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APPROXIMATE ANALYTIC SOLUTIONS FOR THE IONIZATION STRUCTURE OF A PRESSURE EQUILIBRIUM STRÖMGREN SPHERE

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RESUMEN

Presentamos modelos analíticos de una región fotoionizada en equilibrio de presión con el medio ambiente neutro. Los modelos están basados en la suposición de una dependencia lineal entre la fracción de ionización de H y el cuadrado de la velocidad del sonido del gas. Demostramos que bajo estas suposiciones el problema de transporte radiativo "gris" tiene soluciones analíticas, que dan la estructura de ionización y la densidad de la nebulosa en función del radio.

ABSTRACT

We present analytic models for a photoionized region in pressure equilibrium with the surrounding, neutral material. The models are based on the assumption of a linear relation between the H ionization fraction and the square of the sound speed of the gas. We show that under these assumptions the "grey" radiative transfer equation has analytic solutions that provide the ionization structure and the density of the nebula as a function of radius.

Key Words: ISM: HII regions

1. INTRODUCTION

In his classical paper, Strömgren (1939) derived analytic solutions for the radially dependent ionization structure of a homogeneous, photoionized region. He derived both an "inner" and an "outer" approximate solution for the H ionization fraction $x = n_{HII}/n_H$ (where n_{HII} and n_H are the ionized and total H number densities, respectively) as a function of the spherical radius R. Strömgren's "outer solution" is not entirely satisfactory, because it is based on a plane approximation, and therefore does not fix the position of the outer ionization front. Strömgren's "inner solution" shows substantial deviations from the "exact" (i.e., numerical) solution close to the Strömgren radius. Raga (2015) showed that it is possible to use a method similar to the one employed by Strömgren to obtain his "inner solution", to derive piecewise solutions with improved accuracy. In this way, analytic solutions for the full ionization structure of a homogeneous, photoionized region are obtained.

In the present paper, we consider the problem of a pressure equilibrium, photoionized region. This problem might be relevant for compact or ultracompact HII regions (which are embedded in molecular clouds, see e.g. the review of Kurtz 2005) or for giant HII regions, photoionized by a large number of O stars.

A pressure equilibrium HII region has a density that is approximately homogeneous in the inner, fully ionized region. However, towards the outer edge of the photoionized region (in which the temperature has a transition from the $\approx 10^4$K of the photoionized gas to the $\approx 100 \to 1000$ K of the outer, neutral environent), the density has a steep rise. The simplest way to approach this problem is to assume a parametrized relation between the temperature and the H ionization fraction $x = n_{HII}/n_H$, leading (through the condition of pressure equilibrium) to a relation between x and n_H (the total H density).

We show that in this approximation the (grey) radiative transfer+ionization equilibrium problem has different possible analytic solutions, depending on the assumed temperature dependence of the recombination coefficient. The basic equations describing the photoionization of a stratified nebula are derived in § 2. The problem of a pressure equilibrium nebula (in the cases of a temperature-independent recombination coefficient α_H, and of $\alpha_H \propto c^{-2}$, where c is

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the isothermal sound speed) is presented in § 3. The possible applicability of pressure equilibrium HII region models is explored in § 4. Finally, the results are discussed in § 5.

2. THE PHOTOIONIZATION OF A PURE H NEBULA

Let us consider the photoionization equilibrium of a pure H nebula:

$$n_H(1-x)\phi_H = x^2 n_H^2 \alpha_H\,, \tag{1}$$

where $x = n_{HII}/n_H = n_e/n_H$ is the ionization fraction, $n_H = n_{HI} + n_{HII}$ the total H number density (with n_{HI} the neutral H density), n_e the electron density, α_H the recombination coefficient and ϕ_H the photoionization rate of H.

In the "grey approximation of the ISM" (i.e., using a constant, average value σ_H for the photoionization cross section), the H photoionization rate is given by

$$\phi_H = \frac{S_* \sigma_H}{4\pi R^2} e^{-\tau}\,, \tag{2}$$

with

$$\tau = n_H \sigma_H \int_0^R (1-x) dR\,, \tag{3}$$

where R is the spherical radius (measured from the central star) and S_* is the stellar ionizing photon rate. The photoionizations resulting from the diffuse, ionizing photon field have been neglected.

Equation (3) can be written in the form:

$$\frac{1}{f}\frac{df}{dr} = -\lambda(1-x)\left(\frac{n_H}{n_0}\right)\,, \tag{4}$$

with $f \equiv e^{-\tau}$, $r = R/R_S$, and

$$\lambda \equiv R_S n_0 \sigma_H\,, \tag{5}$$

where $n_0 = n_H(R \to 0)$ is the central density of the nebula. The radius has been adimensionalized with the Strömgren radius

$$R_s = \left(\frac{3S_*}{4\pi n_0^2 \alpha_0}\right)^{1/3} \tag{6}$$

of a homogeneous nebula of density n_0 and (constant) recombination coefficient α_0.

Equation (4) has to be integrated with the boundary condition $f(0) = 1$. However, the integration is not necessarily trivial because the ionization fraction x has to be obtained as a function of r and f through equations (1) and (2), which lead to the dimensionless relation

$$\frac{x^2}{1-x} = \frac{\lambda f}{3r^2}\left(\frac{n_0}{n_H}\right)\left(\frac{\alpha_0}{\alpha_H}\right)\,. \tag{7}$$

The derivation of equations (4) and (7) is equivalent to the one presented in § 2 of Raga (2015), but we have allowed here for possible position-dependencies of the density (n_H) and the recombination coefficient (α_H).

3. A PRESSURE EQUILIBRIUM HII REGION MODEL

3.1. *The relation between density and ionization fraction*

In order to proceed analytically, we assume that the square of the isothermal sound speed of the gas is a linear function of the H ionization fraction:

$$c^2(x) = x\,c_i^2 + (1-x)\,c_n^2 \approx x\,c_i^2\,, \tag{8}$$

where $c_i \approx 10$ km s^{-1} and $c_n \approx 1$ km s^{-1} are the isothermal sound speeds of the ionized and neutral media, respectively. The temperature can be calculated as a function of $c(x)$ through the relation

$$c^2(x) = \frac{k(1+x)T}{m_H}\,, \tag{9}$$

where k is Boltzmann's constant and m_H the H mass.

A linear temperature vs. ionization fraction relation has been extensively used in the past to model photoionized regions (see, e.g., Esquivel & Raga 2007). This kind of relation clearly does not reproduce the temperature peaks that are found in more detailed models of the partially ionized, outer boundaries of photoionized regions (see, e.g., O'Dell et al. 2007). However, these peaks (which result from the hardening of the ionizing radiation, obtained in multi-frequency transfer models) have temperatures only ≈ 10 % in excess of the interior temperature of the HII region, and are unlikely to have a strong effect on the associated, pressure equilibrium density stratification.

Then, for a pressure equilibrium nebula (i.e., with constant $P = n_H m_H c^2$), from equation (8) we obtain:

$$n_H = \frac{P}{m_H c^2(x)} = \frac{n_0}{x + (1-x)c_n^2/c_i^2} \approx \frac{n_0}{x}\,, \tag{10}$$

where n_0 is the central density of the nebula. To obtain the third equality, we used the fact that $c_n/c_i \ll 1$ (see above).

Also, inserting the $n_H(x)$ dependence resulting from the pressure equilibrium condition (third equality of equation 10) in the photoionization balance (equation 7) we obtain:

$$\frac{x}{1-x} = \frac{\lambda f}{3r^2}\left(\frac{\alpha_0}{\alpha_H}\right) . \qquad (11)$$

Finally, combining equations (4) and (10) we obtain the differential equation:

$$\frac{1}{f}\frac{df}{dr} = -\lambda \frac{1-x}{x} , \qquad (12)$$

which has to be solved with the x values given by equation (19) as a function of f and r. After an appropriate choice for the temperature-dependence of the recombination coefficient α_H, equation (12) has to be combined with equation (11) to obtain the resulting structure of the photoionized region.

3.2. *Case of constant* α_H

Let us now assume that the recombination coefficient has a temperature-independent value $\alpha_H = \alpha_0$. From equation (11) we then obtain:

$$\frac{1-x}{x} = \frac{3r^2}{\lambda f} . \qquad (13)$$

We then combine this result with the differential equation for f (equation 12) to obtain

$$\frac{df}{dr} = -3r^2 , \qquad (14)$$

which (with the $f(0) = 1$ condition, see above) has the solution

$$f(r) = 1 - r^3 . \qquad (15)$$

Inserting this result in equation (13) we finally obtain the ionization fraction as a function of dimensionless radius $r = R/R_S$:

$$x(r) = \left[1 + \frac{3r^2}{\lambda(1-r^3)}\right]^{-1} . \qquad (16)$$

This solution (which has $x(0) = 1$ and $x(1) = 0$) coincides with the "inner solution" of Strömgren (1939) for a homogeneous HII region. In other words, Strömgren's "inner solution" is the exact analytic solution for a pressure equilibrium HII region with the "temperature law" given by equation (8) (with $c_n = 0$) and for a temperature-independent recombination coefficient. The radial dependencies of the ionization fraction obtained for three different values of λ (10, 100 and 1000) are shown in Figure 1.

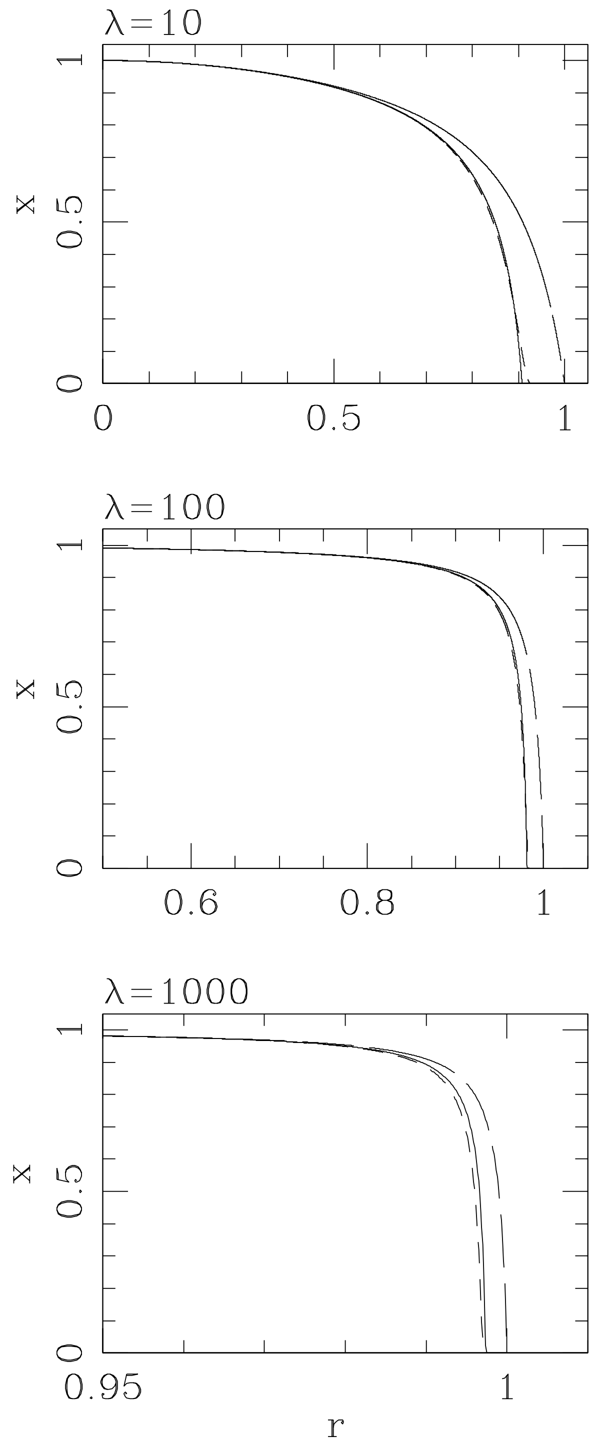

Fig. 1. Ionization fraction vs. dimensionless radius $r = R/R_S$ obtained for three different values of the λ parameter ($\lambda = 10$, 100 and 1000, from top to bottom). The solid curves show the exact (i.e., numerical) solution for the $\alpha_H \propto c^{-1}$ case, and the short dash curves show the corresponding, approximate analytic solutions (see § 3.3). The long dash curves show the constant α_H analytic solutions (see § 3.2).

3.3. *Case of* $\alpha_H \propto c^{-1}$

Let us now assume that the recombination coefficient has a temperature dependence of the form

$$\alpha_H = \alpha_0 \left[\frac{c_i}{c(x)}\right] . \qquad (17)$$

This dependence is similar to an $\alpha_H \propto 1/T^{1/2}$ dependence, except for the fact that the isothermal sound speed squared, $c^2(x)$, has an extra "$1+x$" factor (see equation 9). Combining equations (17) and (10) we then have $\alpha_H = \alpha_0/x$, which we insert in equation (11) to obtain:

$$\frac{1-x}{x^{1/2}} = \frac{3r^2}{\lambda f} \equiv A\,, \qquad (18)$$

which has the solution

$$x(A) = \frac{1}{4}\left[\sqrt{A^2+4} - A\right]^2 . \qquad (19)$$

This equation gives the ionization fraction x as a function of f and r (see the definition of A in equation 18). This result has to be inserted in the right-hand side of equation (12) to obtain the differential equation describing the model (which has to be integrated starting from the $f(0) = 1$ condition at the origin). The resulting equation does not have an exact analytic solution.

An approximate analytic solution can be derived as follows. We first note that equation (19) has the asymptotic limiting behaviors:

$$\frac{1-x}{x} = A\,, \;\; \text{for } A \ll 1\,; \qquad \frac{1-x}{x} = A^2\,, \;\; \text{for } A \gg 1\,. \qquad (20)$$

We then propose the approximate form

$$\frac{1-x}{x} \approx A + 0.7A^2\,, \qquad (21)$$

which in the $A = 0 \to 5$ range has deviations of less than 13% from the results obtained using the exact form for $x(A)$ (see equation 19).

We now combine equations (21), (18) and (12) to obtain:

$$\frac{df}{dr} = -3r^2 - 0.7\frac{9r^4}{\lambda f}\,, \qquad (22)$$

which has to be integrated with the $f(0) = 1$ boundary condition.

An approximate integral of equation (22) can be obtained as follows. For small r the first term on the right-hand-side dominates over the second term, so that we recover Strömgren's solution ($f = 1 - r^3$, see equation 15). We now use this solution for a first iteration, obtained by substituting $f = 1 - r^3$ in the second term on the right-hand-side of equation (22). In this way, the differential equation becomes:

$$\frac{df}{dr} = -3r^2 - 0.7\frac{9r^4}{\lambda(1-r^3)}\,, \qquad (23)$$

which can be directly integrated to obtain:

$$f(r) = 1 - r^3 + 0.7\frac{3}{2\lambda}\left\{3r^2 + \ln\left[\frac{(1-r)^2}{r^2+r+1}\right] + 2\sqrt{3}\left[\tan^{-1}\left(\frac{2r+1}{\sqrt{3}}\right) - \frac{\pi}{6}\right]\right\} . \qquad (24)$$

With this solution, we can calculate $A = 3r^2/(\lambda f)$ and use equation (19) to obtain the radial dependence of the ionization fraction. The ionization structures obtained for three choices of λ (10, 100 and 1000, see equation 5) are shown in Figure 1. The lower values of this λ range correspond to photoionized gas in the narrow line regions of active galaxies (see, e.g., Binette et al. 1993), and the $\lambda = 1000$ value corresponds to an HII region photoionized by an O star (see, e.g., Raga 2015).

In this figure, we also show the $x(r)$ obtained from numerical integrations of equations (12), (18) and (19). It is clear that the approximate, analytic solution (equation 24) reproduces the main features of the "exact" (i.e. numerical) solution. Figure 1 also shows the ionization structure obtained from the constant α_H solution (equation 16).

4. THE APPLICABILITY OF A PRESSURE EQUILIBRIUM HII REGION MODEL

De Pree et al. (1995) pointed out that compact HII regions embedded in molecular clouds would rapidly reach pressure equilibrium with the surrounding neutral material, provided that the environmental density is high enough. The argument is as follows.

Raga et al. (2012a) showed that an HII region within a homogeneous environment reaches the final, pressure-equilibrium configuration at a time

$$t_f \approx 6\times 10^5\,\text{yr} \times \left(\frac{S_*}{10^{49}\text{s}^{-1}}\right)\left(\frac{10^5\text{cm}^{-3}}{n_a}\right)^{2/3}\left(\frac{1\,\text{km}\,\text{s}^{-1}}{c_a}\right)^{7/3}\,, \qquad (25)$$

where n_a and c_a are the number density and the sound speed (respectively) of the neutral environment, and S_* is the stellar ionizing photon rate.

This equation was derived assuming a temperature of 10^4 K for the photoionized gas (see equation 19 of Raga et al. 2012a). Therefore, an HII region driven by an O7 star (with $S_* = 10^{49}\mathrm{s}^{-1}$) in a high density environment (with a density $n_a = 10^5\mathrm{cm}^{-3}$ or higher) will stop expanding in under 10^6 yr, within the lifetime of the central O star. Therefore, it is possible that some (though definitely not all) of the observed ultracompact HII regions might already be in a non-expanding, pressure equilibrium configuration.

This of course will not be true if the wind from the central source has an important effect in pushing out the HII region. The importance of this effect can be evaluated as follows. Raga et al. (2012b) derived a model for a wind-driven HII region expanding into a homogeneous medium. Their analytic model depends on the dimensionless parameter

$$\kappa = 63 \left(\frac{\dot{M}}{5 \times 10^{-7} \mathrm{M}_\odot \mathrm{yr}^{-1}} \right) \left(\frac{v_w}{2500\,\mathrm{km\,s}^{-1}} \right)^2 \times \left(\frac{n_a}{10^5\,\mathrm{cm}^{-3}} \right)^{1/3} \left(\frac{10^{49}\,\mathrm{s}^{-1}}{S_*} \right)^{2/3} \times \left(\frac{1\,\mathrm{km\,s}^{-1}}{c_0} \right)^{1/3} , \tag{26}$$

where $\dot{M}$ is the mass loss rate and v_w the terminal velocity of the wind from the central source (normalized to the values corresponding to an O7 central star). This is equation (29) of Raga et al. 2012b), obtained by setting a temperature of 10^4K for the ionized gas (i.e., an isothermal sound speed of 10 km s^{-1}) and correcting two typos.

Raga et al. (2012b) show that for $\kappa > 1$, the HII region continues expanding in a substantial way due to the outward momentum deposited by the stellar wind. Therefore, a pressure equilibrium model is an appropriate description only for HII regions with $\kappa \leq 1$. From equation (26) we see right away that for an HII region excited by an O7 star, the effect of the stellar wind will be important, and the HII region will never reach a hydrostatic configuration.

Our present pressure equilibrium models are applicable for evolved objects in which t_f is smaller than the main sequence lifetime of the central star (see equation 25) and, at the same time, have $\kappa \leq 1$ (see equation 26). If one looks at the properties of O/B stars (see, e.g., Sternberg et al. 2003), one sees that in order to simultaneously satisfy these two conditions, it is necessary to have HII regions with type B stellar sources (which have much weaker winds, see Babel 1996, Kudritski & Puls 2000 and Raga et al. 2012b).

Another source of departures from a pressure equilibrium configuration is the possible motion of the central star with respect to the surrounding environment. Such a motion can result in complex morphologies for the photoionized region (see, e.g., Zhu & Zhu 2015 and references therein). Therefore, a pressure equilibrium model might be applicable for some specific objects, but definitely not to all compact or ultracompact HII regions!

It is also possible that pressure equilibrium models might be applicable for giant HII regions, photoionized by a large number of O stars. However, the final, hydrostatric configuration of this kind of object is likely to have a pressure stratification resulting from gravitational and radiation pressure forces (see Raga et al. 2015).

5. SUMMARY

We have studied the problem of an HII region in pressure equilibrium with the surrounding, neutral environment. This problem is relevant for compact or ultracompact HII regions, which rapidly evolve to a pressure equilibrium configuration (see de Pree et al. 1995; Raga et al. 2012).

We show that for a linear "temperature law" (relating the square of the isothermal sound speed with the H ionization fraction, see equation 8) it is possible to derive analytic solutions for the "grey ISM radiative transfer" model. For the case of a temperature-independent recombination coefficient, a full analytic solution is obtained. This solution has the same radial dependence of $f = e^{-\tau}$ as the "inner solution" obtained by Strömgren (1939) for a homogeneous HII region (though it has a different radial dependence for the H ionization fraction x, see equations 15 and 16).

We have also studied the case of a recombination coefficient with a temperature dependence of the form $\alpha_H \propto c^{-1}$ (which approximately follows the temperature dependence of the "case B" recombination coefficient, see, e.g., Cantó et al. 1998). In this case, we have been able to obtain an approximate analytic solution, which closely follows the "exact" results (obtained from a numerical integration of the radiative transfer equation). It is clear that it is also possible to obtain approximate analytic solutions for other power-law interpolations of the recombination coefficient.

From the ionization fractions as a function of radius, it is also possible to obtain (through equation 10) the density structure of the nebulae. These structures obviously have a peak at the outer radii of the ionized regions.

Therefore, we have found exact and approximate analytic solutions for the problem of a pressure equilibrium HII region. This is a quite satisfying result in itself. Also, these analytic solutions could be useful for initializing more detailed numerical models of static (e.g., Morisset et al. 2002) or dynamical (e.g., Tremblin et al. 2012) photoionized regions.

We note that our solutions strictly correspond to a region in equilibrium with a surrounding environment of finite pressure, but zero sound speed. In practice, the sound speed of the neutral gas has a small but finite value. The effect of this finite sound speed could be easily incorporated in the analytic models by cutting off the growth in the nebular density (see equation 10) at a maximum density. The effect of this is to introduce the presence of an exponential tail in the ionization fraction vs. radius relation at low values of x (see Raga 2015). We have not discussed this effect in the present paper because we feel that it does not add relevant insights.

To conclude, we point out that the direct applicability of the pressure equilibrium models described in this paper is quite limited (see § 4). The procedures discussed in our paper could be included in more complex HII region models, describing a broader range of the configurations found in the observed nebulae.

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