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ON THE DYNAMICS OF THE UNIVERSE IN D SPATIAL DIMENSIONS

R. F. L. Holanda[1] and S. H. Pereira,[2]

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RESUMEN

En este trabajo se presentan las ecuaciones de la evolución del universo en D dimensiones espaciales, como una generalización de la obra de Lima (2001). También, se discuten las ecuaciones de Friedmann-Robertson-Walker en D dimensiones espaciales para un fluido simple con ecuación de estado $p = \omega_D \rho$. Después de realizar un cambio apropiado de variables, es posible reducir las ecuaciones multidimensionales a la ecuación de una partícula, la cual está sujeta a una fuerza lineal. Esta fuerza puede ser expresada como una ecuación de oscilador, anti-oscilador o una partícula libre. Un resultado interesante es que, en el caso de de Sitter, la evolución es independiente de la dimensión D. También, para el caso plano en D dimensiones, presentamos un caso general con factor Λ. Un resultado interesante es que la reducción de la dimensionalidad implica naturalmente una expansión acelerada.

ABSTRACT

In this paper we present the equations of the evolution of the universe in D spatial dimensions, as a generalization of the work of Lima (2001). We discuss the Friedmann-Robertson-Walker cosmological equations in D spatial dimensions for a simple fluid with equation of state $p = \omega_D \rho$. It is possible to reduce the multidimensional equations to the equation of a point particle system subject to a linear force. This force can be expressed as an oscillator equation, anti-oscillator or a free particle equation, depending on the k parameter of the spatial curvature. An interesting result is the independence on the dimension D in a de Sitter evolution. We also stress the generality of this procedure with a cosmological Λ term. A more interesting result is that the reduction of the dimensionality leads naturally to an accelerated expansion of the scale factor in the plane case.

Key Words: cosmology: theory

1. INTRODUCTION

The cosmological solutions for a relativistic simple fluid in the framework of Friedmann-Robertson-Walker (FRW) models were discussed long ago by Assad & Lima (1988; Lima 2001). In the referred paper, the equation of cosmological dynamics driving the evolution of the scale factor for a simple perfect fluid obeying the equation of state $p = \omega\rho$, was reduced to the one of a point particle subject to a linear force, where p, ρ and ω are the pressure, energy density and equation-of-state parameter describing the cosmic fluid, respectively. It has been demonstrated that the possible non-linear dynamic evolutions predicted by the FRW equations were naturally recovered in such reduction. In particular, closed models behave exactly as simple harmonic oscillators. The full discussion presented by Lima (2001) was restricted to 3+1 space-time dimensions. Nevertheless, after the studies of Ehrenfest (1917), who solved the Kepler problem in arbitrary dimensions, and Kaluza and Klein, in the 1920s, the interest for theories in D spatial dimensions has grown considerably (Duff & Nilsson 1986). Hayashi, Katsuura, & Mendoza (1990) studied the influence of the dimension on physical laws. More recently, cosmological models in higher dimensional space-time have been studied, called D-brane cosmology (Panotopoulos 2005; Gusin 2008; Chingangbam & Deshamukhya 2009; Panigrahi 2004; Polchinski 1995; Polchinski, Chaudhuri, & Johnson 1996). In this paper, we have explored the consequences of the method adopted by

[1]Universidade Estadual da Paraíba, Brazil.
[2]Universidade Federal de Itajubá, Brazil.

Lima for the study of a $D+1$ dimensional FRW cosmology, by studying the scale factor evolution as a function of D for different material contents, e.g., matter, radiation and vacuum dominated universes. We study the generalization when a cosmological Λ-term is also present and we conclude by showing that the reduction of the dimensionality leads to an accelerated phase of expansion in the case of a plane ($k=0$) universe.

2. FRW COSMOLOGIES IN D SPATIAL DIMENSIONS

The space-time metric for multidimensional FRW cosmologies in D spatial dimensions is expressed as follows (Tangherline 1986), (we use $c=1$):

$$ds^2 = dt^2 - a^2(t)\left(1+\frac{kr^2}{4}\right)^{-1}\delta_{ij}dx^i dx^j\,, \qquad (1)$$
$$(i,j = 1,2,3,...,D)$$

where $a(t)$ is the scale factor, k is the curvature parameter of the spatial sections and $r^2 = \sum_i (x_i)^2$. The above expression reduces to the standard form in the 3-dimensional case (Landau & Lifshitz 1989).

In the D-dimensional geometry (equation 1), Einstein's field equations for a relativistic simple fluid and the energy conservation law can be written as:

$$\frac{D(D-1)}{2}\left[\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right] = 8\pi G_D \rho\,, \qquad (2)$$

$$\frac{(D-1)\ddot{a}}{a} + \frac{(D-1)(D-2)}{2}\left[\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right] = -8\pi G_D p\,, \qquad (3)$$

$$\dot{\rho} + D(\rho+p)\frac{\dot{a}}{a} = 0\,, \qquad (4)$$

where G_D, ρ and p are the D-dimensional Newtonian constant, the energy density and pressure of fluid, respectively.

Following standard lines (Turner & White 1997), it will be assumed that the matter content obeys the general equation of state:

$$p = \omega_D \rho\,, \qquad (5)$$

where ω_D is the equation-of-state parameter in D spatial dimensions. For black-body radiation $\omega_D = 1/D$, for matter $\omega_D = 0$ and for vacuum $\omega_D = -1$. An interesting discussion on this equation regarding the adiabatic index $\gamma_D \equiv \omega_D + 1$ can be found in the book by Zel'dovich & Novikov (1996). For simplicity of notation, we will take $\omega_D \equiv \omega$.

By inserting the above expression into equation (4) and integrating it, one may find that the energy density reads:

$$\rho = \rho_0 (\frac{a_0}{a})^{D(\omega+1)}\,, \qquad (6)$$

where the cosmic scale factor, $a(t)$, must be determined from the FRW differential equation (see below). By combining equations (2), (3) and (5), it can be seen that the evolution of the scale factor is driven by the second order differential equation (which correctly reproduces the 3-dimensional case (Assad & Lima 1988; Faraoni 1999; Lima, Moreira, & Santos 1998):

$$a\ddot{a} + \Delta_D \dot{a}^2 + \Delta_D k = 0\,, \qquad (7)$$

where the definition

$$\Delta_D = \frac{D(\omega+1)}{2} - 1 \qquad (8)$$

has been introduced. It is interesting to note that equation (7) does not depend on the Newtonian constant G_D. Thus, we do not need to know its value in order to obtain the evolution.

In principle, the corresponding dynamic behavior must be heavily dependent on the choice of the following three free parameters: (i) the curvature parameter k, (ii) the equation of state parameter ω, and (iii) the spatial dimension D.

Now, let us discuss how the method of solution proposed by Lima (2001) in the 3-dimensional case can be extended for D spatial dimensions. This can be accomplished by using the conformal time η, instead of the cosmological or physical time, $dt = a(\eta)d\eta$. In this case, the equation of motion (equation 7) is expressed as shown below:

$$aa'' + (\Delta_D - 1)a'^2 + \Delta_D k a^2 = 0\,, \qquad (9)$$

where a prime denotes differentiation with respect to conformal time η.

We now employ the auxiliary factor

$$Z(\eta) = \ln a \quad \text{if} \quad \Delta_D = 0\,, \qquad (10)$$

$$Z(\eta) = a^{\Delta_D} \quad \text{if} \quad \Delta_D \neq 0\,, \qquad (11)$$

to obtain, respectively,

$$Z'' = 0 \quad \text{if} \quad \Delta_D = 0\,, \qquad (12)$$

$$Z'' + k\Delta_D^2 Z = 0 \quad \text{if} \quad \Delta_D \neq 0\,. \qquad (13)$$

As expected, although considering that we are treating FRW cosmologies in D spatial dimensions, the equations (12) and (13) are reduced to those found by Lima (2001) for the 3-dimensional case. Note also that equation (13) describes the classical motion of a particle subject to a linear force. This force can be restoring or repulsive depending only on the sign of the curvature parameter. The general solution of equations (12) and (13) can be written as:

$$Z = b_0\eta + c_0 \quad \text{if} \quad \Delta_D = 0\,, \tag{14}$$

$$Z = \frac{z_0}{\sqrt{k}}\sin[\sqrt{k}\Delta_D(\eta+\delta)] \quad \text{if} \quad \Delta_D \neq 0\,, \tag{15}$$

with b_0, c_0, z_0 and δ integration constants. By choosing $\delta = 0$ and determining $z_0 = a_0^{\Delta_D}$, the scale factor evolution can be obtained as the solutions of equations (10) and (11):

$$a(\eta) = a_0 e^{b_0\eta} \quad \text{if} \quad \Delta_D = 0\,, \tag{16}$$

$$a(\eta) = a_0\left(\frac{\sin[\sqrt{k}\Delta_D\eta]}{\sqrt{k}}\right)^{\frac{1}{\Delta_D}} \quad \text{if} \quad \Delta_D \neq 0\,. \tag{17}$$

The range in conformal time η in flat ($k = 0$) and open ($k = -1$) universes is semi-infinite, $+\infty > \eta > 0$, regardless of whether the universe is dominated by radiation ($\omega = 1/D$) or matter ($\omega = 0$). For a closed universe ($k = 1$), η is bounded to $\pi > \eta > 0$ for radiation and to $2\pi > \eta > 0$ for matter dominated universes (Mukhanov 2005).

For a flat space-time ($k = 0$) the system behaves like a free particle and the same happens if $\Delta_D = 0$. However, in the latter case, this free particle behavior holds regardless of the curvature parameter. For $\Delta_D = 0$ equation (16) can be inverted by using $dt = a(\eta)d\eta$, and apart from integration constants, we obtain:

$$a(t) = b_0 t\,. \tag{18}$$

For $\Delta_D \neq 0$, in the limit $k \to 0$, we have for equation (17):

$$a(\eta) = a_0(\Delta_D\eta)^{\frac{1}{\Delta_D}}\,. \tag{19}$$

For $\Delta_D < 0$ the range of η is $-\infty < \eta < 0$. In the specific case of flat universe the parametric solution can also be inverted to give the scale factor as a function of the cosmological time. Apart from an integration constant, we have:

$$a(t) = d_0\big(1+\Delta_D\big)^{\frac{1}{1+\Delta_D}}\ t^{\frac{1}{1+\Delta_D}}\,. \tag{20}$$

It is easy to see that this expression reduces to equation (18) in the limit $\Delta_D = 0$ and identifying $b_0 = d_0$. We can see that, in the case $D = 3$, for $\omega = 0$ and $\omega = 1/3$ we have the correct dependence $a \propto t^{2/3}$ and $a \propto t^{1/2}$ for matter- and radiation-dominated universes respectively.

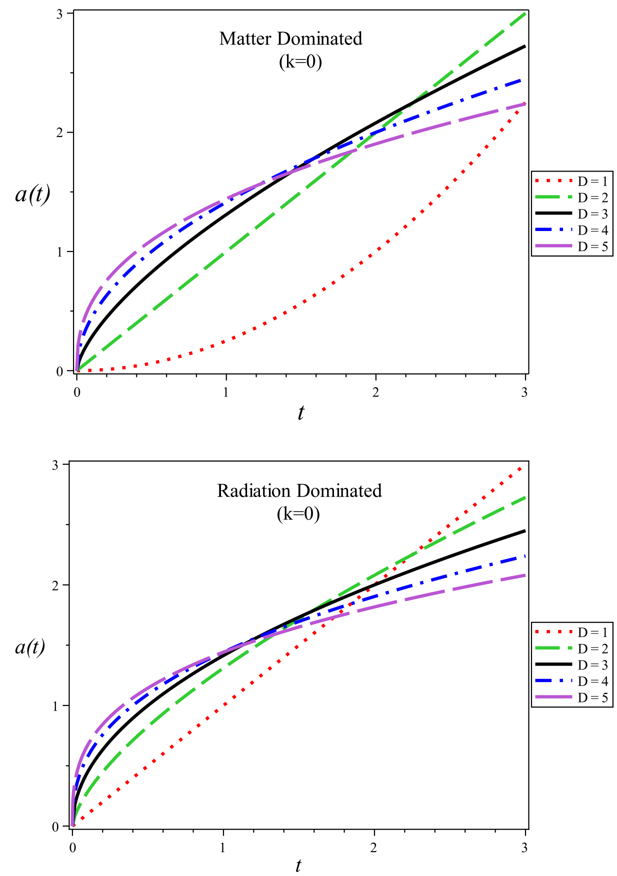

Fig. 1. Scale factor evolution for different values of D in the case of flat universe ($k = 0$), for matter (top) and radiation (bottom) dominated universes. The horizontal time scale is arbitrary. The color figure can be viewed online.

In Figure 1 we represent the scale factor evolution for different values of D in the case of flat universe ($k = 0$), for matter and radiation dominated universes. The expansion rate grows as the dimensionality increases. In the future, however, the opposite behavior is observed, with a reduction of the scale factor as D increases.

To finish the study of the flat case, let us see how the age of the universe depends on the dimensionality. Deriving equation (20) with respect to time and taking $\dot{a}/a$, we obtain for the present time:

$$t_0 = \frac{2}{D(\omega+1)}H_0^{-1}\,, \tag{21}$$

where H_0 is the current Hubble parameter. This shows that the age of the universe decreases as the dimensionality increases. Interestingly, if $D = 3$ and $\omega = 0$ (matter case) we recover $t_0 = 2/3H_0^{-1}$, and for $D = 3$ and $\omega = 1/3$ (radiation case), we have $t_0 = 1/2H_0^{-1}$.

Closed models ($k = 1$) are, for any value of $\Delta_D \neq 0$, analogous to simple harmonic oscillators. The cosmic dynamics in this case is similar to a spring-mass system where the spring constant is determined by the ω-parameter and the number D of spatial dimensions. The solution (17) becomes:

$$a(\eta) = a_0\big(\sin[\Delta_D \eta]\big)^{\frac{1}{\Delta_D}} \,. \qquad (22)$$

Unlike the previous case, it is not possible to write directly the scale factor as function of the physical time t.

For open space-times ($k = -1$), the system behaves as a particle subject to a repulsive force proportional to the distance, or an anti-oscillator. The solution (17) becomes

$$a(\eta) = a_0\big(\sinh[\Delta_D \eta]\big)^{\frac{1}{\Delta_D}} \,. \qquad (23)$$

In both cases, namelly $k = 1$ and $k = -1$, numerical results show that the evolution is always decelerating.

3. VACUUM ($\omega = -1$) DOMINATED UNIVERSE

The case of a vacuum dominated universe ($\omega = -1$) is an interesting one. Note that the parameter Δ_D in equation (8) is independent of dimension in this case, $\Delta_D \equiv -1$. Consequently, the solutions (19), (22) and (23) are all independent of D. For the flat case for instance, we have:

$$a(\eta) = -\frac{a_0}{\eta} \,, \quad -\infty < \eta < 0 \,, \qquad (24)$$

or

$$a(t) = a_0 \mathrm{e}^{\alpha t} \,, \qquad (25)$$

with α a constant, which represents a de Sitter evolution of the scale factor.

4. EVOLUTION WITH A COSMOLOGICAL Λ TERM

As it happens in the 3-dimensional case, we stress that the above method based on the transforming equations (10) and (11) is also convenient when new ingredients are considered, such as the presence of a cosmological Λ term. In this case, the second-order differential equation (7) is as follows:

$$a\ddot{a} + \Delta_D \dot{a}^2 + \Delta_D k = \frac{\Lambda(\omega+1)a^2}{D-1} \,. \qquad (26)$$

One may show that the generalized equation of motion for models with $\Lambda \neq 0$ in terms of the auxiliary scale factor is given by:

$$Z'' + \Delta_D^2 k Z = \frac{2\Delta_D(\Delta_D+1)\Lambda Z^{\frac{\Delta_D+2}{\Delta_D}}}{D(D-1)} \,. \qquad (27)$$

The equation of motion (27) means that closed universes with cosmological constant evolve like anharmonic or non-linear oscillators. The anharmonic contribution to the oscillator is proportional to the cosmological Λ-term and inversely proportional to the number of space dimensions D. Its power index depends uniquely on the equation-of-state ω-parameter. As expected, when $D \to 3$ the corresponding tridimensional result is recovered (see equation 27 in Lima 2001).

In the particular case of a plane universe ($k = 0$), an exact solution of equation (26) can be obtained for arbitrary dimension D and equation of state parameter ω. The solution is

$$a(t) = \left(\frac{D(D-1)}{8\Lambda}\right)^{\frac{1}{D(\omega+1)}} \times \frac{\left[\exp\left((\omega+1)\sqrt{\frac{2\Lambda D}{D-1}}t\right) - 1\right]^{\frac{2}{D(\omega+1)}}}{\exp\sqrt{\frac{2\Lambda}{D(D-1)}}t} \,, \qquad (28)$$

where we have choosed the initial condition $a(0) = 0$. In the case $D = 3$ and $\Lambda \to 0$, this expression has the correct dependence $a \propto t^{2/3}$ and $a \propto t^{1/2}$ for $\omega = 0$ and $\omega = 1/3$ for matter and radiation-dominated universes, respectively. A graphycal analysis for different values of D shows that the expansion starts decelerated but is always accelerated in the future for matter and radiation-dominated universe, as occurs in the tridimensional case. As far as we know, the expression (28) is presented here for the first time.

5. ACCELERATION DRIVEN BY THE REDUCTION OF THE DIMENSIONALITY

Another very interesting feature that we can also observe from Figure 1 for the plane case is that the expansion of the scale factor is decelerated for large values of D, but clearly for some value of the dimension the expansion becomes accelerated, as indicated by the case $D = 1$ in the matter dominated universe. In fact, in the case of matter ($\omega = 0$), the transition from decelerated to accelerated expansion occurs in $D = 2$, indicated by the linear expansion in the figure. Thus, for $D > 2$ the expansion is decelerated, but for $D < 2$ the expansion becomes accelerated. The same behavior occurs in the case of radiation dominated universe ($\omega = 1/3$), where the transition occurs for $D = 1$.

We realize that the reduction of the dimensionality can lead naturally to an accelerated expansion of the scale factor. Looking more closely to equation (20) we see that the transition from decelerated to accelerated phase occurs when the exponent of time becomes greater than 1. Thus for $1/(1+\Delta_D) < 1$ the expansion is decelerated and for $1/(1+\Delta_D) > 1$ the expansion is accelerated. The case $1/(1+\Delta_D) = 1$ represents the transition and can be written in terms of the equation of state parameter ω as $D = 2/(\omega+1)$. For $\omega = 0$ (matter), we obtain $D = 2$, and for $\omega = 1/D$ (radiation) we have $D = 1$, as already noticed before. An interesting consequence that follows is that for $D = 3$ the value of the equation of state parameter for which the transition from a decelerated to an accelerated stage occurs is $\omega = -1/3$. Such an equation of state parameter characterizes the so-called dark energy regime $\omega \leq -1/3$ (Lima 2004). The range $\omega < -1$ represents the phantom regime (Pereira & Lima 2008), and it has been first suggested based on supernova analyses alone which favor $\omega < -1$ rather than a cosmological constant or quintessence (Corasaniti et al. 2004); a more precise observational data analysis allows the equation of state parameter ω in the interval $[-1.38,\ -0,82]$ at 95% confidence level (Melchiorri 2003).

6. CONCLUSION

In this paper, we have studied the influence of the spatial dimension D on the solutions of the FRW equations, as a generalization of the work of Lima (2001). We have shown that for both flat ($k = 0$) and open ($k = -1$) universes, the increase in the dimensionality leads to a growth of the scale factor at the beginning of the evolution, but in the future the opposite behavior is observed, with a reduction of the scale factor as D increases. This occurs for both matter and radiation dominated universe. For a closed universe ($k = 1$) the behavior is that of a simple harmonic oscillator, with the collapse point shifted to small values of t as D increases. A generalized expression for the age of the universe in D-dimensional spaces for a flat universe ($k = 0$) was obtained, and we have shown that the increase in dimensionality implies a smaller value for the age of the universe to reach the actual size of the scale factor. This occurs for both matter and radiation dominated universes.

Another interesting conclusion is that the evolution for a vacuum dominated universe (de Sitter) does not depends on the spatial dimension D. The exact and general solution that describes a universe in the presence of a cosmological Λ-term for arbitrary D and ω is presented for the plane case ($k = 0$), which correctly reproduces the corresponding tridimensional result for the matter and radiation cases.

A very interesting result is that the reduction of dimensionality leads naturally to an accelerated expansion. The dependence of Δ_D with the equation of state parameter ω shows that the transition from a decelerated to an accelerated regime occurs for $\omega < -1/3$, which characterizes a dark energy fluid. Such a result was already known from other studies, but here it has been obtained based only on the dimensional analysis. This is a very important result, given the increasing number of studies in recent years involving the dynamics of the universe in extra dimensions.

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R. F. L. Holanda: Universidade Estadual da Paraíba, Departamento de Física, Rua Baraúnas, 351-58429-500, Bairro Universitário, Campina Grande, PB, Brazil (holanda@on.br).
S. H. Pereira: Universidade Federal de Itajubá, Campus Itabira, Rua Irmã Ivone Drumond, 200-35903-087, Distrito Industrial II, Itabira, MG, Brazil (shpereira@gmail.com).