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LANGE ON MATHEMATICAL EXPLANATION, GROUNDING, AND THE DIVERGENCE THESIS

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SUMMARY: Mark Lange has defended the view that mathematical explanation and grounding explanation diverge. In this paper, I argue that at least one type of grounding relation is immune to his criticisms. After fixing a minimal separatist theory of ground, I show that his arguments mostly rely on another thesis about grounding (I'll call it "atomic grounding thesis"), which is not supported. Furthermore, a second attempt at using purity is not entirely satisfactory.

KEYWORDS: mathematical practice, philosophy of mathematics, grounding relation, grounding explanation, separatism

RESUMEN: Mark Lange ha defendido la opinión de que la explicación matemática y la explicación basada en fundamentación divergen. En este artículo sostengo que al menos un tipo de relación de fundamentación es inmune a sus críticas. Tras introducir una teoría separatista mínima de la fundamentación, muestro que sus argumentos dependen en gran medida de otra tesis sobre la fundamentación (a la cual llamaré "tesis de fundamentación atómica"), que carece de respaldo. Además, un segundo intento utilizando la pureza no resulta completamente satisfactorio.

PALABRAS CLAVE: práctica matemática, filosofía de las matemáticas, relación de fundamentación, explicación basada en fundamentación, separatismo

1. *Introduction*

The question "What is mathematical explanation?" has received some attention in recent analytic philosophy of mathematics. While, according to Mancosu (2000), discussions about mathematical explanation date back to Aristotle, it remains a nascent subject within the analytic tradition of philosophy. Consequently, in this context, only a few theories have been proposed to date regarding this issue.[1]

[1] Examples include the characterizing property theory by Steiner (1978), Kitcher's unification theory developed in Hafner and Mancosu (2008) extracted from Kitcher (1989), and Lange's salience theory (2014). More recently, Poggiolesi (2023) and Poggiolesi and Genco (2023) have developed a theory of mathematical explanation in terms of conceptual grounding.

The subject began to garner wider attention as it turned out that some mathematical proofs are more explanatory than others. Some proofs merely verify that something holds, while others provide a better explanation—a better understanding—of the fact under study. However, this is not just an aesthetic aspect of mathematical proofs. To borrow Detlefsen's words, "a prime goal of proof is explanation" (2008, p. 17). With this in mind, we assume that mathematical explanation is an integral part of mathematical practice, especially when it comes to mathematical research. When encountering a problem, such as the distribution of prime numbers within the natural numbers, mathematicians begin to offer explanations of the mathematical fact under study. Mathematical explanation thus becomes a crucial aspect of mathematical practice, particularly in mathematical research.

On the other hand, philosophers have recently paid considerable attention to the topic of grounding. Metaphysical grounding is a non-causal form of determination that is naturally tied to the theory of dependence,[2] explanation,[3] and notions of modality.[4] Given the apparent ontic connection between the *explanandum* and *explanans*,[5] one may think that mathematical explanation is a form of grounding explanation. The primary focus of this research is to explore the nature of mathematical explanation and its relationship to metaphysical grounding. In his work, Marc Lange (2019) argues that mathematical explanation and grounding explanation are distinct, a position I refer to as the "Divergence Thesis".

Evaluating the Divergence Thesis ultimately hinges on our specific view of grounding and its connection with grounding explanation. This paper primarily aims to defend the viability of at least one version of the grounding relation that can withstand Lange's arguments. However, while providing a structured approach, Lange's view of grounding oversimplifies the grounding relation by placing undue emphasis on logical profiles. A more robust understanding of grounding, as developed by Litland[6] and Krämer,[7] aligns better with the everyday practice of mathematics. I advocate the view that mathematical explanation involves a metaphysical determination relation that traces the truth of the items in the *explanandum* (what

[2] See Schnieder 2020.
[3] See Glazier 2020.
[4] See Skiles 2020.
[5] See Pincock 2015, and D'Alessandro 2020.
[6] See Litland 2023.
[7] See Krämer 2018.

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is to be explained) back to the *explanans* (what explains). This is best captured by a particular theory of grounding (to be discussed shortly in subsection 3.2). This view distinguishes grounding explanation from the grounding relation, where the former is backed by the latter, resulting from a proper understanding of how the truth of the items in the grounded is determined by the truth of the items in the grounds. This view is further elaborated in a separate work.

Here is an outline of the paper. In section 2.2, I pinpoint a version of the grounding relation that I claim is immune to Lange's arguments. In section 3, I critically assess Lange's two main arguments for the Divergence Thesis. I will show that his first argument relies on an unjustified thesis, the Atomic Grounding Thesis, which I discuss in section 3. I will argue that the minimal assumptions made in section 2.2 (i.e., DeT_G, DuT, and TrT) provide arguments against this thesis. Moving on to Lange's second argument, in subsection 3.2, I argue that his appeal to the purity of ground-revealing proofs is also unsuccessful. I conclude that a single mathematical fact can have multiple ground-revealing proofs, with each proof providing a different explanation. In section 4, I criticize Lange's additional arguments in sections 3 and 4 of Lange 2019. In subsection 4.2, I show that opponents of the Divergence Thesis can also account for coincidences in mathematics. In subsection 4.3, I show that they can also account for the context-sensitivity of mathematical explanation.

2. *Preliminaries on the Divergence Thesis and Grounding Explanation*

A major challenge in assessing the relationship between mathematical explanation and grounding explanation stems from the lack of broadly accepted definitions for these two notions. As Marc Lange notes, "there is currently no widely accepted account of either mathematical explanation or grounding" (2019, p. 1). Given this ambiguity, it is essential to understand the meanings attributed to mathematical explanation and grounding relation, as well as various positions on their relationship, before delving into the discussion.

2.1. What is the Divergence Thesis?

Let us begin by exploring the notion of mathematical explanation. This type of explanation within the practice of mathematics is notably diverse.[8] For instance, not all explanations within mathematics

[8] For an introduction to the subject, see Mancosu 2011 and Mancosu 2001.

are derived from mathematical proofs.[9] However, for the purposes of this paper, our discussion will be limited to explanations offered by mathematical proofs. These mathematical explanations feature two principal components. The first component, which we will refer to as the *objective element* of mathematical explanation, pertains to a dependency relationship between the elements presented in the *explanandum* and those in the *explanans*.[10] To illustrate, in the explanations provided by a mathematical proof, the mathematical fact that is proved depends on the mathematical facts that serve as premises. The second component, termed the *subjective aspect* of mathematical explanation, underscores how explanatory proofs produce a better understanding of the facts being explained. This dual perspective provides a framework necessary for a *bona fide* theory of mathematical explanation.

The main focus of Lange's work (2019) is the relationship between mathematical explanation and grounding explanation. To use Lange's own words, "it is natural to wonder whether mathematical explanation is a variety of grounding explanation. This paper will offer several arguments that it is not" (2019, p. 1). I call this the "Divergence Thesis" (henceforth DT), which asserts that mathematical explanations are not a variety of grounding explanations. Let us pin it down here:

Mathematical explanation is not a variety of grounding explanation. (DT)

Acknowledged as a metaphysical thesis, one only needs one successful example to support (DT). In other words, one only requires a single example, including a ground-revealing proof that is not explanatory *per se*. Specifically, as (DT) is primarily a metaphysical thesis, the example in question should show that the ground-revealing proof of a mathematical fact does not provide any explanation. So, an explanation from the perspective of the everyday mathematician may appear unsatisfactory or superficial. This will not disqualify it from being considered an explanation.

For clarity, one should carefully distinguish (DT) and the separatism thesis about grounding (discussed in the next subsection). While both are metaphysical statements, they differ in that (DT) focuses on whether mathematical and grounding explanations are the

[9] Lange (2018) and D'Alessandro (2017) argue that not all of mathematical explanations are carried out by mathematical proofs.

[10] D'Alessandro (2020) lists some of the different conceptions of dependency between the *relata* of explanation.

same, whereas separatism about grounding states that the relation of grounding is different from the explanation that it backs. So, discussions about separatism and unionism about grounding relations are independent of the debates surrounding (DT). Therefore, while separatism broadly addresses the grounding relation independently of its connection to mathematical explanations, (DT) specifically examines the link between grounding and mathematical explanations without presupposing their equivalence. Thus, although separatism relates to a form of divergence, it does not directly correspond to the assertions made by (DT). Here is a diagram illustrating this point (ST denotes the separatism thesis):

$$\text{Grounding Relation} \xrightarrow{ST} \text{Grounding Explanation} \xrightarrow{DT} \text{Mathematical Explanation.}$$

Nonetheless, opponents of (DT) have differing views on the relationship between mathematical and grounding explanations, depending on their particular conception of the grounding relation. For example, Poggiolesi and Genco (2023), as well as Poggiolesi (2023), more directly favor the idea that mathematical explanation is a variety of grounding explanation. However, these theories construe grounding as a conceptual relation. It is also worth noting the historical contributions of Bernard Bolzano to this discussion. His account of objective proof is "one that indicates the objective dependence of a given truth on other truths, where this dependence is ultimately to be cashed out in terms of grounding" (Rusnock 2022, p. 367).[11] In particular, he draws a distinction between proofs that merely certify that a proposition is true and proofs that provide answers to why-questions (i.e., ground-revealing proofs). According to Rusnock (2022, p. 367), Bolzano assumed that ground-revealing proofs are explanatory—an early contribution to discussions of mathematical explanation in terms of grounding.[12]

With these preliminaries out of the way, I delve into the grounding relation in the next subsection, as it is pivotal to our discussion.

2.2. A Minimal Separatist Theory of Ground

Many grounding theorists take the grounding relation as primitive. Roughly speaking, the grounds for the fact that X consist of the

[11] Bolzano's ideas on the notion of the purity of proofs are also relevant here and will be briefly addressed in subsection 3.2.

[12] See Lange's work (2022) for an overview of Bolzano's contribution to the theory of force composition and objective grounding.

answers to the question "In virtue of what is it the case that X?". The fact that A is denoted by $[A]$. Say $[A]$ is a *full* ground for $[B]$ if no other fact needs to be added to $[A]$ to obtain $[B]$. Second, suppose that $[A]$ is grounded in $[B]$, and $[B]$ is grounded in $[C]$. Then we say that $[A]$ is *immediately* grounded in $[B]$, and is *mediately* grounded in $[C]$. An immediate full ground is denoted by $\prec$. Additionally, for the purposes of this paper, grounding is considered *factive*. This means that if $[A] \prec [B]$, then it is the case that A and it is the case that B. In addition, for the purposes of this paper, the grounding relation holds between facts. However, we do not distinguish between a true proposition and a fact. So, when $[A] \prec [B]$, the fact that A determines the fact that B, or the truth of the proposition that A determines the truth of the proposition that B. I adopt the following thesis regarding the grounding relation as a metaphysical form of constitution:

> A grounding relation posits a determination relation between the truth of the items appearing in the grounded and the truth of the items appearing in the ground. In other words, the facts appearing in the ground determine the facts appearing in the grounded. (DeT$_G$)

The thesis outlined above aligns well with what we might call "explanatory realism". According to Roski, this is the view that "all explanations provide information about relations of productive determination such as, *inter alia*, causation and grounding" (2021, p. 14121). Moreover, a relation of determination in the sense of DeT$_G$ is also a dependency relation; it highlights that the fact under study depends on the facts appearing in the ground.

In addition, DeT$_G$ is compatible with the view that Lange adopts in Lange 2019. According to Lange, the ground for a fact is "whatever it is in virtue of which that fact obtains, and a truth-bearer (such as a proposition) is grounded in its truth-makers" (2019, p. 2). If $[B]$ is whatever it is in virtue of which $[A]$ obtains, then $[A]$ is determined by $[B]$. Moreover, if $[B]$ determines $[A]$, then $[B]$ is whatever it is in virtue of which $[A]$ obtains. Let us see an example from Fine (2012):

> The fact that the ball is red and round is the case in virtue of the fact that the ball is red is the case, and the fact that the ball is round is the case. (1)

To use the notation presented earlier, let A := "the ball is red", B := "the ball is round", and let C := "the ball is red and round". Then, we have:

$$[A], [B] \prec [C]. \qquad (2)$$

As Glazier puts it (2020), the connection between grounding and explanation is one of the few uncontroversial ideas in the debates about grounding. In other words, it is almost granted that the items in the *explanans* provide items in the *expalanandum* with some sort of explanation. Call this *grounding explanation*. To use Raven's terminology (2015), two main frameworks address the relation between grounding and explanation. According to *unionism*, grounding is a form of explanation.[13] Alternatively put, this view suggests that the grounding relation and the grounding explanation are one and the same. On the other hand, *separatism* sees grounding as a determination relation that "backs" an explanation.[14] The motivating idea behind separatist theories of grounding is the similarity between grounding and causation. Just as causation is a form of determination in the physical world that produces an explanation, separatism treats grounding as a form of determination, albeit in a non-causal domain.

The present paper adopts a separatist view of grounding. I assume that the grounding relation itself pertains to a determination relation in a non-causal domain, while grounding explanations represent what the grounding relation, once established, will provide:

For any instance of $[A] \prec [B]$, the left-hand side provides the right-hand side with a grounding explanation. In particular, the relation of grounding is different from the explanation that it offers. (DuT)

Finally, following Lange (2019), I will assume that the grounding relation is transitive. Here is the statement:

If $[A] \prec [B]$, and $[B] \prec [C]$, then $[A] \prec [C]$. (TrT)

[13] See Sjölin 2020 and Trogdon 2018.
[14] See Skiles and Trogdon 2021 and Schaffer 2016.

While the grounding relation and the grounding explanation are metaphysically distinct, they are not unrelated. Using standard terminology, the relationship between metaphysical grounding and grounding explanation is called the *backing* relation. A genuine case of grounding backs a grounding explanation. However, the nature of the backing relation is a matter of some debate. Some theorists define backing in terms of representation,[15] while others use explanation itself to clarify the backing relation.[16] For the present research, I stick to an intuitive notion of backing.

As a remark, the DuT serves as a resource to argue that knowledge of the existence of a ground *per se* does not guarantee an understanding of the grounding explanation. This is analogous to causal domains, where one may know a cause exists for a physical fact without fully comprehending the explanation. Combined with the DeT_G, DuT allows for cases where a ground is known to exist, but it is not properly revealed. These theses indicate that merely mentioning grounds (i.e., ensuring there are grounds for a fact) is insufficient for explaining them.[17]

The previous remark points towards a more general framework. As Litland suggests, there is a distinction between metaphysical, logical, and conceptual notions of grounding (2023, p. 17). The framework outlined here provides a metaphysical version of the grounding relation that cannot be reduced to logical entailment or mere deduction. On this view, the transition between the grounded and the grounds requires establishing a determination relation between the truth of the items in the grounded all the way back to the truth of the items in the ground. Applying this view to mathematical facts, the criterion for a ground-revealing proof is that a proof is explanatory if it traces the truth of the items in the *explanandum* all the way back to the truth of the items in the *explanans*, answering the why-questions regarding the fact under study via a proper understanding of the determination relation between the grounded and the ground.

With these preliminaries, let us examine Lange's arguments in favor of DT in the next section.

[15] See Trogdon 2018.
[16] See Kovacs 2020.
[17] See section 4.1 and the proof of the Fundamental Theorem of Algebra using the Liouvel's Theorem.

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3. *Two Main Arguments for DT*

3.1. Grounds Often Don't Explain

In a nutshell, Lange's first argument for DT is that grounds of a mathematical fact often do not explain it. In other words, his first argument suggests that ground-revealing proofs are non-explanatory. Lange supports this claim with various examples. Let us consider one of his examples along with two proofs mentioned in Lange 2019. Consider the list of natural numbers from 1 to 99,999.

Fact 3.1.1. There are $50,000$ occurrences of 7 in the list from 1 to $99,999$.

Proof. Start writing down all of the numbers from 1 to $99,999$ and count each occurrence of the digit 7. For example, there is one occurrence of the digit 7 in the number 7, one in 17 (totaling two occurrences so far), and one in $99,997$ (totaling $50,000$ occurrences up to this point). Hence, there are 50,000 occurrences of the digit 7 in the list from 1 to 99,999. □

Proof. Add 0 to the list, which contains no occurrences of the digit 7. The new set becomes $\{0, 1, 2, \ldots, 99999\}$. The number of 7s in this new list is the same as in the original list. Next, for every number with fewer than five digits, add leading zeros to make each number five digits long. This new set looks like $\{00000, 00001, 00002, \ldots, 99999\}$. This set also has the same number of 7s as the original list. This set represents all possible five-digit combinations of the digits 0 through 9. There are $100,000$ numbers, and each digit occurs equally often. Therefore, the total number of digits in this set is $500,000$. Since each digit appears with equal frequency, one-tenth of these digits are 7s, resulting in $50,000$ occurrences of the digit 7. □

Let us delve into further details of the first argument for DT. Using the example above, Lange claims that the first proof identifies the fact's grounds but does not provide as much explanation as the second proof. He argues that this example shows that grounds alone do not automatically yield an explanation. As Lange puts it, "If mathematical explanation worked by tracing back the explanandum to its grounds, then it would be puzzling why the first proof does not qualify as a mathematical explanation" (2019, p. 3).

As mentioned earlier, under a metaphysical understanding of DT, Lange only needs a single successful example to defend DT. Nevertheless, *en route* to do so, he should first establish the following three statements:

1. That the first proof does not explain at all.

2. That the second proof, while more explanatory, does not include the grounds of the fact in question.

3. Other explanatory proofs, often, do not contain the grounds of the fact in question.

In what follows, I will show that Lange's arguments are inadequate to support any of the three statements. Regarding 1, it is unclear why he believes the first proof does not explain anything at all. Lange himself acknowledges that "explanatory power is a matter of degree; it is not all or nothing" (2014, p. 511, footnote 21). Although the second proof is granted to have greater explanatory power than the first, the first proof still provides minimally informative grounds. This is not particularly a problem for the opponent of DT because, as Litland observes, "a proposition may have many distinct full immediate grounds" (2023, p. 3) and different grounds have different explanatory powers. Moreover, the first proof has at least two different variants, each with distinct grounds, conveying different information, and possessing different explanatory values. While the explanatory value of the grounds in both variants is limited, they offer different explanations.

To illustrate, on the one hand, the first proof relies on identifying a series of partial grounds $[A_i]$ that produce a full ground for the Fact 3.1.1. For example, $[A_1]$ is the fact that there is one occurrence of the digit 7 in the list from 1 to 7, and $[A_2]$ is the fact that there is one occurrence of the digit 7 in the list from 8 to 17. So, in the list from 1 to 99,999, the ground consists of $[A_1]$, $[A_2]$, ... , $[A_{50000}]$.

On the other hand, one might say that $[A_1]$ is itself a mathematical fact, grounded in a series of grounds $[A_1^i]$ (where $i = 1, \ldots, 7$), indicate that there is no occurrence of 7 in the list from 1 to 7 except in the number 7. For example, $[A_1^1]$ says that there is no occurrence of 7 in the number 1, $[A_1^2]$ is the ground for the fact that there is no occurrence of 7 in the number 2, and so on, up to $[A_1^7]$ which serves as the ground for the fact that there is an occurrence of the digit 7 in the number 7. So, $[A_1^i]$ (where $i = 1, \ldots, 7$) forms the ground for $[A_1]$.

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The first variation uses $[A_i]$s, while the second employs $[A^i_j]$s as grounds for the Fact 3.1.1. Although both variations are not considered the best explanations in everyday mathematical practice, it is not clear they offer no explanation whatsoever. For example, these two variations differ in the information they convey and their overall explanatory power. To illustrate, $[A^i_j]$ includes information regarding the absence of the digit 7 in certain members of the list, whereas $[A_i]$ reports the number of occurrences of 7s in sub-lists (e.g., from 1 to 7, from 8 to 17, etc.). Given the metaphysical undertone of the present debate about DT, it is not clear that the difference between the first and second proofs concerns explanation *per se* (which is required for Lange's argument).

To further illustrate the explanatory aspect of the first proof, imagine a hypothetical scenario where humans, including the best mathematicians, could only count a finite number of things and had only basic finite arithmetic knowledge. In such a scenario, the only grounds people could identify would be those in the first proof. Consequently, the first proof, with its minimal information, would be the sole explanatory proof available, as the second proof would be incomprehensible due to its more complex grounds. The main issue with the first proof is that it uses grounds with minimal mathematical information (e.g., whether a number on the list contains the digit 7) and, therefore, merely provides a minimal explanation.

As a final remark on 1, opponents of DT, in particular, those who adopt a separatist view of grounding, as outlined in subsection 2.2, are not committed to the idea that all mathematical proofs are explanatory. The resources provided by DeT_G and DuT allow us to imagine cases in which a mathematical proof only verifies that a theorem is true, i.e., a proof that only ensures the existence of a ground without properly revealing it. However, in the case of the first proof for Fact 3.1.1, the issue is that the ground itself is minimally informative.

Regarding 2 and 3, Lange needs to show that the second proof is not ground-revealing. The second proof employs a different set of concepts to establish the fact in question. Nonetheless, the second proof does provide a basis for why Fact 3.1.1 holds. It determines that the number of 7s in the list from 1 to $99,999$ is $50,000$. Why, then, is it not justified to say that the second proof provides a ground for Fact 3.1.1 according to DeT_G? One cannot argue solely from the minimal separatist theory of ground (i.e., theses DeT_G, DuT, and TrT) that the second proof does not involve the fact's grounds.

Therefore, Lange introduces an additional assumption regarding the grounding relation. Due to its similarities with the logical structure of sentences, I call this statement the "Atomic Grounding Thesis" (Lange 2019, p. 2):

> I am making the rough presupposition that a mathematical fact is grounded by the atomic (or negated atomic) truths to which one is led if one starts with that fact and moves "downward" to logically simpler truths in an obvious way, such as from universal facts (i.e., facts expressed by generalizations) to their instances and from conjunctions to their conjuncts. (AGT)

This statement is the most controversial claim Lange makes about the grounding relation. Furthermore, the role of AGT in Lange's argument needs further clarification. On the one hand, AGT may be used to show that the first proof includes the ground for Fact 3.1.1. On the other hand, it is used to argue that the second proof does not include the grounds for Fact 3.1.1. Therefore, AGT can play two significant roles, which I will call the *designatory* and *discriminatory* roles, referred to as designatory AGT and discriminatory AGT, respectively. Since Lange did not identify these roles, I will illustrate them through examples.

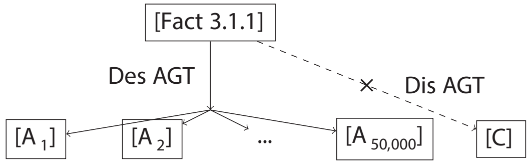
As a designatory statement, AGT aims to identify a particular ground. In the example above, designatory AGT identifies the individual occurrences of 7s in the list from 1 to $99,999$ as the ground for Fact 3.1.1. On the other hand, discriminatory AGT prevents an alleged ground from being considered an actual ground. For example, in the case above, discriminatory AGT says that the second proof does not involve the grounds for Fact 3.1.1. Indeed, Lange implicitly endorses the discriminatory role. For instance, in his comments on the second proof of 3.1.1, Lange says, "This proof does not specify wherein the list of numbers each 7 appears; it does not give the result's ground" (2019, p. 2). This shows that AGT not only designates the fact's grounds in the first proof but also prevents the second proof from being considered as identifying the grounds for Fact 3.1.1. I will argue that the discriminatory role of AGT is unjustified.

First and foremost, the discriminatory role of AGT is essential to defend DT. Otherwise, opponents of DT could argue that what the two proofs of Fact 3.1.1. show is that not every ground enjoys the same level of explanatoriness. According to opponents of DT, especially those opponents of DT who embrace the separatist theory of grounding, various ground-revealing proofs provide different explanations for a mathematical fact. Hence, examples like the above could even be seen as evidence in favor of the view that identifies mathematical explanation with grounding explanation. Hence, every ground-revealing proof is explanatory, at least to a minimal degree. Just as the goal of science is to find better answers, mathematicians pursue a similar goal. So, nothing in the mere designatory version of AGT provides sufficient evidence for DT. Consequently, the discriminatory AGT is crucial to defend DT; one must argue that the other proofs do not contain grounds for the mathematical fact in question to defend DT.

Although he does not use the specific terms mentioned here (i.e., discriminatory vs. designatory), Lange has addressed the issue as a possible objection, stating,

> It might be objected that although the first proof displays the result's grounds, this fact does not preclude the second proof from also doing so; a given fact can have many complete sets of grounds. For instance, a given fact's grounds can themselves have grounds, and the latter may then qualify as grounds of the given fact. (2019, p. 2)

Lange provides three responses to this objection. First, he asserts that the example aims to demonstrate that the grounds of a particular mathematical theorem do not inherently explain the theorem. We have already discussed this initial point. Second, he contends that the second proof in the example does not identify the grounds of the fact in question. Opponents of DT might argue that the second proof designates a different ground for the fact in question. What prevents opponents of DT from acknowledging that the second proof involves a different ground? Lange's response relies on the discriminatory aspect of AGT. Finally, Lange's third answer is that proofs identifying grounds are pure, a point which will be discussed in subsection 3.2. So, let us discuss discriminatory AGT both from the viewpoint of the minimal separatist theory of grounding and the everyday mathematical practice. The role of AGT in evaluating the example mentioned earlier is clarified in the following diagram ($[C]$ refers to the ground(s) revealed by the second proof, and Dis AGT and Des AGT represent the Discriminatory AGT and Designatory AGT, respectively):



From a purely grounding-theoretic point of view, Lange ties the grounding content to the logical profile of propositions, suggesting that a mathematical fact "is grounded by the atomic (or negated atomic) truths to which one is led if one starts with that fact and moves 'downward' to logically simpler truths in an obvious way" (2019, p. 2). However, relying solely on the logical profile for grounding content is questionable. This is because the grounding relation is hyperintensional, meaning that even logically equivalent propositions may have different grounds. As Krämer states, "Ground is accordingly sensitive to features of a truth that go beyond its logical profile" (2018, p. 786). Grounding should reveal how the truth of the *explanandum* depends on the *explanans*, involving more than just logical entailment but also the substantive connections underpinning mathematical truths. Therefore, a more comprehensive understanding of grounding requires a ground-first approach, going beyond logical forms to metaphysical relations and providing a more natural view of mathematical explanation.

One major issue with discriminatory AGT is what Litland (2023, section 4.2) calls the "commonality problem". This problem arises in theories of grounding that fail to specify what is common to all instances of grounding. Discriminatory AGT only exacerbates this issue. Lange defines grounding for a fact as "whatever it is in virtue of which that fact obtains" (2019, p. 2), making it difficult to justify the discriminatory AGT. According to discriminatory AGT, the transition from a fact to its grounds occurs only " 'downward' to logically simpler truths in an obvious way" (2019, p. 2). This approach contradicts Lange's own earlier statement on the nature of the grounding relation. The core of this problem lies in not adopting a ground-first approach. A metaphysical understanding of the grounding relation can address the commonality objection by providing a clear metaphysical definition of the grounding relation.

The theory of grounding advocated here exemplifies a ground-first approach, avoiding the grounding-theoretic problems mentioned above. The separatist theory of grounding, along with DeT_G, DuT, and TrT, provides a framework that specifies the nature of the grounding relation through DeT_G, which is a metaphysical notion. Additionally, DuT allows this theory to avoid reductionist views of grounding. Furthermore, this view naturally poses a criterion for ground-revealing proofs. A proof, according to this view, is ground-revealing if one can trace the truth of the items within the grounded all the way back to the truth of those within the ground. Thus, the explanation backed by the determination relation results from a proper understanding of the determination relation itself. It involves comprehending how the items in the ground determine the truth of those in the grounded, without falling into any reductionism.

In addition to the purely grounding-theoretic issues, discriminatory AGT is also a dubious statement in terms of everyday mathematical practice, where mathematicians often provide several mathematically equivalent conditions for proving a mathematical theorem. These statements typically take the form, "The following conditions are equivalent", followed by several mathematically equivalent conditions. These equivalent conditions, which we can call *definition-lemmas*, serve as lemmas offering different criteria for the same mathematical fact. The *raison d'être* of definition-lemmas is that mathematicians prefer to use the most appropriate condition for proving that a mathematical fact holds. Each criterion is connected to a specific family of concepts, and each equivalent condition provides a different representation of a mathematical fact. Nevertheless, AGT (especially in its discriminatory version) overly relies on a particular logical profile.

Let us consider an example of a definition-lemma. To prove that a set S is *infinite* one should show that the cardinality of S (denoted by $|S|$) is larger than any finite number. Another way to prove that S is infinite is to show that an infinite set (say a copy of $\mathbb{N}$ or $\mathbb{Q}$) is included in S. Here is Definition-Lemma:

Definition-Lemma 3.1.2. Let S be a set. Then, to show that S is infinite, one can proceed with one of the following equivalent conditions:

1. *The cardinality of S is larger than every natural number. Formally, for all $n \in \mathbb{N}$, $|S| \geq n$.*

2. *An infinite set is included in S. More formally, there is an infinite set S', and a function f such that $f : S' \to S$ is an injection.*

Definition-Lemma 3.1.2 suggests, among other things, that the property of being infinite can be understood as a disjunction. Therefore, to prove that a particular set S is infinite, one can either show that $|S|$ is larger than n for any natural number, i.e., $|S| \geq 1, |S| \geq 2, \ldots$ Another approach is to show that an infinite set is injected in S. It is worth noting that Definition-Lemma 3.1.2 does not imply that the definition of infinite is a disjunction. Rather, it shows that to prove the infinity of a particular set, say S, one can proceed by either of the two conditions. As we will see in the example below, each of the two conditions provided above produces a different understanding of why a particular set is infinite—either by showing that S cannot be finite or by showing that it is large enough to contain an infinite set.[18] Let "Inf" represent the property of being infinite. Additionally, let "Card$n$" and "Inj$f$" denote the first and second conditions of Definition-Lemma 3.1.2, respectively. Thus, we have the following:

$$\mathrm{Inf}(S) \Leftrightarrow (\mathrm{Card}n(S) \vee \mathrm{Inj}f(S)) \tag{3}$$

Let us see how the existence of the definition-lemmas in the practice of mathematics provides counter-examples to discriminatory AGT. A field F is a structure of the form $(F, +, \times, 1_F, 0_F)$ containing two group structures: a multiplicative group and an additive group with 1_F and 0_F indicating the neutral elements of the multiplicative and additive groups, respectively. Say that F has *characteristic* 0, if for all n and $a \in F$, $a + a + \cdots + a$ (n-times) $\neq 0$. Using the definition-lemma above, we prove that every field with characteristic 0 is infinite.

Fact 3.1.3. Let F be a field with characteristic 0. *Then, F is infinite.*

[18] This is particularly important because equivalent conditions in a definition-lemma avoid the issues encountered by Genco and Poggiolesi in a similar context. Consider the case of colors. Genco and Poggiolesi (2023), in their view of conceptual grounding based on conceptual complexity, treat red as the set of all red shades. Thus, they define the color red as "the set of all types of red-crimson, scarlet, [. . .] —and hence can be seen as composed of them. In this case, the color red will count as more complex than the color crimson". However, as Litland (2023) points out, this view is intuitively problematic. The color red is not merely a collection of its shades, such as crimson or scarlet. Instead, crimson can be seen as a variant of red. In the context of definition-lemmas, I circumvent this by proposing that the equivalent conditions offer multiple ways to demonstrate that a set is infinite. From a purely grounding-theoretical perspective, this aligns with what Krämer (2018) refers to as "modes of verification". This is a point that Lange's view of grounding does not capture, as we will see shortly.

Proof. Let F be a field with characteristic 0. Then, for any positive integer n, consider the elements $0, 1_F, 2 \cdot 1_F, 3 \cdot 1_F, \ldots, n \cdot 1_F$. If the field had a finite size, then for some n, the elements would eventually repeat, i.e., $m \cdot 1_F = k \cdot 1_F$ for some $m \neq k$. This implies $(m - k) \cdot 1_F = 0$, which contradicts the assumption that the characteristic is 0 (since $m - k$ would be a positive integer). Therefore, the set $\{0, 1_F, 2 \cdot 1_F, 3 \cdot 1_F, \ldots\}$ contains distinct elements. Hence, F must be infinite. □

Proof. Let F be a field with characteristic 0. The field must contain an element 1_F. For any integer n, $n \cdot 1_F$ (sum of 1_F with itself n times) is distinct because the characteristic is 0. Construct the set $\mathbb{Z} \cdot 1_F = \{n \cdot 1_F \mid n \in \mathbb{Z}\}$, which is isomorphic to the integers $\mathbb{Z}$. Form the field of fractions $\mathbb{Q} \cdot 1_F = \left\{\frac{m}{n} \cdot 1_F \mid m, n \in \mathbb{Z}, n \neq 0\right\}$, which is isomorphic to $\mathbb{Q}$. Since $\mathbb{Q}$ is infinite, F must also be infinite. □

The first proof explains why F is infinite by showing that for any natural number n, the field F as a set should contain more members than n; i.e., $|F| \geq n$ for all natural numbers n. In other words, this proof explains the fact that F is infinite by showing that since F has characteristic 0, F cannot be finite. So, with the aid of the first clause of Definition-Lemma 3.1.2, it answers the question "Why is F infinite?". Thus, it demonstrates that because the cardinality of F should be larger than any natural number, it must be infinite. In particular, the first proof shows how the truth of the Fact 3.1.3 is traced back all the way to the truth of the premises, including Definition-Lemma 3.1.2.

The second proof, in contrast, explains the infinity of F by showing that F should contain an isomorphic copy of $\mathbb{Q}$, which is infinite. Thus, it shows that under the conditions where F has characteristic 0, it should be "large enough" to contain an infinite set, say $\mathbb{Q}$ or $\mathbb{N}$.[19] In particular, as the second proof does not depend on the choice of F, it shows that all fields with characteristic 0 are structurally similar, i.e., they all contain a copy of the rational numbers. Finally, the second proof relies on another condition of Definition-Lemma 3.1.2, as it makes use of the fact that F is infinite if there is an injection from $\mathbb{Q}$ to F. Again, this proof traces the truth of the fact that is proven to be the truth of the items appearing as premises.

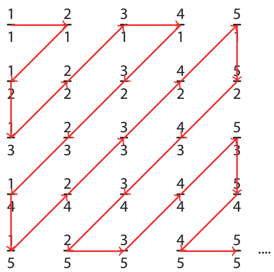
[19] As a technical note, in this case, there is no difference whether the injected set is $\mathbb{N}$ or $\mathbb{Q}$ as these two sets have the same cardinality. This is illustrated by the following diagram, with the arrows suggesting how the set of rational numbers could be counted by natural numbers (for simplicity, we only count $\mathbb{Q}^+$):

Discriminatory AGT fails to account for examples like this one. To illustrate, the first proof uses the first clause of the Definition-Lemma 3.1.2. The first proof successfully proves that for any natural number $n \in \mathbb{N}$, the cardinality of $|F|$ is bigger than n, i.e., that $|F| \geq n$ for all $n \in \mathbb{N}$. So, it is in virtue of the fact that $|F| \geq n$ for all of the natural n that F is infinite. Thus, the facts expressed by the sentences $|F| \geq n$ for each $n \in \mathbb{N}$ form a ground for the fact that F is infinite. On the other hand, the second proof uses the second clause of the Definition-Lemma 3.1.2. It proves that an infinite set (say $\mathbb{N}$ or $\mathbb{Q}$) is injected in F. To put the previous two conclusions together, we have:

$$[\text{Card}n(F)] \prec [\text{Inf}(F)], \text{ and } [\text{Inj}f(F)] \prec [\text{Inf}(F)]. \tag{4}$$

This serves as a counter-example to discriminatory AGT. Recall that Lange understands the ground of a fact as "whatever it is in virtue of which that fact obtains, and a truth-bearer (such as a proposition) is grounded in its truth-makers" (2019, p. 2). According to this proof, F is infinite in virtue of the fact that $|F|$ is larger than any natural number. To be more specific, Lange's criterion implies that the grounds in the first proof are $|F| \geq 1$, $|F| \geq 2$, ... which constitute $[\text{Inf}(F)]$. However, discriminatory AGT implies that $[\text{Inj}f(F)]$ is not a ground of $[\text{Inf}(F)]$. On the other hand, if we proceed with the second clause of Definition-Lemma 3.1.2, the ground of the fact is that an infinite set is included in F, which is $[\text{Inf}(F)]$. Nonetheless, discriminatory AGT implies that $[\text{Card}n(F)]$ is not a ground of $[\text{Inf}(F)]$. In either case, discriminatory AGT contradicts 4.

That said, opponents of DT who embrace the separatist theory of grounding (i.e., DeT_G, DuT, and TrT) can account for the above case. First, in both proofs, the truth of the items within the premises



determines the truth of the fact that F is infinite. Thus, by DeT_G, we can see that a ground has been mentioned in both proofs. In addition, the explanations that are backed by the determination relation are properly revealed in both cases. This is because, via these proofs, the truth of the items within the fact being proved is traced back to the truth of the items occurring as premises, rendering them explanatory proofs of Fact 3.1.2. Thus, these proofs are ground-revealing, as a single mathematical fact could have multiple grounds in this framework.

This way, the existence of definition-lemmas in everyday practice of mathematics provides a counter-example to the discriminatory version of AGT. However, this version of AGT is crucial for Lange's main line of argument. Returning to the example of 7s, as you recall, to show that the combinatorial proof does not contain Fact 3.1.1's ground, Lange appealed to discriminatory AGT, which is an unjustified statement. I conclude that Lange's argument that the second proof does not include the Fact 3.1.1's ground is not convincing.

3.2. Metabasis Eis Allo Genos

This subsection focuses on Marc Lange's second argument, which suggests that ground-revealing proofs are pure and that this purity undermines their explanatory power. The argument from purity can be seen either as a second argument for DT or as an argument for a justification of the discriminatory role of AGT. However, deciding between these two interpretations does not affect the counterarguments presented here. After briefly reviewing the notion of purity, I attempt to extract Lange's criterion for purity of proofs and then I show that these attempts are unsuccessful. Finally, as a counterexample to the link between purity and explanatory value, I present Bernard Bolzano's proof of the mean value theorem. According to both Lange's and Bolzano's criteria, this proof is pure; however, it notably diverges from Lange's statement as it is not a brute-force proof and, hence, remains highly explanatory.

According to Detlefsen and Arana (2011) and Detlefsen (2008b), debates about purity can be traced back to Aristotle. Aristotle's notion of purity is metaphysical. He believed that purity is a way to prevent what he called "metabasis eis allo genos" (meaning crossing from one genus to another). He argued that all categories are metaphysically distinct, so moving from one category to another is prohibited. Another metaphysical version of purity is presented in the 19th century by Bolzano. Bolzano's idea of the purity of proofs was influenced by his view on the role of geometry in mathematics. He held that geometry, as a subordinate discipline of mathematics to algebra, arithmetic, and analysis, should not intermingle with universal mathematics (Rusnock 2022, p. 365). For Bolzano, the former is not as general as the latter. Consequently, he believed that introducing geometric concepts into proofs of universal mathematics (e.g., algebra or analysis) compromises their clarity and precision. This belief in maintaining a strict separation ensured that proofs remained pure—free from extraneous concepts not essential to the theorem itself. By doing so, contrary to Lange's view of pure proofs, Bolzano argued that only pure proofs are explanatory. On the other hand, Detlefsen and Arana (2011) describe "topical purity" as an epistemological view of purity pertaining to the stability of solutions in mathematical problems. Topical purity is considered an epistemic virtue, balancing the resources used in the solution and the concepts used in the statement of the problem.

Considering the metaphysical implications of DT, a metaphysical version of purity is essential to either justify a discriminatory role for AGT or to argue that the grounds of a mathematical fact should be pure, with pure proofs lacking explanatory power. Therefore, it seems necessary to derive a metaphysical criterion for purity from Lange's 2019, section 2. However, Lange's version of purity is not fully developed. Additionally, Lange proposes various versions of purity, which cannot all be evaluated in the same way. In one version, he employs the contentious notion of the "essences" of the items in a theorem. Consider the sentence "$n!/k!(n-k)!$ for $(n \geq k)$, is an integer". Lange assumes that the essence of the items in the statement is arithmetical. Consequently, a pure proof of the statement should be arithmetical. Thus, we have a preliminary stance:

> Arguably, a pure proof is roughly a proof that proceeds entirely from the facts about the essences of the mathematical items figuring in the theorem being proved. (2019, p. 7) (5)

Interpreting 5 as a metaphysical criterion or a definition of purity requires a clear understanding of what constitutes the essence of a mathematical fact. As a metaphysical statement, 5 should provide a criterion for identifying the essence of a given mathematical fact. However, this is not a straightforward task. Steiner (1978) used the term "characterizing property" to avoid the controversial notion of an essence of a mathematical fact. Many theorems in mathematics

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connect algebraic facts to geometric facts. For instance, Hilbert's *Nullstellensatz*[20] establishes a bridge between algebra and geometry. It is fundamental to modern algebraic geometry, linking ideals (an algebraic concept) with varieties (a geometric concept). Many proofs in algebraic geometry rely on these connections, demonstrating that algebraic facts about sets relate to facts about polynomial ring ideals. Proofs using these links are as valid as other proofs. Even if there is a *bona fide* notion of the essence of a mathematical fact, the connection between a pure proof and grounds remains unclear. Therefore, 5 is underdeveloped as a definition and cannot serve as a metaphysical criterion.

Lange's paper also discusses an alternative conception of purity. According to this version, a pure proof is not limited to using only the concepts explicitly stated in the theorem. Instead, the concepts used in a pure proof can extend beyond those necessary for merely understanding the theorem. For example, in a discussion about the Taylor series, Lange states that "a proof that appeals to imaginary numbers should still be regarded as pure if it provides information about the theorem's grounds" (2019, p. 6). This means that any concept contributing to the understanding of the grounds of a theorem is considered intrinsic to the theorem and can be included in a pure proof. In other words, a proof is pure if it uses concepts that, despite being potentially extraneous to the theorem's statement, provide significant insight into its grounds. This approach allows for a broader range of concepts to be utilized in pure proofs as long as they illuminate the grounds of a mathematical theorem.

> For a theorem T let $\mathcal{C}$ be the set of concepts occurring in T's statement. Let c be a particular concept. Then c can occur in a pure proof of T if c gives information about some of T's grounds. (6)

Criterion 6 appears to offer an improved version of purity compared to the previous one. It avoids the contentious notion of the essence of a mathematical fact and is more modest in its scope, permitting the use of concepts not directly stated in the theorem. However, this criterion has its own limitations. By appealing to the grounds of a mathematical fact, 6 might trivialize the distinction between pure and impure proofs. It essentially puts the cart before the horse, as determining whether a proof is pure according to this criterion

[20] See Lang 2005, p. 380, Theorem 1.5.

requires first identifying the grounds for the fact in question. In this sense, 6 cannot provide AGT with any discriminatory power since it relies on the concept of grounding, which AGT itself seeks to elucidate. Therefore, the criterion's applicability is limited because it presupposes a clear understanding of what constitutes the grounds of a mathematical fact. Furthermore, under a more liberal interpretation of the grounding relation, 6 might be seen as trivially true for any concept used in any correct proof. If any concept contributing to understanding the theorem's grounds is considered intrinsic and permissible in a pure proof, the criterion becomes overly inclusive. This inclusivity risks erasing the distinction between pure and impure proofs altogether, making the criterion less effective in discriminating between different types of proofs based on their explanatory power.

Beyond the underdevelopment of the concept of purity in Lange's work, the primary issue with his argument is that it fails to show that pure proofs often do not explain.[21] For any *bona fide* understanding of purity, Lange must demonstrate that pure proofs frequently fail to explain or often resort to brute-force methods, which lack explanatory value. However, neither of these claims appears to be substantiated. Moreover, some mathematicians would contest both assertions regarding the relationship between the purity of a proof and its explanatory power. For instance, Bolzano believed that a pure proof of the mean-value theorem is the most explanatory proof. In what follows, I will present Bolzano's proof of the mean value theorem. I will then show that although Bolzano's proof is pure, it is highly explanatory and, in particular, it is not a brute-force proof.

Bolzano's proof of the mean-value theorem, as outlined in subsection 13.3.4 of Rusnock 2022, starts with two continuous real-valued functions $f(x)$ and $g(x)$, defined on a closed interval $[a, b]$. The initial conditions are set as $f(a) < g(a)$ and $f(b) > g(b)$. Bolzano employs a precise definition of continuity for his proof: a function $f(x)$ is deemed continuous on $[a, b]$ if, for any point x within that interval, the difference $f(x+\epsilon)-f(x)$ can be made arbitrarily small by making

[21] In footnote 11 of his work (2019, p. 7), Lange says, "Although a proof's purity need not contribute to its explanatory power, purity is nevertheless a feature that mathematicians often seek and prize in proofs." So, according to Lange, a proof's purity does not automatically link with its explanatory value. However, according to Lange, pure proofs often resort to brute-force methods and, consequently, are not explanatory. Hence, according to Lange, while these notions are semantically separate, in practice, grounding-revealing proofs are often pure proofs that use brute-force methods. Below I discuss Bolzano's proof of the mean-value theorem to challenge these claims from the viewpoint of mathematical practice.

ϵ sufficiently small. Then, the mean value theorem, as presented by Bolzano, says that there is a $c \in (a, b)$ such that $f(c) = g(c)$.

Here is the proof sketch of the mean-value theorem by Bolzano. Assuming the continuity of f and g, there exists an $\epsilon > 0$ such that for any i in $(0, \epsilon)$, the inequalities $f(a + i) < g(a + i)$ hold. This is because the values of f and g can be made arbitrarily close to $f(a)$ and $g(a)$, respectively, within a sufficiently small interval around a. Since $f(b) > g(b)$, there must be a smallest $U \in (a, b)$ such that $f(a + U) = g(a + U)$. This U marks the point where the function values transition from $f(x) < g(x)$ to $f(x) > g(x)$. By continuity, at this critical point U, the functions f and g must be equal. Therefore, there exists a point $c \in (a, b)$ where $f(c) = g(c)$, completing the proof.

Bolzano's method emphasizes an analytical approach, relying strictly on the defined mathematical properties rather than geometrical notions. In particular, the concepts used in the proof, such as the continuity of functions, are intrinsic to the mean-value theorem. This proof is explanatory because it addresses the underlying reasons why f and g must be equal at some point within the interval $[a, b]$. Moreover, the proof does not resort to brute-force methods. Instead, it employs the definition of continuity to highlight the high explanatory power of the mathematics involved.

The example presented here highlights a deeper flaw in Lange's approach to grounding and explanation, specifically, a deeper issue in his conception of grounding. His emphasis on the logical profile of propositions as the sole determinant of their grounds leads him to undervalue the explanatory power of pure proofs, categorizing them as lacking explanatory value and employing a brute-force method. A revised understanding of grounding in mathematics results in a more faithful approach to pure proofs in mathematics.

I conclude that Lange's argument from purity is unsuccessful either in showing that AGT has a discriminatory power or showing that ground-revealing proofs are pure and that pure proofs do not explain. I discuss Lange's complementary arguments for DT in what follows.

4. *Further Arguments for DT*

This section will discuss Lange's two additional arguments in favor of DT. These two arguments are, at least in part, motivated by Lange's theory of mathematical explanation. A full analysis of Lange's account of mathematical explanation is beyond the scope of the present research. However, as Lange suggests, "a sketch of such an account would support my arguments in the previous section. In addition, the account I will now sketch suggests two further arguments that mathematical theorems are not automatically explained by their grounds" (2019, p. 11). Therefore, for these purposes, I briefly outline his account and then present my counterarguments.

4.1. Salience Theory and Mathematical Explanation

Lange's theory of mathematical explanation is presented briefly in Lange 2019 and in more detail in Lange 2014. According to Lange, a mathematical explanation comes from exploiting the salient features of its *explanandum*. More specifically, a proof is explanatory if it addresses the salient features of the fact under study. For instance, in the example of 7s, the second proof is explanatory because it accounts for the salient feature that $50,000$ (the number of 7s appearing in the list) is equal to half of $100,000$ (which is larger than $99,999$ by one). In contrast, the first proof does not address the salient feature and is therefore not explanatory.

As a corollary to Lange's theory, a brute-force proof is not explanatory because, according to Lange, these proofs are not sensitive to salient features and, hence, are not explanatory. To borrow his own words: "an explanation must (on this proposal) pick out particular features of the setup that are similar to the result's salient features, tracing the result's salient features back to them" (2019, p. 13).[22] Subsequently, Lange provides two complementary arguments to support DT, i.e., an argument from the coincidences in mathematics and an argument from context-shift. These two further arguments are discussed separately in the next two subsections. However, a

[22] The explanatory role of proofs using brute-force methods requires a deeper study. While some proofs employing brute force are not explanatory as they sometimes fail to properly reveal the grounds, the view that such proofs are never explanatory seems too hasty. For example, consider the proof of the infinity of fields with characteristic 0 (the first proof for Fact 3.1.3 in subsection 3.1). In this proof, a series of partial grounds determining that $|F| \geq n$ for each n produce a full ground for Fact 3.1.3. Although every partial ground uses a method similar to brute force, the overall proof explains why F should be larger than any finite number and, consequently, infinite.

From a ground-theoretic point of view, this could be combined into a general framework of bilateralism about the ground as discussed in Litland (2023, p. 20). In some cases, like the one discussed in the first proof of Fact 3.1.3, individual grounds could be viewed as taking on the task of what Litland describes as *explanatory rejection*. As every individual case is rejecting one case, they overall produce a positive explanation of the fact that F is infinite.

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brief critical survey of salience theory seems appropriate as salience theory is the motivating idea behind these two arguments.

A major problem with salience theory is that it fails to show how the items within the *explanandum* depend for their truth on the truth of the items within the *explanans*. It is common to assume that, in a genuine case of explanation, there should be a dependency relation between both *relata* of the explanation. Consequently, the salience theory fails to identify the differences between a proof that properly establishes such a dependency relation and one that does not. Here is an example. To prove a mathematical theorem, mathematicians sometimes cite another theorem. In some cases, the cited theorem is essential in understanding why the proved theorem holds.

Theorem 4.1.1. (Fundamental Theorem of Algebra) Every polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

There are many proofs for the Fundamental Theorem of Algebra 4.1.1, including topological proofs or algebraic proofs. One of the straightforward proofs of Theorem 4.1.1 comes from Liouville's Theorem. Liouville's Theorem states that:

Theorem 4.1.2. (Liouville's Theorem) Let f be an entire function (analytic everywhere in the complex plane) and bounded. Then, f is constant.

Recall that a polynomial equation $p(x)$ in $\mathbb{C}$ is something of the following form, where x is a variable over $\mathbb{C}$ and a_is $\in \mathbb{C}$:

$$p(x) = a_1x^n + a_2x^{n-1} + \ldots + a_{n+1} = 0.$$

Polynomials with coefficients in $\mathbb{C}$ have a striking feature. Polynomials are combinations of algebraic expressions that depend on x for their value. Therefore, as x varies, $p(x)$ also varies. However, notably, polynomial equations show similar patterns of behavior under certain conditions. For example, for $x \to \infty$, or $x \to -\infty$, $|p(x)| \to \infty$. Moreover, the Fundamental Theorem of Algebra addresses the same issue, suggesting that such polynomials display a similar behavior regarding solvability in $\mathbb{C}$: they all have a solution in $\mathbb{C}$.

Here is a proof of the fundamental theorem of algebra using Liouville's theorem. Let $p(x)$ be an arbitrary polynomial equation over $\mathbb{C}$ such that for all $x \in \mathbb{C}$, we have $p(x) \neq 0$. As we have chosen a non-identically zero polynomial, we can construct $p(z) = 1/p(x)$, which is well-defined. However, as $x \to \infty$, then $|p(x)| \to \infty$ and hence $p(z) \to 0$. This shows that $p(z)$ is bounded. Nonetheless, applying Liouville's Theorem 4.1.2. to $p(z)$ implies that $p(z)$ should be a constant polynomial equation, but this is a contradiction. As a result, $p(x)$ should have a solution for some $x \in \mathbb{C}$. This is a restatement of the Fundamental Theorem of Algebra 4.1.1.

According to Ahlfors, "Liouville's theorem leads to an almost trivial proof of the *fundamental theorem of algebra*" (1979, p. 122). It does not adequately explain why every polynomial equation must at least have a root in $\mathbb{C}$. The reason is that the theorem that plays a significant role in the proof functions as a "black box", something we do not fully understand. Of course, the proof above could be modified to incorporate the proof of Liouville's Theorem 4.1.2 such that the resulting proof becomes explanatory. However, we have a proof via Liouville's Theorem 4.1.2 that fails to explain why Algebra's Fundamental theorem holds properly. The proof sketched above addresses a salient feature of the polynomials and traces it back to the premises. While the proof above might be considered explanatory according to salience theory, it does not seem to be the case.

This example points towards a more significant issue with salience theory: it fails to capture the objective aspect of mathematical explanation. In line with explanatory realism, a *bona fide* instance of explanation should provide a determination relation between the items within the *explanandum* and the items within the *explanans*. In mathematical explanations, such determination relation is between the truth of the proposition being proven and the truth of the items proving it. However, salience theory fails to capture this significant aspect of mathematical explanations. Consequently, salience theory fails to capture that the whole proof relies on an unexplained "black box".[23]

With this critical overview of salience theory, let us begin discussing two complementary arguments for DT.

[23] The separatist grounding theory can differentiate between this proof and a full proof of the Fundamental Theorem of Algebra. The proof using Liouville's Theorem only verifies that some grounds exist without revealing those grounds. While it indicates that the Fundamental Theorem of Algebra *could* be proved using Liouville's Theorem, it falls short of offering a fully explanatory proof of the Fundamental Theorem of Algebra as it fails to reveal the specific grounds involved—it merely verifies the existence of such grounds without revealing their nature. So, although there are cases where salient features of a theorem could motivate research or even be considered indicative of a proper explanation, it does not entirely capture how the truth of an *explanandum* holds, in virtue of the truth of the *explanans*.

4.2. It Happens Because it Happens!

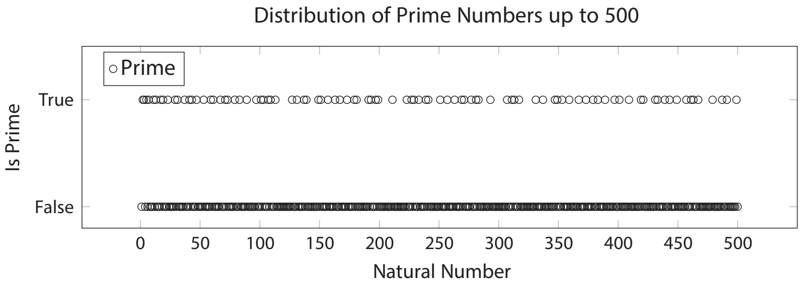
The first complementary argument revolves around the existence of mathematical coincidences. According to Lange, mathematical coincidence refers to facts that are coincidentally true and lack a common explanation[24] (for further details, see Lange 2010). Lange argues that if mathematical explanation is to explain some fact by tracing it back to its grounds, then every mathematical fact should have an explanation. Furthermore, Lange considers the existence of mathematical coincidences as evidence in favor of the salience theory. This is because if a mathematical phenomenon enjoys no salient features, there will be no explanation for it.

Opponents of DT can also account for the existence of coincidences in mathematics. They would argue that, in these cases, we have mathematical facts that do not warrant explanation within everyday mathematical practice. Moreover, explanations of these phenomena are minimal because they are not typically considered *de jure* explanations (i.e., fruitful explanations from the viewpoint of everyday mathematical practice), although in some of these cases, we do have a *de facto* explanation that is not considered a good explanation in terms of everyday mathematical practice. For opponents of DT who also advocate for a separatist view of grounding relations, another resource for addressing the existence of coincidences in mathematics is the duality thesis, which states that the grounding relation is not the same as the explanation that it backs. The former is a determination relation in a non-causal domain, whereas the latter is an explanation backed by the former. In certain coincidence cases, while we are aware of the existence of a determination relation, we lack a full understanding of it. In these cases, these facts are considered coincidental until a full explanation is available.

A mathematical fact that seems coincidental is not unexplainable forever. A conjunction may appear to be a mere coincidence. However, once connections to other parts of mathematics are found, we might come up with explanations of the *prima facie* coincidental mathematical facts. The absence of a common explanation is not evidence of absolute unexplainability. Lack of a common explanation indicates that we are yet to find a ground. The point is illustrated below through the distribution of the prime numbers among the natural numbers.

[24] I agree with Lange that there are coincidences in mathematics, but some mathematicians reject the existence of any coincidences in mathematics. For example, Davis (1981) believes that any seemingly coincidental fact pertains to a general law.

Consider the distribution of prime numbers among the natural numbers. As Zagier puts it, "the prime numbers belong to the most arbitrary and ornery objects studied by mathematicians" (Zagier 1977, p. 7). To illustrate, when examining the distribution of prime numbers among the natural numbers, no common pattern or explanation emerges. Consider the following plot, which shows the distribution of prime numbers up to 500:



As we can see, the gap between the two prime numbers exhibits no general pattern or a striking feature. Furthermore, we know that for any natural number n, there exist n consecutive natural numbers that are not prime. To illustrate, consider $(n + 1)! + 2$, $(n + 1)! + 3$, $\ldots$, $(n + 1)! + n + 1$. The first element $(n + 1)! + 2$ is divisible by 2, the second element $(n + 1)! + 3$ is divisible by 3, and the nth element is divisible by $n + 1$. Therefore, these successive numbers are not primes. This suggests that there are arbitrarily large gaps between prime numbers. To borrow Zagier's metaphor, prime numbers "grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout" (1977, p. 7).

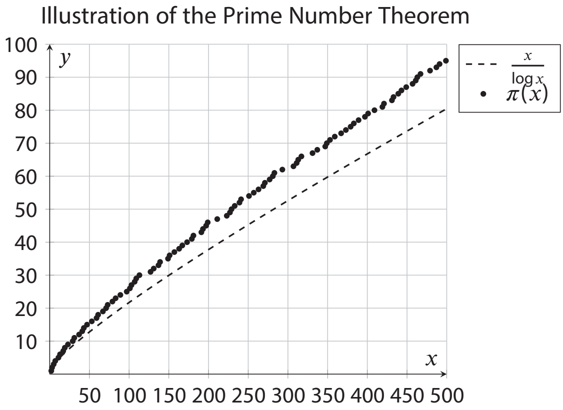
However, this seemingly coincidental phenomenon turns out to point toward a deeper mathematical fact. According to Diamond 1982, no detailed information was known about the distribution of prime numbers in ancient times; however, "at the end of the 18th century, [...] the 'right question' about the distribution of primes was asked and a conjectured answer offered by A.M. Legendre [Leg] and by C.F. Gauss [Gau]" (p. 553). To be more specific, let $\pi(x)$ denote the number of prime numbers less than x. Then the prime number theorem states:

$$\pi(x) \sim \frac{x}{log(x)} \quad (x \to \infty). \tag{7}$$

So, as x increases, the following ratio converges to 1:

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1$$

This repeats the Prime Number Theorem, which was conjectured based on empirical observations by mathematicians like Carl Friedrich Gauss and Adrien-Marie Legendre in the late 18th and early 19th centuries and was proved in 1896 (Diamond 1982, p. 554). This convergence is illustrated in the following diagram, with the dotted curve intended to represent $\pi(x)$ and the dashed curve representing $\frac{x}{\log x}$ illustrating how it approximates the other curve as x grows:



The prime number theorem suggests that the distribution of prime numbers, initially assumed to be a purely coincidental phenomenon, is not entirely random. Further analysis shows that as x increases, $\pi(x)$ shows some predictable behavior. Nonetheless, as Diamond emphasizes, these studies were prompted by the "right questions" about the distribution of the prime numbers. Indeed, the observations made by Gauss and Legendre, who "recast the prime distribution question in a statistical form" (Diamond 1982, p. 553), motivated the studies in the later century that led to the proof of the prime number theorem. However, we saw earlier that the distribution of the prime numbers, *prima facie*, shows no pattern and thus lacks any striking feature except for its complete randomness.

This example shows not only that a seemingly coincidental mathematical fact may have an explanation, but also that, for the purposes of mathematical research, salient features of a mathematical fact may even be a wrong call. The "right questions", however, are those prompting mathematical research, consequently leading to a slight improvement in the direction of problem solving and finally offering a genuine explanation.

4.3. Context-Shift and Mathematical Explanation

Another complementary argument arises from the context-shift. Lange states, "Which feature of a theorem is salient (and whether it has any salient feature at all) depends on the context. Indeed, as the conversational context shifts, a feature that had been salient can retreat into the background as a new feature becomes salient." (2019, p. 14). Lange argues that proofs identifying a fact's grounds, such as those using brute-force methods, lack flexibility in adapting to different contexts. To illustrate, Lange remarks,

> A brute-force approach is not selective in its focus; it simply plugs everything in and calculates everything out. In contrast, an explanation must (on this proposal) pick out particular features of the setup that are similar to the result's salient features, tracing the result's salient feature back to them. (2019, p. 13)

Steiner discusses a similar issue in 1978 (p. 149), noting that some explanatory proofs contain a "nub of the explanation" (i.e., the core of the proof). He addresses this in the context of discussing the explanatory nature of Galois groups in a proof.

Opponents of DT can account for what Lange refers to as context-shift and Steiner as the nub of explanation. According to the duality thesis DuT, the metaphysical relation of grounding differs from the grounding explanation it supports. While the existence of the grounding relation is a metaphysical issue (a determination relation in a non-causal domain), and is not context-dependent, the grounding explanation it backs depends on the context. Furthermore, what Steiner considers the "nub of the explanation" also depends on the context. A proof sketch (i.e., a short survey of the main steps of a proof) might suffice for an expert, while a detailed proof offers a better understanding for a lay reader. This discussion touches on the subjective aspect of mathematical explanation, which involves producing an enhanced understanding of a mathematical fact (i.e., the *explanandum*) based on other mathematical facts (i.e., the *explanans*). Thus, opponents of DT can account for the context shift, arguing that the subjective aspect of the explanation is context-sensitive, among other factors.

In addition, opponents of DT who embrace the separatist theory of grounding (i.e., DeT_G, DuT, and TrT) can account for cases of *interpretations* in the practice of mathematics. The interpretations' *raison d'être* is to study a structure in terms of another structure because the latter is simpler or easier to understand.[25] For instance, model theorists sometimes interpret a complicated first-order structure in a simpler one. According to Marker (2002), any structure in a countable language can be interpreted in a graph. To borrow Hodges' words "When a structure B is interpreted in a structure A, every first-order statement about B can be translated back into a first-order statement about A, and in this way, we can read off the complete theory of B from that of A" (2022, section 2). Therefore, whenever we interpret a structure into another, the sentences regarding one structure will be translated into those pertaining to the other (with possibly a different language). This shows that, in a more general framework, the separatist theory of grounding can account for what Lange describes as context-shift. The following definition is from Marker (2002):

Definition 4.3.1. An $\mathcal{L}_0$-structure $\mathcal{N}$ is *definably interpreted* in an $\mathcal{L}$-structure $\mathcal{M}$ if and only if we can find a definable $X \subseteq M^n$ for some n and we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions on $X \ldots$ so that the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{N}$.

Here is an example illustrating how such interpretations change the context. Consider the multiplicative group of invertible matrices with entries in K, i.e., $(GL_n(K), \cdot, e)$. This structure is definably interpreted in $(K, +, \cdot, 0, 1)$ for a field K. The language for the former structure differs from the latter (the pure field structure). They enjoy different structures and different settings. However, the former is definably interpreted as the latter.

Let us see how the separatist theory of grounding (i.e., DeT_G, DuT, and TrT) can account for cases of interpretations and context shifts. Let A be a true sentence in $(GL_n(K), \cdot, e)$. Let $[B]$ be a ground for $[A]$ in $(GL_n(K), \cdot, e)$. By DeT_G, the truth of $[B]$ determines

[25] Sometimes, mathematicians choose to interpret one structure into another not just for simplicity but also because such interpretations yield a mathematical fact that connects with better-understood mathematical facts. In these cases, mathematicians describe a fact up to the level of definable interpretation.

the truth of $[A]$ in an explanatory way. Interpret $(GL_n(K), \cdot, e)$ in $(K, +, \cdot, 0, 1)$ and translate A to C in $(K, +, \cdot, 0, 1)$. We obtain a different ground for $[C]$, say $[D]$. Since $[C]$ is the interpreted version of $[A]$ in $(K, +, \cdot, 0, 1)$, and due to the transitivity of grounding TrT, we have

$$[D] \prec [C] \prec [A].$$

Therefore, $[D] \prec [A]$. This shows that the separatist theory can account for context shifts, such as from the context of invertible matrices to the context of fields.

Finally, the main reason Lange claims that grounding is not compatible with context-shift is his specific view of the grounding relation, particularly the role of discriminatory AGT in his arguments. In contrast, a separatist theory of grounding, like the one presented here, is compatible with context-shifts and is generally a more suitable notion for describing the complexities of mathematical practice. This includes the existence of multiple proofs for a single theorem, context shifts, and the overall practice of mathematical research.

5. *Conclusion*

In summary, Lange's defense of the Divergence Thesis DT, which posits that mathematical explanations are not a variety of grounding explanations, relies on several arguments that ultimately prove insufficient. Through a minimal separatist theory of grounding, AGT lacks the discriminatory power necessary to sustain Lange's claims. Furthermore, discriminatory AGT is a dubious statement both from a grounding-theoretic point of view and the viewpoint of everyday mathematical practice. The arguments from purity and salience fail to adequately support DT, as they either introduce circular reasoning or fail to provide an adequate account of the intricacies of mathematical practice. Finally, Lange's additional arguments motivated by his salience theory are insufficient to support DT, as they seem to be corollaries to the two main arguments. Finally, this research could offer new insights into how we understand the connection between grounding explanation and mathematical explanation. Considering the current discussion, Lange's arguments, although insufficient to support DT, suggest that if we are to understand mathematical explanation in terms of grounding, it must be through a separatist theory of grounding.[26]

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