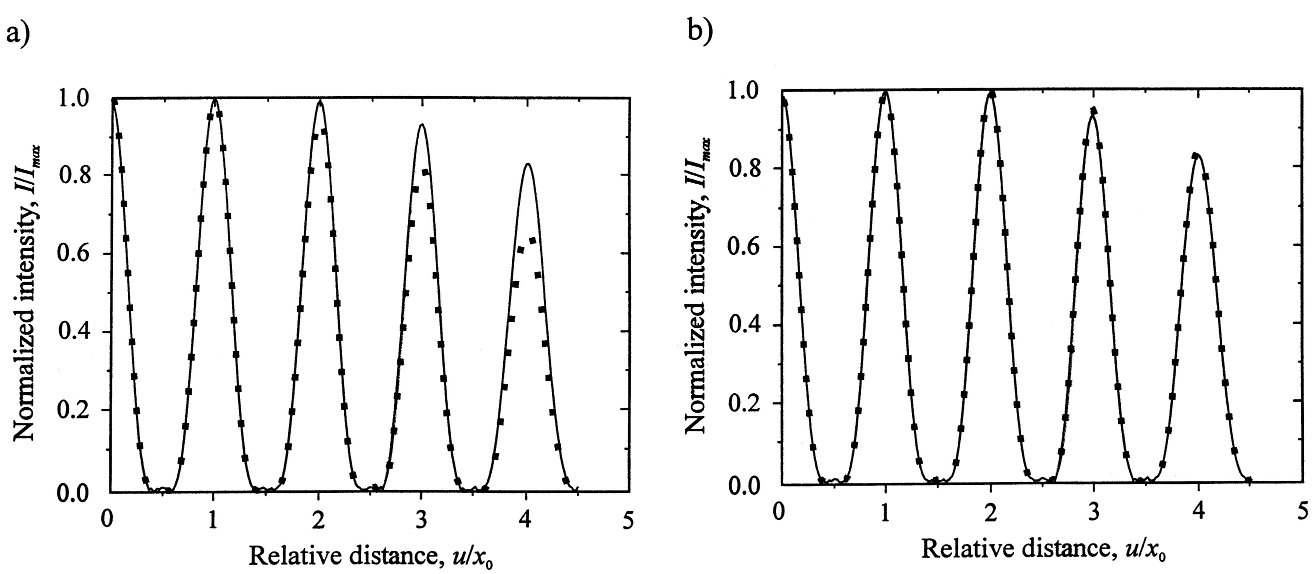

FIGURE 3. Results of calculating the image intensity distribution in accordance with Eq. (29) for: (a) $M = 1$; (b) $M = 4$. Theoretical intensity distribution, obtained according to Eq. (27), is shown by solid curves.

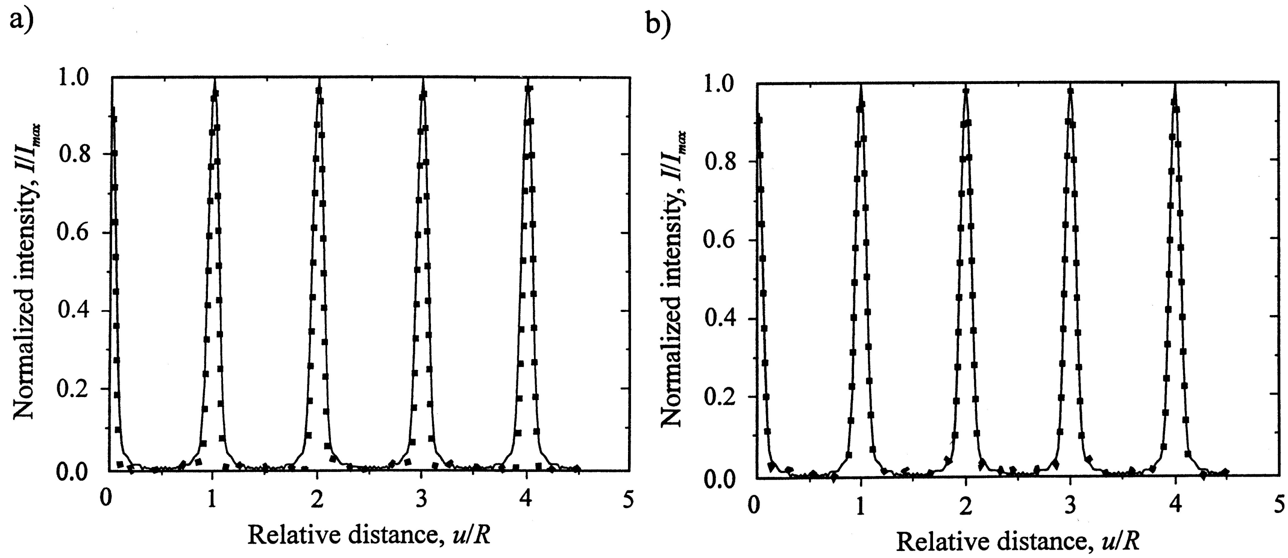

FIGURE 4. Results of calculating the intensity distribution of Fourier spectrum in accordance with Eq. (34) for: (a) $M = 1$; (b) $M = 4$. Theoretical intensity distribution, obtained according to Eq. (32), is shown by solid curves.

5. Concluding remarks

The FBLT algorithm allows to reduce considerably the computational effort needed for calculating the intensity distribution at the output of partially coherent optical system and its efficiency is larger the more coherent the illumination in a global sense. It must be noted that the application of this algorithm requires the knowledge of the coherent-mode representation of illuminating field (eigenvalues λ_n and eigenfunctions φ_n). Unfortunately, the analytical expression of the coherent modes is known for a small number of cases such as Gaussian Schell-model sources [7], twisted Gaussian Schell-model sources [8] and Bessel-correlated Schell-model sources [9]. In general case, the evaluation of coherent modes entails the numerical solution of the integral equation (11) that is not an easier computational task than the proper calculation of BLT. However, it should be taken into account that once φ_n and λ_n have been calculated for the given illumination, they can be stored and applied to the calculation of BLT for any object and any optical system. Thus, the FBLT algorithm can be considered as an indispensable tool for the analysis and computer simulating of optical systems with partially coherent illumination. We hope to report on other possible applications of the FBLT algorithm in the nearest future.

Acknowledgments

This work was supported by the National Council for Science and Technology of México (CONACyT) under the projects 25841-E and 36875-E.

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