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EXPLANATORY PROOFS AND GROUNDING IN MATHEMATICS: A REPLY TO MAAREFI

MARC LANGE
Rutgers University
Department of Philosophy
New Brunswick, New Jersey
USA
marc.lange@rutgers.edu
https://orcid.org/0000-0002-0704-7376

SUMMARY: Do explanatory proofs of theorems in mathematics derive their explanatory power by virtue of providing information about the grounds of the theorems they explain? I have argued that they do not (Lange 2019), and I have offered my own account of what makes certain mathematical proofs but not others able to explain why some theorem holds (Lange 2017). Recently, Maarefi (2025) has critiqued both these arguments and that account in support of a grounding-based conception of explanatory proofs. Here I respond to some of his critiques.

KEYWORDS: explanation in mathematics, ground, mathematical coincidence, mathematical practice, proof

RESUMEN: ¿Las demostraciones explicativas de teoremas en matemáticas derivan su poder explicativo en virtud de que proporcionan información sobre los fundamentos de los teoremas que explican? He argumentado que no (Lange 2019), y ofrezco mi propia explicación sobre lo que hace que algunas demostraciones matemáticas, y otras no, sean capaces de explicar por qué un teorema es válido (Lange 2017). Recientemente, Maarefi (2025) ha criticado tanto estos argumentos como esa explicación en apoyo de una concepción de las demostraciones explicativas basada en fundamentos. A continuación respondo a algunas de sus críticas.

PALABRAS CLAVE: explicación en matemáticas, fundamento, coincidencia matemática, práctica matemática, demostración

I am grateful to Maarefi for the attention that his paper (Maarefi 2025) devotes to my work—especially to Lange 2019. A few of his arguments merit brief responses.

Regarding the grounds of mathematical facts, I agree with Maarefi (p. 36) that "[g]rounding [. . .] involv[es] more than just logical entailment but also the substantive connections underpinning mathematical truths". However, I believe that fortunately, we do not need to settle difficult questions about those "substantive connections" in order to argue cogently against the view that explanatory proofs in mathematics acquire their explanatory power by virtue of the information that they supply about the explanandum's grounds. Rather, to make the arguments against that view in Lange 2019, I needed to appeal only to minimal, widely accepted presumptions regarding grounding—principally to what Maarefi (p. 34) terms "AGT", which he associates with the following passage:

> I am making the rough presupposition that a mathematical fact is grounded by the atomic (or negated atomic) truths to which one is led if one starts with that fact and moves 'downward' to logically simpler truths in an obvious way, such as from universal facts (i.e., facts expressed by generalizations) to their instances and from conjunctions to their conjuncts. (Lange 2019, p. 2)

I chose my mathematical examples and constructed my arguments in Lange 2019 with the aim of needing only something minimal like "AGT" to underwrite them, since such presuppositions are relatively uncontroversial among grounding theorists (despite their many disagreements about grounding).

I was therefore surprised that Maarefi (p. 34) described "AGT" as "the most controversial claim Lange makes about the grounding relation". Now perhaps this remark is true, strictly speaking, because my arguments in Lange 2019 appeal to very few claims about the grounding relation! In any event, "AGT" seems uncontroversial. Correia and Schneider (2012, p. 17) write, "The guiding idea of the existing approaches [. . .] is that truth-functional compounds are connected to their component parts by grounding ties. Thus, a true disjunction is grounded in its true disjunct(s), a true conjunction in its conjuncts, etc." Correia and Schneider (2012, p. 7) cite Bolzano (to whom Maarefi (pp. 27, 41) defers regarding ground) as having endorsed the same view. Likewise, Fine (2012, p. 63) remarks, "We naturally want to say that any grounds for A&B should be mediated through A and B". These ideas about grounding seem sufficient to motivate my claim (Lange 2019, p. 2) about the grounds of the fact that there are exactly 50,000 instances of the digit "7" in the list of whole numbers from 1 to 99,999 as represented in base 10. This fact serves as the explanandum in one of my examples (Lange 2019, pp. 1–3). The total number of 7's in the list is grounded in the conjunction of the various contributions to the total (that is, in the conjunctive fact that there is no 7 in the first number on the list, none in the second number, none . . . none in the sixth number, one

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in the seventh number, . . . ,and none in the 99,999$^{\text{th}}$ number), while this conjunctive fact, in turn, is grounded in its various conjuncts.[1]

This appeal to AGT to support my claim about the explanandum's grounds does not presuppose that AGT captures *everything* that there is to grounding mathematical facts. AGT pertains only to facts expressed in a natural way by truth-functional compounds, generalizations over finitely many cases, and the like. The grounds of other kinds of mathematical fact may not be logically simpler than what they ground; it may be that neither a given ground nor what it (perhaps together with some other facts) grounds is naturally expressed as a conjunction, a disjunction, a generalization, or the like. Something more "substantive" must then determine what grounds what. That is why I can agree with Maarefi that grounding involves "substantive connections". But it is not the case that in making my claims about the grounds of the fact that there are 50,000 7's in the list, my "approach contradicts Lange's own earlier statement on the nature of the grounding relation" (Maarefi 2025, p. 36), namely, that a fact's ground is "whatever it is in virtue of which that fact obtains" (Lange 2019, p. 2). My strategy in Lange 2019 was to strengthen my arguments by making use solely of minimal, relatively uncontroversial presuppositions about grounding and by selecting my mathematical examples (such as an explanation of the fact concerning the number of 7's in the list) so that such minimal presuppositions largely suffice to resolve what grounds what.

It seems to me widely accepted that the fact that exactly five of the marbles in the urn are black is grounded in the fact that this marble in the urn is black, that one is not black (or, specifically, is white), that one is black, [. . .], and that those are all of the marbles in the urn. Frankly, I am more confident of this claim about how the urn fact is grounded (and that it is widely accepted) than I am of any general thesis about grounding that purportedly covers this example, such as "AGT". But this claim about the ground of the fact that exactly five of the marbles in the urn are black seems closely analogous to my claim about the ground of the fact that the total number of 7's on the list is 50,000. So the proof of this fact that locates and tallies all of the 7's on the list, one by one, should be explanatory (it seems to me) if an explanation in mathematics derives

[1] I do not mean to suggest that AGT is accepted by every grounding-enthusiast. For instance, Bhogal (2017) proposes that a generalization that states a law of nature is not grounded in its instances. But Bhogal emphasizes that this is an innovation on his part and that even on his view, natural laws are the only universal generalizations that are not grounded in their instances.

its explanatory power by virtue of supplying information about the explanandum's grounds.

In Lange 2019, I gave two proofs of the fact that the total number of 7's on the list is 50,000: a proof that gives the explanandum's ground by tallying every appearance of a 7 in the list and a proof that operates by symmetry without locating and tallying each (or, in fact, any) of the 7's in the list. Maarefi (p. 32) insists that "it is unclear why [Lange] believes the [proof that tallies each appearance of a 7] does not explain" why there are 50,000 7's. Later Maarefi (p. 33) repeats that "it is not clear that [proofs that locate and tally the 7's] offer no explanation whatsoever". In support of his claim, Maarefi (p. 32) says that the tallying "proof still provides minimally informative grounds" of the explanandum. It seems to me that Maarefi's argument here—that the tallying proof's providing the explanandum's grounds is a reason to believe that the tallying proof explains (at least to some extent)—presupposes that explanatory proofs derive their explanatory power by virtue of providing information about the explanandum's grounds. But the point of my paper was to argue against this presupposition. So Maarefi's argument that the tallying proof is explanatory, since it provides information about the explanandum's grounds, is question-begging in this context.

In further support of his claim that the tallying proof is (at least somewhat) explanatory, Maarefi (p. 33) asks us to imagine what would be the case if no human being could grasp the (more complex) non-tallying proof, but human beings could grasp the (simpler) tallying proof.

> In such a scenario, the only grounds people could identify would be those in the [tallying] proof. Consequently, the [tallying] proof, with its minimal information, would be the sole explanatory proof available, as the [non-tallying] proof would be incomprehensible due to its more complex grounds.

But this argument cannot show that the tallying proof is explanatory (even to some degree). This argument again presupposes that explanatory proofs acquire their explanatory power by providing information about grounds—and so again begs the question. Perhaps Maarefi's argument is fundamentally that if human beings cannot grasp the non-tallying proof, then the tallying proof is the only remaining way (for them, at least) to prove the explanandum, and so the tallying proof must be explanatory (for human beings, at least). This argument presupposes that any mathematical fact with a proof must have an explanatory proof (indeed, one that is graspable by human beings). I (Lange 2009, p. 14) have argued against this presupposition both by citing mathematicians who have denied it in specific cases and by interpreting mathematical coincidences (and mathematical facts possessing no salient feature) as having no explanations despite having proofs (see Lange 2017).

Also in support of his claim that the tallying proof is (at least somewhat) explanatory, Maarefi (p. 32) correctly points out that on my view, explanatoriness is a matter of degree rather than all or nothing. But I do not see why its being a matter of degree suggests that the tallying proof is (at least somewhat) explanatory; even when some property is a matter of degree, there can be entities that lack that property entirely. In any case, it makes little difference to my main argument whether the tallying proof is "minimally" explanatory or not at all explanatory; an account of explanatory proofs as deriving their explanatory power by virtue of providing information about the explanandum's grounds will still have difficulty accounting for the tremendous explanatory superiority of the non-tallying proof over the tallying proof. (My (2017) account of what makes certain proofs more explanatory than others nicely explains this difference, as I will briefly describe.) After all, the non-tallying proof provides much less information about the explanandum's grounds than the tallying proof does.

What (on my view) does the non-tallying proof do that the tallying proof does not do, making the non-tallying proof so explanatorily superior? Firstly, the non-tallying proof adds to the list a number (0) that falls outside of the explanandum's scope (which is the numbers from 1 to 99,999), giving us exactly 100,000 numbers on the list. Secondly, the non-tallying proof adds some digits ("0"s) preceding each of the numbers having fewer than five digits in base 10 (e.g., turning 1 to 00001). Doing so makes them all five-digit numbers. Both of these additions fall outside of the explanandum. So the facts about what grounds the explanandum (whatever those facts are) do not account for the explanatory utility of making these exogenous additions.

In contrast, by using my proposal regarding what makes certain proofs (more) explanatory, we can account for the explanatory utility of these additions. The explanandum's salient feature (in the given context) is that the total number of 7's (50,000) equals $\frac{1}{2}$ of 100,000, where strikingly, 100,000 is only one greater than the number (99,999) of numbers in the explanandum's scope. The additions that appear so foreign from the grounding perspective enable the explanandum to be proved in a way that shows where the explanandum's salient features (the $\frac{1}{2}$ and the 100,000) come from. The explanatory proof traces the explanandum's salient $\frac{1}{2}$ back to a $\frac{1}{2}$ in the setup: 5 digits per number (now that we have added enough "0" digits to bring each of the listed numbers up to five digits) divided by 10 possibilities for each digit. The explanatory proof traces the explanandum's salient 100,000 back to a 100,000 in the setup (now that we have added 0 to the 99,999 listed numbers, making 100,000 numbers in total). Multiply 100,000 numbers by five digits per number, divided by 10 possibilities for each digit, yields 50,000 appearances of each possible digit (such as 7).

My claim that the non-tallying proof is much more explanatory than the tallying proof is expressed by the mathematicians Dreyfus and Eisenberg (1986, p. 3), from whom I borrowed this example. Maarefi cites no passages from the current mathematics literature corroborating his claims that various proofs are (or are not) explanatory, such as his claim (Maarefi 2025, pp. 37–40) that various proofs that a given set is infinite are explanatory and his claim (2025, p. 48) that a well-known proof of the Fundamental Theorem of Algebra by Liouville's theorem is not explanatory.

Furthermore, I have provided (Lange 2017) many examples that are similar to the 7's example. Each of these analogous examples involves a proof that gives unified, uniform treatment to all of the cases in the explanandum's scope (i.e., treating all of them together, rather than one-by-one, and in the same way), where that proof is deemed by mathematicians to be explanatory—whereas a proof of the same explanandum that treats each of its cases separately, failing to take advantage of some salient symmetry in the explanandum, is deemed by mathematicians to be not at all explanatory. (See, for instance, the "calculator number" example (Lange 2017, pp. 276, 280, 289). In the wake of a case-by-case proof of the calculator-number theorem, mathematicians asked whether the theorem is a coincidence; if that proof had been even somewhat explanatory, then presumably it would have shown the theorem not to be entirely coincidental.) These analogous examples supply additional reason to believe that the tallying proof is not at all explanatory.

Maarefi rightly remarks that even if explanatory proofs derive their explanatory power by virtue of providing information about the explanandum's grounds, an explanatory proof does not need to give all of an explanandum's grounds. (This seems to me analogous to the familiar idea that although causal explanations in science derive their

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explanatory power from providing information about the explanandum's causal history, a causal explanation does not need to give the explanandum's complete causal history back to the Big Bang!) Not all facts about the explanandum's grounds will be explanatorily relevant in a given context. This leaves room for context-sensitivity in whether various proofs are explanatory, even if there is no context-sensitivity in what grounds what. (Again, this seems to me like the familiar idea that there is context-sensitivity in which of the features of an event's causal history count as helping to causally explain it, even if there is no context-sensitivity in which past events were causes of the explanandum event. Hanson (1958, p. 54) famously asserted that a motorist's death might be explained "by a physician as 'multiple haemorrhage', by the barrister as 'negligence on the part of the driver', by a carriage-builder as 'a defect in the brakeblock construction', by a civic planner as 'the presence of tall shrubbery at that turning'.") Maarefi (pp. 52–53) proposes that accounts of explanatory mathematical proofs as ground-revealing can thereby capture the examples that I have given where a proof's explanatory power in math changes when the context changes. (As I am about to discuss, my account attributes these changes in explanatory power to changes in which of the explanandum's features are salient.)

However, Maarefi's proposal remains just a promissory note in the absence of any account of the ways in which context influences which facts about the explanandum's grounds are explanatorily relevant. Consider my example (Lange 2019, pp. 15–17) of explaining why a particular "mathemagical" trick succeeds for all pairs of initial real numbers on which the trick has been tried. Initially (since the trick's past success was in fact no fluke), an explanation would have to be a proof covering together and in the same way all possible pairs of initial real numbers (x and y) and showing that for each pair, the grand total is 11 times the number in row 7 of the table employed in the trick (Lange 2019, p. 15), ensuring the trick's success. But once such an explanation has been given, the context shifts so that an explanatory proof would then have to cover together and in the same way both the x-sum and the y-sum of the table's entries, unifying the two sums in the course of showing that for any possible (x,y), the x-coefficient in the grand total is 11 times the x-coefficient in row 7 and the y-coefficient in the grand total is 11 times the y-coefficient in row 7. The proof that was initially explanatory is no longer explanatory after the context shifts. I (Lange 2019, pp. 15–17) explained how a feature of the explanandum (a similarity between the x- and y-sums) that could not have been salient before the first proof was discovered is rendered salient by the first proof. (The mathematics textbook that I quote (Lange 2019, p. 16) highlights this shift.) By contrast, I do not see how the shift in what it takes for a proof to explain the trick's success is accounted for by thinking of this shift as a shift in which facts about the explanandum's grounds are explanatorily relevant. I do not see what a reference to grounding adds here.

Maarefi (p. 47) alleges that there is "a major problem with salience theory" (that is, with an account of explanatory proofs that appeals to the salience, in a given context, of some feature of the explanandum). The alleged problem is that any account invoking salience "fails to capture the objective aspect of mathematical explanation" (Maarefi 2025, p. 48). According to Maarefi (p. 48), that is because "a *bona fide* instance of explanation" (whether mathematical, scientific, or any other kind of explanation, I think Maarefi intends) "should provide a determination relation between the items within the explanandum and the items within the explanans. [. . .] However, salience theory fails to capture this significant aspect of mathematical explanations". But my account does incorporate an "objective aspect of mathematical explanation": in all of the examples discussed in my 2019, the explanation is a proof of the explanandum, and being a proof is an objective matter. Perhaps "a determination relation" (as Maarefi uses this term) must be some relation more metaphysically robust than deductive entailment, such as causation or grounding. If so, then my account admittedly does not incorporate such a relation. But then Maarefi begs the question against my account by presupposing that every explanation in mathematics must appeal to "a determination relation".

Of course, along with proof, salience also plays an important role in my account. Which of the explanandum's features (if any) is salient depends on the context. But as we have seen, Maarefi's own proposal is like mine in requiring that context influence a proof's explanatory power by somehow influencing which facts regarding an explanandum's grounds are explanatorily relevant. So Maarefi cannot object to the inclusion of something context-sensitive in an account of what makes certain proofs explanatory.

Furthermore, that salience is context-sensitive and therefore not "objective" does not mean that salience is entirely in the eye of the beholder. It is intersubjective; mathematicians generally agree (in a given context) on what features of the explanandum are striking, just as they generally agree on whether a proof is "brute force" and on whether a given mathematical fact would be a coincidence if its various cases cannot be proved together and in the same way.

In Lange 2019, I argued that mathematicians recognize many impure proofs as explanatory; recognize that they are explanatory not despite their impurity, but precisely because of the explanatory contributions made by whatever renders them impure; and recognize that an account of a proof's explanatory power as arising from its providing information about the explanandum's grounding has difficulty in accounting for the explanatory contributions sometimes made by impurity in a proof. Maarefi (p. 42) complains that "Lange's version of purity is not fully developed. Additionally, Lange proposes various versions of purity." However, my aim in Lange 2019 is not to endorse any particular specific account of what purity consists in. Rather, my aim in entertaining various precisifications of "purity" is to argue that my claim that impure proofs can be explanatory holds under various specific conceptions of purity—conceptions that differ in how broadly to construe the sense in which a concept that figures in a pure proof must be "intrinsic" to what is being proved. There is no such thing in Lange 2019 as "Lange's criterion for purity of proofs" (Maarefi 2025, p. 41) nor would the argument be strengthened by one.

Maarefi (p. 44) says that "the primary issue with [Lange's] argument is that it fails to show that pure proofs often do not explain". My aim is to argue that purity is not necessary for explanation and that the explanatory power of some impure proofs is difficult to account for on a grounding-based account of why some mathematical proofs but not others are explanatory. That many pure proofs do not explain (which Maarefi decries my failure to show) would seem to me to do little or nothing to support my claim that some impure proofs do explain (unless any theorem, even one with no pure proof that explains it, must have an explanatory proof, which is a claim that I reject). Furthermore, that many pure proofs do not explain should be common ground between my account of explanation and any grounding-based account. Frequently, there are many pure proofs of a given theorem, but only some select few of them explain it (to any significant degree). In Lange 2017, I give many examples of proofs that are pure and explanatory (as well as many examples of proofs that are pure and not explanatory). In arguing that some impure proofs of various theorems explain them, I do not deny the existence of pure proofs of various theorems that explain them. I do not argue for the claim that Maarefi (p. 41) attributes to me: that the "purity [of ground-revealing proofs] undermines their explanatory power". Not all pure proofs are brute-force proofs or otherwise incapable of explaining what they prove.

Maarefi (pp. 47–48) proposes a counterexample to my "salience theory". He cites a proof of the Fundamental Theorem of Algebra (FTA) that appeals to Liouville's theorem in complex analysis. He says that this proof counts by my lights as explanatory: it "addresses a salient feature of the polynomials" that are the FTA's subject "and traces it back to the premises". However, he says, this proof is not explanatory because "the theorem that plays a significant role in the proof" (Liouville's theorem) "functions as a 'black box', something we do not fully understand. [. . .] [S]alience theory fails to capture that the whole proof relies on an unexplained 'black box'."

It seems to me, first of all, that sometimes we can use one theorem to give an explanatory proof of another theorem even if we have no explanatory proof of the former theorem and, for that matter, even if it has no explanation. After all, we have long recognized that scientists can use one fact to explain another fact even if they have no explanation of the former fact and, for that matter, even if it has no explanation (Hempel 2001, p. 329). (Admittedly, however, scientific explanation is not always a good guide to explanation in mathematics.) So even if the FTA's proof by Liouville's theorem is not explanatory, I am dubious of Maarefi's account of why it is not explanatory.

Secondly, let's consider what Maaarefi takes to be "the salient feature of the polynomials" that the proof "traces [. . .] back to the premises". Maarefi (p. 47) identifies what he calls "a striking [i.e., salient] feature" of Liouville's theorem: that "notably", it identifies a respect in which "polynomial equations", despite their diversity in many respects, "show similar patterns of behavior". (Liouville's theorem says roughly that as x approaches infinity, every polynomial p(x)'s absolute value goes to infinity—a similarity in every polynomial's behavior.) Let's grant that a striking feature of Liouville's theorem is that it identifies a behavior common to all polynomials. Maarefi (p. 47) says that "the Fundamental Theorem of Algebra addresses the same issue, suggesting that such polynomials" (namely, those p(x) where x ranges over $\mathbb{C}$ and with coefficients in $\mathbb{C}$) "display a similar behavior regarding solvability in $\mathbb{C}$: they all have a solution in $\mathbb{C}$". Let's grant that, too. So Maarefi has argued that in this proof, the putative explanans (Liouville's theorem) has a striking feature that is similar to a feature of the explanandum (FTA). However, this reverses the condition that (according to my account) makes a proof explanatory. On my account, an explanatory proof traces a salient feature *of the explanandum* back to a similar feature of the *explanans*. So far, then, Maarefi has not shown that my account deems this proof of the FTA using Liouville's theorem to be explanatory.

Maarefi might reply by asserting that a salient feature of the explanans (FTA) is that it identifies something common to all polynomials p(x) with coefficients in $\mathbb{C}$ and x ranging over $\mathbb{C}$. I don't know whether this feature is indeed salient; whether it is salient depends on the context in which the FTA is being considered, and Maarefi identifies no particular context in which the FTA's explanation is being demanded. Furthermore, it is easy to imagine mathematical contexts where other features of the FTA are salient instead of its identifying something common to all polynomials p(x) with coefficients in $\mathbb{C}$ and x ranging over $\mathbb{C}$. Often what is salient about the FTA is that it identifies something special about $\mathbb{C}$. Whereas polynomials with coefficients in $\mathbb{Z}$ do not all have roots in $\mathbb{Z}$, polynomials with coefficients in $\mathbb{Q}$ do not all have roots in $\mathbb{Q}$, and polynomials with coefficients in $\mathbb{R}$ do not all have roots in $\mathbb{R}$, the FTA says that all polynomials with coefficients in $\mathbb{C}$ have roots in $\mathbb{C}$. Why is that—why are the complex numbers "algebraically closed", unlike the integers, the rationals, and the reals? What's so special about $\mathbb{C}$?

That is a familiar context in which the FTA is encountered. In this context, $\mathbb{C}$'s role in the FTA (rather than the feature that Maarefi emphasizes) is salient. Liouville's theorem has the same feature; it also privileges $\mathbb{C}$. (It applies only to functions over the entire complex plane; there is no analogous theorem for $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.) Perhaps in such a context, my account would deem the FTA's proof from Liouville's theorem to be explanatory—and perhaps that proof would genuinely be explanatory. (In other contexts, some other feature of the FTA might be salient, such as that it applies only to polynomials of finite degree, not to power series.) That is, perhaps $\mathbb{C}$ alone is algebraically closed because (roughly speaking) for $\mathbb{C}$ alone do all polynomials shoot off to infinity. That this proof appeals to Liouville's theorem without explaining why that theorem holds seems not to preclude this proof's possessing explanatory power.

As Maarefi recognizes, I appeal throughout (Lange 2019) to the existence of mathematical coincidences as supporting my account of explanatory proofs over a grounding-based account. A proof specifying some theorem's grounds may portray that theorem as coincidental when it actually is not. Maarefi (p. 49) offers the following summary of my account of what mathematical coincidences are: "According to Lange, mathematical coincidence refers to facts that are coincidentally true and lack a common explanation." Even as a rough sketch, this is a very unhelpful characterization. To use "coincidentally true" to cash out "mathematical coincidence" makes the account seem empty. Furthermore, a mathematical fact that "lacks a common explanation" may have no salient feature, in which case (by my lights) it does not qualify as a mathematical coincidence. (For instance, that 31 is prime and the 7^{th} digit of π is 2 has no salient feature and so is not a mathematical coincidence.) Roughly, on my account (Lange 2017, p. 277), a mathematical coincidence is a mathematical fact having as a salient feature that it identifies a natural mathematical property that is common to every case in its scope and where the mathematical fact has no explanation because it has no proof that exploits some other respect in which all of these cases are alike and that proves the fact by treating all of its cases together (rather than one-by-one) and in the same way. For instance, it is widely regarded as a mathematical coincidence that (in base 10) the 13^{th} digit of π is the same as the 13^{th} digit of e (Lange 2017, p. 42).

Responding to my arguments that attempt to use mathematical coincidences against a grounding-based account of explanatory proofs, Maarefi (p. 49) suggests that a mathematically coincidental fact has grounds and so (contrary to my view) has an explanation, but "do[es] not warrant explanation within everyday mathematical practice" because its explanations are not "fruitful". I (Lange 2019, p. 14) cited the mathematician Spivak as insisting that a mathematical fact (even one having as a salient feature that it reveals a natural property common to every case in the fact's scope) may have no explanatory proof at all, rather than merely no "fruitful" explanatory proof. But perhaps Maarefi would reply that what Spivak meant was that such a mathematical fact might have no fruitful explanatory proof, not that it might have no explanatory proof at all. It seems to me that a proof's "fruitfulness" (e.g., in the sense of providing a common proof of various known results, or in the sense of pioneering a novel strategy that can then be used to prove new theorems, or . . .) is one (or more than one!) of the many distinct mathematical virtues that some but not all proofs possess (alongside purity, brevity, visualizability, . . .) perhaps having little or no connection to explanatoriness (another such virtue).

Maarefi (p. 52) then pursues the matter of fruitfulness by arguing that "for the purposes of mathematical research, salient features of a mathematical fact may even be a wrong call." He describes a famous episode in the history of research into the distribution of prime numbers. It was long recognized that "the distribution of the prime numbers, prima facie, shows no pattern and thus lacks any striking feature except for its complete randomness" (Maarefi 2025, p. 51). Yet mathematicians later discovered the Prime Number Theorem, which specifies a function of N that is what the number of primes less than or equal to N approaches in the limit of large N. Maarefi concludes that although the distribution of primes "lacks any striking feature", there is an explanation, contrary to my account.

I agree with Maarefi that among the most fruitful questions about the distribution of primes have turned out to be questions about how many natural numbers less than or equal to N are prime rather than questions about whether this or that particular number is prime. But I do not see this discovery as revealing a counterexample to my account of explanation in mathematics—that is, as identifying a mathematical fact having no salient feature but having an explanation. As I mentioned, Maarefi emphasizes that the sequence of primes P and composites C (P,P,C,P,C,P,C,C,C,P, . . .) has no salient feature and no explanation. By contrast, the sequence (for the natural numbers N) of the total number of primes less than or equal to N has a salient feature, namely, that all of the sequence's sufficiently large entries are fairly well approximated by the same function of N—i.e., that there is, as Maarefi (p. 51) says, "a predictable pattern" here (which is why the Prime Number Theorem was conjectured in the first place). Maarefi apparently thinks that some proof of the Prime Number Theorem would explain why (and not merely prove that) a given function approximates the sequence's large entries. If he is correct, then this example would nevertheless not involve a mathematical fact having an explanation but lacking any salient feature.

The numbers 7, 37, 337, 3337, 33337, and 333337 are all prime—a striking similarity among them! But (mathematicians believe) this fact has no explanation (Gardner 1980, p. 18); it is a coincidence. (Roughly, there is no way to prove that they are all prime by treating them all together and alike; any proof addresses each of these numbers individually.) Since the search for such an explanation would be fruitless, searching for one would be "a wrong call", in Maarefi's (p. 52) terms. But there is no tension between the fact "that, for the purposes of mathematical research, salient features of a mathematical fact may even be a wrong call" (as in this example) and my claim that every mathematical fact having an explanation must also have a salient feature (because a mathematical fact's salient feature in a given context is responsible for the distinction, in that context, between explanatory and non-explanatory proofs of that fact). Our understanding of what is salient about the primality of 7, 37, 337, 3337, 33337, and 333337 is what informs our understanding of what a proof of this fact would have to be like in order to explain it. An explanatory proof of this fact could not prove the primality of these various numbers one-by-one. Rather, it would have to prove their primality together from the form (only in base 10!) that they all share. Even to deny that this fact has an explanation and to recognize that it is a coincidence, we have to recognize the difference between explaining it and merely proving it, which requires that we recognize this fact's salient feature.

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