Mathieu functions revisited: matrix evaluation and generating functions

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An updated didactical approach to Mathieu functions is presented as an alternative to the orthodox treatment. The matrix evaluation of the angular Mathieu functions is not only pertinent, but long over due, after seventy five years of matrix quantum mechanics and with the availability of computing tools. Plane waves are identified as generating functions of the Mathieu functions, showing their explicit expansions. Some mathematical and physical applications are illustrated.

Keywords: Mathieu functions; matrix evaluation; generating functions

Se presenta un enfoque actualizado y didáctico para las funciones de Mathieu como una alternativa a la presentación ortodoxa. La evaluación matricial de las funciones de Mathieu angulares es no solamente pertinente sino retrasada después de setenta y cinco años de mecánica cuántica matricial y con la disponibilidad de herramientas computacionales. Se identifican las ondas planas como funciones generadoras de las funciones de Mathieu, mostrando sus desarrollos explícitos. Se ilustran algunas aplicaciones matemáticas y físicas.

Descriptores: Funciones de Mathieu; evaluación matricial; funciones generadoras

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1. Introduction

Mathieu functions were first investigated by that author in his "Memoire sur le movement vibratoire d'une membrane de forme elliptique" in 1868 [1], evaluating the lowest order characteristic numbers and corresponding angular functions in ascending powers of the intensity parameter. A decade later, Heine defined the periodic Mathieu angular functions of integer order as Fourier cosine and sine series, without evaluating the corresponding coefficients; obtained a transcendental equation for the characteristic numbers expressed in terms of an infinite continued fraction; and also demonstrated that one set of periodic functions of integer order could be expanded in a series of Bessel functions [2]. Floquet published his mathematical work "Sur les equations différentielles linéaires á coefficients periodiques" containing the theorem named after him [3], and Hill his memoir "On the path of motion of the lunar perigee" introducing the determinant named after him [4], both works having played relevant roles in subsequent investigations of the Mathieu functions and their extensions.

For the purposes of the revisiting in the title of this paper, reference is made to Chapter 20 on Mathieu functions by G. Blanch in Ref. 5. The reader is also referred to some of the classical books on mathematical methods [6–10] and specific books on Mathieu functions [11–16] whose authors made original contributions to establish mathematical properties, to develop physical applications and to evaluate numerical values of such functions. Humbert's monograph contains the bibliography up to 1924 [11], and McLachlan's book provides a historical introduction and updated bibliography through 1947 [15].

Recent physical applications of the Mathieu functions have been discussed by Ruby [17]. His opening sentence is "with few exceptions, notably Morse and Feshbach, and Mathews and Walker, most authors of textbooks on mathematics for science and engineering choose to omit any discussion of the Mathieu equation". His review of the solution of the Mathieu equation follows the orthodox treatment [5], involving the evaluation of the characteristic numbers from the continued fraction equation and then the evaluation of the corresponding Fourier coefficients.

The authors of the present article have become involved with Mathieu functions in connection with two recent works. The first one was the review of the doctoral thesis "Formal analysis of the propagation of invariant optical fields with elliptic symmetries" [18], in which the orthodox method of evaluating the Mathieu functions was also used. The second one is about the bidimensional hydrogen molecular ion confined inside an ellipse, which can be solved in a matrix form following the methods of quantum mechanics [19].

The last two paragraphs contain the motivation to revisit the Mathieu functions. Section 2 is devoted to the separation of the Helmholtz equation in elliptical coordinates leading to the canonical and modified forms of the Mathieu equations. The orthodox method to construct the angular Mathieu functions is briefly described as a point of comparison. Its extension for the construction of the radial Mathieu functions is also mentioned. In Sec. 3, their matrix evaluation is formulated, recognizing that the output from the matrix diagonalization provides simultaneously the values of both characteristic numbers and Fourier coefficients. In Sec. 4 another gap in the study of the Mathieu functions is also filled in. It is well known that the majority of the special functions can be obtained from generating functions [5], but the Mathieu functions have been so far an exception. We have identified that plane waves are also their generating functions showing the corresponding series expansions in products of radial and angular Mathieu functions. In the final section a discussion is made of illustrative mathematical and physical applications of some of our results.

2. Helmholtz equation in elliptical coordinates

Elliptical coordinates $(0 \leq u < \infty, 0 \leq v \leq 2\pi)$, are defined through the transformation equations to Cartesian coordinates:

$$x = c\cosh u \cos v,$$
$$y = c\sinh u \sin v, \tag{1}$$

where c is the common semifocal distance of confocal ellipses with eccentricities $1/\cosh u$, and confocal hyperbolas with eccentricities $1/\cos v$.

Helmholtz equation in these coordinates takes the form

$$\left\{\frac{2}{c^2(\cosh 2u-\cos 2v)}\left[\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right]+k^2\right\}\psi(u,v)=0, \tag{2}$$

and is separable. In fact, it admits factorizable solutions

$$\psi(u,v) = U(u)V(v), \tag{3}$$

where the respective factors satisfy the ordinary differential equations

$$\left[\frac{d^2}{du^2} + 2q\cosh 2u\right]U(u) = aU(u), \tag{4}$$

$$\left[\frac{d^2}{dv^2} - 2q\cos 2v\right]V(v) = -aV(v). \tag{5}$$

Here $q = k^2c^2/4$ is the intensity parameter and a is the separation constant. Equation (5) is the canonical form of Mathieu's equation, and Eq. (4) is the modified form. They are connected through the substitutions $v \rightarrow iu$ and $V \rightarrow U$, so that the elliptical or radial Mathieu functions U can be obtained by analytical continuation of the hyperbolic or angular Mathieu functions V.

The periodic angular Mathieu functions can be expressed as Fourier series of four different types:

$$\mathrm{ce}_{2r+p}(v,q) = \sum_{s=0}^{\infty} A_{2s+p}^{2r+p}(q)\cos\left[(2s+p)v\right], \tag{6}$$

$$\mathrm{se}_{2r+p}(v,q) = \sum_{s=0}^{\infty} B_{2s+p}^{2r+p}(q)\sin\left[(2s+p)v\right]; \tag{7}$$

when $p = 0$ their period is π, and when $p = 1$ their period is 2π. The characteristic numbers for the even solutions of Eq. (6) are represented as $a_{2r+p}(q)$, and for the odd solutions of Eq. (7) as $b_{2r+p}(q)$.

The substitution of Eq. (6) or (7) in the differential Eq. (5) leads to three-term recurrence relations for the Fourier coefficients $A_{2s+p}^{2r+p}(q)$ or $B_{2s+p}^{2r+p}(q)$. Their explicit forms appear as Eqs. 20.2.5–20.2.11 in Ref. 5. The ratios of consecutive coefficients of the same parity then satisfy two types of continued fractions, appearing as Eqs. 20.2.12–20.2.20 in Ref. 5. The convergence of the Fourier series of Eqs. (6) and (7) requires that the coefficients A_m and B_m vanish as $m \rightarrow \infty$, which implies the vanishing of the respective infinite continued fractions. The corresponding roots are the characteristic numbers $a_{2r}, a_{2r+1}, b_{2r}, b_{2r+1}$ from Eqs. 20.2.21–20.2.24 in Ref. 5. Ince developed the numerical methods to evaluate the characteristic numbers accurately, and from them the ratios of the Fourier coefficients, and finally the coefficients themselves taking into account the normalization of Eqs. (6) and (7) [20]. Ince's method became the orthodox method to evaluate Mathieu functions.

The orthonormalization for the Mathieu functions in Eqs. (6) and (7) is the same as that of the cosine and sine functions, namely,

$$\int_0^{2\pi} \mathrm{ce}_m(v,q)\mathrm{ce}_n(v,q)\; dv = \pi\delta_{mn}, \tag{8}$$

$$\int_0^{2\pi} \mathrm{se}_m(v,q)\mathrm{se}_n(v,q)\; dv = \pi\delta_{mn}. \tag{9}$$

This normalization convention leads to the conditions on the respective Fourier coefficients:

$$2(A_0^{2r})^2 + (A_2^{2r})^2 + \ldots + (A_{2s}^{2r})^2 + \ldots = 1, \tag{10}$$

$$(A_1^{2r+1})^2 + (A_3^{2r+1})^2 + \ldots + (A_{2s+1}^{2r+1})^2 + \ldots = 1, \tag{11}$$

$$(B_2^{2r})^2 + (B_4^{2r})^2 + \ldots + (B_{2s}^{2r})^2 + \ldots = 1, \tag{12}$$

$$(B_1^{2r+1})^2 + (B_3^{2r+1})^2 + \ldots + (B_{2s+1}^{2r+1})^2 + \ldots = 1. \tag{13}$$

Notice the factor of 2 in the first term of Eq. (10), associated with the normalization to 2π of $\cos mv$ for $m = 0$.

In the limit in which $q \rightarrow 0$, Eqs. (6) and (7) guarantee that the Mathieu functions tend to the respective cosine and sine functions of the same order. Mathieu made the analysis of the vibrations of an elliptic membrane for small values of q obtaining power series in this parameter for the characteristic values and the periodic functions, corresponding to Eqs. 20.2.25–20.2.26 and 20.2.27–20.2.28 in Ref. 5, respectively. In the context of quantum mechanics these expansions can be understood as the result of applying perturbation theory in different orders of approximation, taking the circular membrane as the nonperturbed starting problem.

The radial Mathieu functions are obtained by analytical continuation from those of Eqs. (6) and (7):

$$\mathrm{Ce}_{2r+p}(z,q) = \mathrm{ce}_{2r+p}(iz,q)$$
$$= \sum_{s=0}^{\infty} A_{2s+p}^{2r+p}(q)\cosh\left[(2s+p)z\right], \tag{14}$$

and

$$\mathrm{Se}_{2r+p}(z,q) = -i\,\mathrm{se}_{2r+p}(iz,q) = \sum_{s=0}^{\infty} B_{2s+p}^{2r+p}(q)\sinh\big[(2s+p)z\big], \tag{15}$$

involving the same expansion coefficients.

The radial Mathieu Eq. (4) can be rewritten in terms of the alternative arguments $2\sqrt{q}\cosh u$ or $2\sqrt{q}\sinh u$. Then it can be solved as series of ordinary Bessel functions. The expansion coefficients turn out to satisfy three-term recurrence relations closely connected with the corresponding ones appearing in the angular Mathieu functions. Here we simply borrow Eqs. 20.6.3–20.6.6 in Ref. 5 to illustrate the point and as a reference for the analysis in Sec. 4:

$$\mathrm{Ce}_{2r}(z,q) = \frac{\mathrm{ce}_{2r}(\frac{\pi}{2},q)}{A_0^{2r}}\sum_{k=0}^{\infty}(-)^k A_{2k}^{2r} J_{2k}(2\sqrt{q}\cosh z) = \frac{\mathrm{ce}_{2r}(0,q)}{A_0^{2r}}\sum_{k=0}^{\infty} A_{2k}^{2r} J_{2k}(2\sqrt{q}\sinh z), \tag{16}$$

$$\begin{aligned}\mathrm{Ce}_{2r+1}(z,q) &= \frac{\mathrm{ce}'_{2r+1}(\frac{\pi}{2},q)}{\sqrt{q}A_1^{2r+1}}\sum_{k=0}^{\infty}(-)^{k+1} A_{2k+1}^{2r+1} J_{2k+1}(2\sqrt{q}\cosh z)\\ &= \frac{\mathrm{ce}_{2r+1}(0,q)}{\sqrt{q}A_1^{2r+1}}\coth z\sum_{k=0}^{\infty}(2k+1) A_{2k+1}^{2r+1} J_{2k+1}(2\sqrt{q}\sinh z),\end{aligned} \tag{17}$$

$$\mathrm{Se}_{2r}(z,q) = \frac{\mathrm{se}'_{2r}(\frac{\pi}{2},q)}{qB_2^{2r}}\tanh z\sum_{k=1}^{\infty}(-)^k 2k B_{2k}^{2r} J_{2k}(2\sqrt{q}\cosh z) = \frac{\mathrm{se}'_{2r}(0,q)}{qB_2^{2r}}\coth z\sum_{k=0}^{\infty} 2k B_{2k}^{2r} J_{2k}(2\sqrt{q}\sinh z), \tag{18}$$

$$\begin{aligned}\mathrm{Se}_{2r+1}(z,q) &= \frac{\mathrm{se}_{2r+1}(\frac{\pi}{2},q)}{\sqrt{q}B_1^{2r+1}}\tanh z\sum_{k=0}^{\infty}(-)^k (2k+1) B_{2k+1}^{2r+1} J_{2k+1}(2\sqrt{q}\cosh z)\\ &= \frac{\mathrm{se}'_{2r+1}(0,q)}{\sqrt{q}B_1^{2r+1}}\sum_{k=0}^{\infty} B_{2k+1}^{2r+1} J_{2k+1}(2\sqrt{q}\sinh z).\end{aligned} \tag{19}$$

3. Matrix evaluation of angular Mathieu functions

In this section the matrix solution of Eq. (5) is formulated. The Fourier series of Eqs. (6) and (7): divided by $\sqrt{\pi}$, in order to have basis functions normalized to one, are the starting point. Then it is straightforward to construct the matrices of Eq. (5), with a change of sign in its LHS and RHS, in the respective bases. The results are:

$$\begin{bmatrix} 0 & \sqrt{2}q & 0 & 0 & \dots & & & & \\ \sqrt{2}q & 4 & q & 0 & \dots & & & & \\ 0 & q & 16 & q & \dots & & & & \\ \vdots & \vdots & \vdots & \ddots & & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ & & & & & & q & (2s)^2 & q \\ & & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \sqrt{2}A_0 \\ A_2 \\ A_4 \\ \vdots \\ \vdots \\ \vdots \\ A_{2s} \\ \vdots \end{bmatrix} = a_{2r} \begin{bmatrix} \sqrt{2}A_0 \\ A_2 \\ A_4 \\ \vdots \\ \vdots \\ \vdots \\ A_{2s} \\ \vdots \end{bmatrix}, \tag{20}$$

$$\begin{bmatrix} 1+q & q & 0 & 0 & \dots & & & & \\ q & 9 & q & 0 & \dots & & & & \\ 0 & q & 25 & q & \dots & & & & \\ \vdots & \vdots & \vdots & \ddots & & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ & & & & & & q & (2s+1)^2 & q \\ & & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \\ \vdots \\ \vdots \\ \vdots \\ A_{2s+1} \\ \vdots \end{bmatrix} = a_{2r+1} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \\ \vdots \\ \vdots \\ \vdots \\ A_{2s+1} \\ \vdots \end{bmatrix}, \tag{21}$$

$$\begin{bmatrix} 4 & q & 0 & 0 & \dots & & & & \\ q & 16 & q & 0 & \dots & & & & \\ 0 & q & 36 & q & \dots & & & & \\ \vdots & \vdots & \vdots & \ddots & & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ & & & & & q & (2s)^2 & q & \\ & & & & & & & \ddots & \end{bmatrix} \begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ \vdots \\ \vdots \\ \vdots \\ B_{2s} \\ \vdots \end{bmatrix} = b_{2r} \begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ \vdots \\ \vdots \\ \vdots \\ B_{2s} \\ \vdots \end{bmatrix}, \tag{22}$$

$$\begin{bmatrix} 1-q & q & 0 & 0 & \dots & & & & \\ q & 9 & q & 0 & \dots & & & & \\ 0 & q & 25 & q & \dots & & & & \\ \vdots & \vdots & \vdots & \ddots & & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ & & & & & q & (2s+1)^2 & q & \\ & & & & & & & \ddots & \end{bmatrix} \begin{bmatrix} B_1 \\ B_3 \\ B_5 \\ \vdots \\ \vdots \\ \vdots \\ B_{2s+1} \\ \vdots \end{bmatrix} = b_{2r+1} \begin{bmatrix} B_1 \\ B_3 \\ B_5 \\ \vdots \\ \vdots \\ \vdots \\ B_{2s+1} \\ \vdots \end{bmatrix} \tag{23}$$

For the sake of simplicity coefficients is not written. Notice the $\sqrt{2}$ factor in the first element of the eigenvector of Eq. (20) needed to compensate for the extra factor $1/\sqrt{2\pi}$ in the normalization of the respective basis function. Also notice the tridiagonal and symmetric character of the matrices representing the Mathieu equation operator. The common and different elements in Eqs. (20) and (22), and in Eqs. (6) and (7), are also worth noticing.

Of course, these equations are equivalent to the three-term recurrence relations behind the orthodox method of evaluating the Mathieu functions. A quotation from Blanch [5] is pertinent in connection with the numerical implementation of the orthodox method:

> "…continued fractions can be converted to determinant form.
>
> …a determinant with an infinite number of rows– a special case of Hill's determinant. Although the determinant has actually been used in computations where high speed computers were available the direct use of the continued fraction seems much less laborious."

After three decades and a half, the availability of computing tools is so widespread that the last sentence of the quotation is no longer valid. In fact, our didactic and practical proposition is to use the matrix form of Eqs. (20)–(23) to solve Eq. (5). Conceptually, the eigenvalue nature of the problem is emphasized in this approach. Practically, the associated secular determinant is simpler and more familiar than the Hill determinant. Computationally, the availability of reliable, fast and accurate matrix diagonalization programs allows the evaluation of N characteristic numbers and its respective N eigenvectors, in a single run for an $N \times N$ matrix.

As an illustration of the matrix evaluation of the Mathieu function by using 10×10 matrices for $q = 0 - 5$ and 30×30 matrices for $q = 5$–25, the respective characteristic numbers and eigenvectors are obtained, matching and improving the accuracy of their corresponding values tabulated in Ref. 5.

We close this section by pointing out that Floquet solutions

$$F_v(z) = \sum_{k=-\infty}^{\infty} c_{2k} \mathrm{e}^{i(\nu+2k)z} \tag{24}$$

of Eq. (4) can also be formulated and evaluated in matrix form:

$$\begin{bmatrix} & \ddots & & & & & & & & \\ q & (-2n+\nu)^2 & q & & & & & & & \\ & & \ddots & & & & & & & \\ & 0 & q & (-2+\nu)^2 & q & 0 & & & & \\ & & 0 & q & \nu^2 & q & 0 & & & \\ & & & 0 & q & (2+\nu)^2 & q & 0 & & \\ & & & & & & \ddots & & & \\ & & & & & & q & (2n+\nu)^2 & q & \\ & & & & & & & & \ddots & \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-2n} \\ \vdots \\ c_{-2} \\ c_0 \\ c_2 \\ \vdots \\ c_{2n} \\ \vdots \end{bmatrix} = a \begin{bmatrix} \vdots \\ c_{-2n} \\ \vdots \\ c_{-2} \\ c_0 \\ c_2 \\ \vdots \\ c_{2n} \\ \vdots \end{bmatrix} \tag{25}$$

where ν is the characteristic exponent [5].

4. Plane waves as generating functions of Mathieu functions

The plane waves

$$\psi(x,y) = \mathrm{e}^{i\vec{k}\cdot\vec{\rho}} = \mathrm{e}^{i(k_x x + ik_y y)} \tag{26}$$

and circular waves

$$\psi(\rho,\varphi) = J_m(k\rho)\mathrm{e}^{im\varphi} \tag{27}$$

are regular solutions of the Helmholtz equation satisfying orthogonality and completeness properties in their common domain $(\infty < x < \infty, -\infty < y < \infty)$ and $(0 \le \rho < \infty, 0 \le \varphi \le 2\pi)$. Such properties allow the expansions of plane waves as superpositions of circular waves:

$$\mathrm{e}^{ikx} = \mathrm{e}^{ik\rho\cos\varphi} = \sum_{m=-\infty}^{\infty} i^m J_m(k\rho)\mathrm{e}^{im\varphi}, \tag{28}$$

$$\mathrm{e}^{iky} = \mathrm{e}^{ik\rho\sin\varphi} = \sum_{m=-\infty}^{\infty} (-1)^m J_m(k\rho)\mathrm{e}^{im\varphi}. \tag{29}$$

These equations also mean that the plane waves are generating functions for the circular waves.

The perusal of the literature on Mathieu functions shows that there are not explicitly recognized generating functions for them [1–16]. Since Mathieu functions are the solutions of the Helmholtz equation in elliptical coordinates, the existence of such generating functions could and should be expected. Indeed, in this section plane waves are identified as generating functions of elliptical waves, showing their explicit expansions as series of products of regular radial and angular Mathieu functions.

The starting point is Eq. (28) using the elliptical coordinate representation of the Cartesian coordinate, Eq. (1), with the result

$$\mathrm{e}^{ikx} = \mathrm{e}^{ikc\cosh u\cos v} = \sum_{m=-\infty}^{\infty} i^m J_m(kc\cosh u)\mathrm{e}^{imv}. \tag{30}$$

Next, the even and odd terms in the summation are grouped together to obtain

$$\mathrm{e}^{i2\sqrt{q}\cosh u\cos v} = J_0(2\sqrt{q}\cosh u) + 2\sum_{s=1}^{\infty}(-1)^s J_{2s}(2\sqrt{q}\cosh u)\cos(2sv)$$
$$+2i\sum_{s=1}^{\infty}(-1)^s J_{2s+1}(2\sqrt{q}\cosh u)\cos[(2s+1)v], \tag{31}$$

where the replacement $kc = 2\sqrt{q}$ has been made.

Since the Fourier and angular Mathieu function bases are both complete and orthogonal in the same domain, the first ones can be written as linear superpositions of the second ones:

$$\cos[(2s+p)v] = \sum_{r=0}^{\infty} \bar{A}_{2r+p}^{2s+p}(q)\mathrm{ce}_{2r+p}(v,q), \tag{32}$$

$$\sin[(2s+p)v] = \sum_{r=0}^{\infty} \bar{B}_{2r+p}^{2s+p}(q)\mathrm{se}_{2r+p}(v,q). \tag{33}$$

These are the inverse transformations of Eqs. (6) and (7). The corresponding transformation coefficients are connected by a simple transposition

$$\bar{A}_{2r+p}^{2s+p}(q) = \frac{1}{\pi}\int_0^{2\pi} \mathrm{ce}_{2r+p}(q,v)\cos[(2s+p)v]\,dv$$
$$= A_{2s+p}^{2r+p}(q), \tag{34}$$

$$\bar{B}_{2r+p}^{2s+p}(q) = \frac{1}{\pi}\int_0^{2\pi} \mathrm{se}_{2r+p}(q,v)\sin[(2s+p)v]\,dv$$
$$= B_{2s+p}^{2r+p}(q); \tag{35}$$

except for $\bar{A}_{2r}^{0} = 2A_0^{2r}$, associated with the different normalization of $\cos(2sv)$ for $s = 0$.

Then the successive substitutions of Eqs. (34) and (32), and the result in Eq. (31) leads to the result

$$\mathrm{e}^{i2\sqrt{q}\cosh u\cos v} = 2\sum_{s=0}^{\infty}(-1)^s J_{2s}(2\sqrt{q}\cosh u)\sum_{r=0}^{\infty} A_{2s}^{2r}(q)\mathrm{ce}_{2r}(v,q)$$
$$+2i\sum_{s=0}^{\infty}(-)^s J_{2s}(2\sqrt{q}\cosh u)\sum_{r=0}^{\infty} A_{2s+1}^{2r+1}(q)\mathrm{ce}_{2r+1}(v,q). \tag{36}$$

The exchange of the summations over the indices r and s gives

$$\mathrm{e}^{i2\sqrt{q}\cosh u\cos v} = 2\sum_{r=0}^{\infty}\mathrm{ce}_{2r}(v,q)\sum_{s=0}^{\infty}(-)^s A_{2s}^{2r}(q) J_{2s}(2\sqrt{q}\cosh u)$$
$$+2i\sum_{r=0}^{\infty}\mathrm{ce}_{2r+1}(v,q)\sum_{s=0}^{\infty}(-1)^s A_{2s+1}^{2r+1}(q) J_{2s+1}^{2r+1}(2\sqrt{q}\cosh u), \tag{37}$$

The final step is to recognize that the summations over the index s are the even and odd order regular radial Mathieu functions, except for the normalization factor, according to Eq . (16) and (17), respectively, with the final result

$$\mathrm{e}^{i2\sqrt{q}\cosh u\cos v}=2\sum_{r=0}^{\infty}\frac{A_0^{2r}}{\mathrm{ce}_{2r}(\frac{\pi}{2},q)}\mathrm{Ce}_{2r}(u,q)\mathrm{ce}_{2r}(v,q)-2i\sum_{r=0}^{\infty}\frac{\sqrt{q}A_1^{2r+1}}{\mathrm{ce}'_{2r+1}(\frac{\pi}{2},q)}\mathrm{Ce}_{2r+1}(u,q)\mathrm{ce}_{2r+1}(v,q). \tag{38}$$

Similarly, if we start with the plane wave of Eqs. (29) and (2) for y, we obtain

$$\mathrm{e}^{i2\sqrt{q}\sinh u\sin v}=\sum_{m=-\infty}^{\infty}(-1)^m J_m(2\sqrt{q}\sinh u)\mathrm{e}^{imv}=J_o(2\sqrt{q}\sinh u)+2\sum_{s=1}^{\infty}J_{2s}(2\sqrt{q}\sinh u)\cos(2sv)$$
$$+2i\sum_{s=0}^{\infty}J_{2s+1}(2\sqrt{q}\sinh u)\sin[(2s+1)v]. \tag{39}$$

By writing the Fourier basis functions in terms of the Mathieu functions, Eqs. (32) and (33), and using the relations between the transformation coefficients of Eqs. (34) and (35), Eq. (39) becomes

$$\mathrm{e}^{i2\sqrt{q}\sinh u\sin v}=2\sum_{r=0}^{\infty}\mathrm{ce}_{2r}(q,v)\sum_{s=0}^{\infty}A_{2s}^{2r}(q)J_{2s}(2\sqrt{q}\sinh u)+2i\sum_{r=0}^{\infty}\mathrm{se}_{2r+1}(q,v)\sum_{s=0}^{\infty}B_{2s+1}^{2r+1}(q)J_{2s+1}(2\sqrt{q}\sinh u). \tag{40}$$

Now the sums over s are identified with the corresponding regular radial Mathieu functions of Eqs. (16) and (19), respectively, with the final expansion

$$\mathrm{e}^{i2\sqrt{q}\sinh u\sin v}=2\sum_{r=0}^{\infty}\frac{A_0^{2r}}{\mathrm{ce}_{2r}(0,q)}\mathrm{Ce}_{2r}(u,q)\mathrm{ce}_{2r}(v,q)+2i\sum_{r=0}^{\infty}\frac{\sqrt{q}B_1^{2r+1}}{\mathrm{se}'_{2r+1}(0,q)}\mathrm{Se}_{2r+1}(u,q)\mathrm{se}_{2r+1}(v,q). \tag{41}$$

Equations (38) and (41) represent plane waves as generating functions of the Mathieu functions. Next we analyze some of their specific values and of their derivatives with respect to the hyperbolic coordinate for the special values of $v=0$ and $\pi/2$, in order to establish some of the other relationships and representations of the radial Mathieu functions of Eqs. (16)–(19).

Equation (38) for $v=\pi/2$ and Eq. (41) for $v=0$, give the same result

$$1=2\sum_{r=0}^{\infty}A_0^{2r}(q)\mathrm{Ce}_{2r}(u,q), \tag{42}$$

analogous to 9.1.46 for Bessel functions in Ref. 5.

The first derivative of Eq. (31) with respect to v is

$$-i2\sqrt{q}\cosh u\sin v\mathrm{e}^{i2\sqrt{q}\cosh u\cos v}=-2\sum_{s=1}^{\infty}(-)^s J_{2s}(2\sqrt{q}\cosh u)2s\sin(2sv)$$
$$-2i\sum_{s=0}^{\infty}(-)^s J_{2s+1}(2\sqrt{q}\cosh u)(2s+1)\sin[(2s+1)v]. \tag{43}$$

Then the use of Eqs. (33) and (35) allows rewriting it as

$$-i2\sqrt{q}\cosh u\sin v\mathrm{e}^{i2\sqrt{q}\cosh u\cos v}=-2\sum_{r=1}^{\infty}\mathrm{se}_{2r}(v,q)\sum_{s=1}^{\infty}(-1)^s 2sB_{2s}^{2r}J_{2s}(2\sqrt{q}\cosh u)$$
$$-2i\sum_{r=0}^{\infty}\mathrm{se}_{2r+1}(v,q)\sum_{s=0}^{\infty}(-)^s(2s+1)B_{2s+1}^{2r+1}J_{2s+1}(2\sqrt{q}\cosh u). \tag{44}$$

Now the value of the exponential for $v=\pi/2$ can be rewritten as

$$1=\sum_{r=0}^{\infty}\frac{\mathrm{se}_{2r+1}(\frac{\pi}{2},q)}{\sqrt{q}\cosh u}\sum_{s=0}^{\infty}(-1)^s(2s+1)B_{2s+1}^{2r+1}J_{2s+1}(2\sqrt{q}\cosh u). \tag{45}$$

The first derivative of Eq. (41) with respect to v is

$$i2\sqrt{q}\sinh u\cos v e^{i2\sqrt{q}\sinh u\sin v} = 2\sum_{r=0}^{\infty}\frac{A_0^{2r}}{\mathrm{ce}_{2r}(0,q)}\mathrm{Ce}_{2r}(u,q)\mathrm{ce}'_{2r}(v,q) + 2i\sum_{r=0}^{\infty}\frac{\sqrt{q}B_1^{2r+1}}{\mathrm{se}'_{2r+1}(0,q)}\mathrm{Se}_{2r+1}(u,q)\mathrm{se}'_{2r+1}(v,q). \quad (46)$$

Now the value of the exponential for $v=0$ becomes

$$1 = \sum_{r=0}^{\infty}\frac{B_1^{2r+1}(q)}{\sinh u}\mathrm{Se}_{2r+1}(u,q). \quad (47)$$

The comparison of Eq. (45) and (47) leads to the representation of the radial Mathieu functions,

$$\mathrm{Se}_{2r+1}(u,q) = \frac{\mathrm{se}_{2r+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}B_1^{2r+1}}\tanh u\sum_{s=0}^{\infty}(-)^s(2s+1)B_{2s+1}^{2r+1}J_{2s+1}(2\sqrt{q}\cosh u), \quad (48)$$

as an alternative to the one used in going from Eq. (40) and (41), and which is the other form of Eq. (19).

Similarly, the first derivative of Eq. (39) with respect to v is

$$i2\sqrt{q}\sinh u\cos v e^{i2\sqrt{q}\sinh u\sin v} = -2\sum_{s=1}^{\infty}J_{2s}(2\sqrt{q}\sinh u)2s\sin(2sv) + 2i\sum_{s=0}^{\infty}J_{2s+1}(2\sqrt{q}\sinh u)(2s+1)\cos[(2s+1)v], \quad (49)$$

and can be rewritten in terms of angular Mathieu functions as

$$i2\sqrt{q}\sinh u\cos v e^{i2\sqrt{q}\sinh u\sin v} = -2\sum_{r=1}^{\infty}\mathrm{se}_{2r}(v,q)\sum_{s=1}^{\infty}(2s)B_{2s}^{2r}J_{2s}(2\sqrt{q}\sinh u) + 2i\sum_{r=0}^{\infty}\mathrm{ce}_{2r+1}(v,q)\sum_{s=0}^{\infty}(2s+1)B_{2s+1}^{2r+1}J_{2s+1}(2\sqrt{q}\sinh u). \quad (50)$$

Now the value of the exponential function for $v=0$ is represented as

$$1 = \sum_{r=0}^{\infty}\frac{\mathrm{ce}_{2r+1}(0,q)}{\sqrt{q}\sinh u}\sum_{s=0}^{\infty}(2s+1)B_{2s+1}^{2r+1}J_{2s+1}(2\sqrt{q}\sinh u). \quad (51)$$

The first derivative of Eq. (38) with respect to v is

$$-i\sqrt{2}\cosh u\sin v e^{i2\sqrt{q}\cosh u\cos v} = 2\sum_{r=0}^{\infty}\frac{A_0^{2r}}{\mathrm{ce}_{2r}\left(\frac{\pi}{2},q\right)}\mathrm{Ce}_{2r}(u,q)\mathrm{ce}'_{2r}(v,q) - 2i\sum_{r=0}^{\infty}\frac{\sqrt{q}A_1^{2r+1}}{\mathrm{ce}'_{2r+1}\left(\frac{\pi}{2},q\right)}\mathrm{Ce}_{2r+1}(u,q)\mathrm{ce}'_{2r+1}(v,q). \quad (52)$$

Now the value of the exponential function for $v=\pi/2$ takes the form

$$1 = \sum_{r=0}^{\infty}\frac{A_1^{2r+1}(q)}{\cosh u}\mathrm{Ce}_{2r+1}(u,q). \quad (53)$$

Then the comparison of Eqs. (51) and (53) leads to the representation of the radial Mathieu function,

$$\mathrm{Ce}_{2r+1}(u,q) = \frac{\mathrm{ce}_{2r+1}(0,q)}{\sqrt{q}A_1^{2r+1}}\coth u\sum_{s=0}^{\infty}(2s+1)B_{2s+1}^{2r+1}J_{2s+1}(2\sqrt{q}\sinh u), \quad (54)$$

as an alternative of the one going from Eqs. (37) and (38) corresponding to the other form of Eq. (17).

On the other hand, the reader may notice that in Eqs. (38) and (41) the Mathieu functions of even order and sine type do not appear. Next, we establish their representations by taking their second derivatives with respect to v via Eq. (44) and (50), respectively,

$$[-i2\sqrt{q}\cosh u\cos v+(-i2\sqrt{q}\cosh u\sin v)^2]\mathrm{e}^{i2\sqrt{q}\cosh u\cos v}=-2\sum_{r=1}^{\infty}\mathrm{se}'_{2r}(v,q)\sum_{s=1}^{\infty}(-)^s(2s)B_{2s}^{2r}J_{2s}(2\sqrt{q}\cosh u)$$
$$-2i\sum_{r=0}^{\infty}\mathrm{se}'_{2r+1}(v,q)\sum_{s=0}^{\infty}(-1)^s(2s+1)B_{2s+1}^{2r+1}J_{2s+1}(2\sqrt{q}\cosh u), \quad (55)$$

$$[-i2\sqrt{q}\sinh u\sin v+(i2\sqrt{q}\sinh u\cos v)^2]\mathrm{e}^{i2\sqrt{q}\sinh u\sin v}=-2\sum_{r=1}^{\infty}\mathrm{se}'_{2r}(v,q)\sum_{s=1}^{\infty}(2s)B_{2s}^{2r}J_{2s}(2\sqrt{q}\sinh u)$$
$$+2i\sum_{r=0}^{\infty}\mathrm{se}'_{2r+1}(v,q)\sum_{s=0}^{\infty}(2s+1)B_{2s+1}^{2r+1}J_{2s+1}(2\sqrt{q}\sinh u). \quad (56)$$

Now the exponential of Eq. (55) for $v=\pi/2$ takes the form

$$1=\sum_{r=1}^{\infty}\frac{\mathrm{se}'_{2r}(\frac{\pi}{2},q)}{2q\cosh^2 u}\sum_{s=1}^{\infty}(-1)^s 2sB_{2s}^{2r}J_{2s}(2\sqrt{q}\cosh u). \quad (57)$$

Correspondingly the exponential of Eq. (56) for $v=0$ gives

$$1=\sum_{r=1}^{\infty}\frac{\mathrm{se}'_{2r}(0,q)}{2q\sinh^2 u}\sum_{s=1}^{\infty}2sB_{2s}^{2r}J_{2s}(2\sqrt{q}\sinh u). \quad (58)$$

The comparison of Eqs. (57) and (58) leads to the equivalent representations for the radial Mathieu functions $\mathrm{Se}_{2r}(u,q)$ of Eq. (18) with the conventional normalization. Also both Eq. (57) and (58) can be rewritten as

$$1=\sum_{r=1}^{\infty}\frac{B_2^{2r}(q)}{\sinh 2u}\mathrm{Se}_{2r}(u,q). \quad (59)$$

To sum up this section, Eqs. (38) and (41) are generating functions of the Mathieu functions. From the analysis of some of their values and of their derivatives, different properties of the radial Mathieu functions have been explicitly established. Specifically, their alternative representations of Eqs. (16)–(19) and alternative representations of one, Eqs. (42), (47), (53) and (59). The reader can verify that Eq. (47) is analogous to $9.1.47$ and Eq. (53) to $9.1.48$ for Bessel functions Ref. 5. Equation (59) is the corresponding result for $\sinh 2u$. The four equations can also be understood as particular cases of Eqs. (32) and (33) with $(s=0,p=0)$ and $(s=0,p=1)$, and $(s=0,p=1)$ and $(s=0,p=0)$, and $v\to iu$, respectively.

5. Some applications and discussion

The analysis of the last part of the previous section illustrates some of the immediate applications of the generating functions of Eqs. (38) and (41). Other applications lead to integral representations of the radial Mathieu functions. In fact, the orthogonality properties of the angular Mathieu functions of Eqs. (8) and (9), allow the evaluation of the expansion coefficients of the respective generating functions of Eq. (38) and (41) with the results

$$\mathrm{Ce}_{2r}(u,q)=\frac{\mathrm{ce}_{2r}(\frac{\pi}{2},q)}{2\pi A_0^{2r}}\times\int_0^{2\pi}\cos[2\sqrt{q}\cosh u\cos v]\mathrm{ce}_{2r}(v,q)\,dv, \quad (60)$$

$$\mathrm{Ce}_{2r+1}(u,q)=\frac{-\mathrm{ce}'_{2r}\left(\frac{\pi}{2},q\right)}{2\pi\sqrt{q}A_1^{2r+1}}\times\int_0^{2\pi}\sin[2\sqrt{q}\cosh u\cos v]\mathrm{ce}_{2r+1}(v,q)\,dv, \quad (61)$$

$$\mathrm{Ce}_{2r}(u,q)=\frac{\mathrm{ce}_{2r}(0,q)}{2\pi A_0^{2r}}\times\int_0^{2\pi}\cos[2\sqrt{q}\sinh u\sin v]\mathrm{ce}_{2r}(v,q)\,dv, \quad (62)$$

$$\mathrm{Se}_{2r+1}(u,q)=\frac{\mathrm{se}'_{2r+1}(0,q)}{2\pi B_1^{2r+1}}\times\int_0^{2\pi}\sin[2\sqrt{q}\sinh u\sin v]\mathrm{se}_{2r+1}(v,q)\,dv. \quad (63)$$

The integrals involved are equivalent to four times the integrals from 0 to $\pi/2$. Similar integral representations can be obtained starting from the derivatives of the generating functions. Additional results follow from analytical continuation of the variable u. The reader can check that they correspond to Eqs. $20.7.20$–$20.7.24$ in Ref. 5.

Concerning the physical applications of the Mathieu functions, the reader is referred to $[11, 15, 17–19]$. The faster and reliable matrix method to evaluate the characteristic numbers and Fourier coefficients for the Mathieu functions may be easily implemented by those interested in developing other physical applications. The explicit identification of plane waves as generating functions of elliptical waves has been of special interest in our group in order to investigate optical diffraction and electron transport in bidimensional structures with elliptical-hyperbolic boundaries.

From a didactical point of view, we recommend to teachers and students to incorporate some of the methods and results of Sec. 3 and 4 in their study of the Mathieu functions. The revisiting in this paper does not pretend to be exhaustive, and we anticipate additional insights and extensions.

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