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THE PHILOSOPHICAL SIGNIFICANCE OF THE REPRESENTATIONAL THEORY OF MEASUREMENT —RTM AS SEMANTIC FOUNDATIONS

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SUMMARY: The Representational Theory of Measurement (RTM), especially the canonical three volume *Foundations of Measurement* by Krantz et al., is a landmark accomplishment in our understanding of measurement. Despite this, it has been far from easy to pinpoint what exactly we can learn about measurement from RTM, and who the target audience for RTM's formal results should be. In what sense does RTM provide foundations of measurement, and what is the philosophical significance of such foundations? I argue that RTM provides semantic foundations of measurement, and that their philosophical significance lies in a shift towards structural representation.

KEY WORDS: structuralism, structural representation, structural similarity, meaningfulness, invariance

RESUMEN: La Teoría Representacional de la Medición (RTM, por sus siglas en inglés), especialmente la canónica obra en tres volúmenes *Los fundamentos de la medición* de Krantz et al., es un logro histórico en nuestra comprensión de la medición. A pesar de esto, no ha sido nada fácil determinar qué podemos aprender exactamente de RTM acerca de la medición y quién debería ser el público objetivo de los resultados formales de RTM. ¿En qué sentido RTM proporciona fundamentos de la medición y cuál es el significado filosófico de tales fundamentos? Argumento que RTM proporciona fundamentos semánticos de la medición, y que su significado filosófico radica en un cambio hacia la representación estructural.

PALABRAS CLAVE: estructuralismo, representación estructural, similitud estructural, significado, invariancia

1. *Introduction*

The Representational Theory of Measurement (RTM) is often presented as a foundational theory of measurement, particularly in the canonical three volume presentation *Foundations of Measurement* (Krantz et al. [1971] 2007; Suppes et al. [1989] 2007; Luce et al.

[1990] 2007). The claim to being a foundational theory of measurement has been challenged: does RTM even offer a complete theory of measurement? After all, many important aspects of measurement, most notably uncertainty and error, receive little treatment by RTM, and much of the discussion is too abstract to be considered useful for practical applications (Savage and Ehrlich 1992). For the practice of measurement, RTM can easily seem irrelevant (Cliff 1992). For philosophers, however, the significance of RTM might very well lie in its foundational aspirations. Philosophers of science are often concerned with the foundational aspects of the sciences they investigate, viz. the foundations of physics or the foundations of biology, or even the foundations of particular theories, e.g., axiomatic quantum field theory. This is not to say that philosophers of science are *only* concerned with foundational questions, but it might suggest that RTM, as a foundational project, is of more interest to philosophers of measurement, than to some practitioners of measurement. To understand the philosophical significance of RTM, then, we need to ask *in what sense RTM provides foundations of measurement and what these foundations are*. These are the questions I'm concerned with in this essay. I argue that RTM fails to establish epistemic foundations yet provides more than formal foundations of measurement; instead, it provides semantic foundations by providing a formal theory of numerical representation as structural representation. While my reading draws on interpretive hints provided by the authors of *Foundations*, my main aim is to show that such semantic foundations are indeed provided by RTM, regardless of whether this was the main intention of all contributors to the programme.

The paper is organised as follows: section 2 provides a brief overview of the representational theory of measurement. Section 3 investigates whether RTM can provide epistemic foundations for measurement and agrees with recent critics that it cannot. Section 4 looks at the converse move by defenders of RTM, which seeks to restrict RTM to purely formal foundations. I argue that this perspective is too limited: while RTM provides formal foundations, its contributions go beyond that. Section 5 accordingly explores what I take to be the semantic foundations of measurement provided by RTM. Section 6 argues that RTM as semantic foundations is both successful and philosophically significant. Section 7 concludes.

2. *The Representational Theory of Measurement*

Representationalism about measurement goes back to the 19th century (Díez 1997a; 1997b), with the core idea that the use of numbers in measurement is purely *representational*. This contrasts with the idea that measurement is of properties that are themselves somehow "numerical". The representational theory of measurement is best known through the three volume treatise *Foundations of Measurement*, wherein the authors develop a detailed formal theory of different types of measurement structures and their numerical representations. On the face of it, this formal theory is a theory of measurement insofar as measurement is understood "as the construction of homomorphisms (scales) from empirical relational structures of interest into numerical relational structures that are useful" (Krantz et al. [1971] 2007, vol. 1, p. 9). With this apparent definition of measurement in hand, RTM proceeds to prove, for a variety of such "empirical relational structures", first a *representation theorem*, and second, a *uniqueness theorem*.

To prove such representation and uniqueness theorems, the relevant "empirical relational structures" have to be characterised axiomatically. Accordingly, the *Foundations of Measurement* presents a range of such axiomatically characterised structures —most notably, additive extensive structures, difference structures, and conjoint measurement structures. For each type of structure, representation and uniqueness theorems are proven. It is this axiomatic characterisation that, at least initially, seems to be what makes RTM *foundational*. The axioms, broadly speaking, characterise a set of entities and relations over them, in particular ordering relations and, for some structures, a concatenation operation as well. These axioms can be interpreted as holding of particular empirical relational structures, as well as of relevant numerical structures. For example, a set of weights and a beam balance can be interpreted as an additive extensive measurement structure: for two weights, w_1 and w_2, w_1 is heavier than w_2 if and only if, when each is placed in a pan, then pan with w_1 in it is lower than the pan with w_2 in it; if the two pans are balanced, w_1 and w_2 are of equal weight. Placing w_1 and w_2 together in one pan is equivalent to the combined weight of w_1 and w_2. Here, certain empirical relations and operations are interpreted as satisfying the axioms for additive extensive measurement structures. What makes numerical structures suitable representations for certain kinds of empirical structure is precisely that we can find interpretations of axioms that make them hold of the empirical structure on the one hand, and the numerical structure on the other. Constructing a homomorphism from the empirical relational structure to the numerical structure is to demonstrate that the axiomatically characterised structure is preserved in the mapping.

The representation theorem establishes the existence of a homomorphism from the empirical relational structure to a suitable numerical structure. An empirical relational structure, for example a collection of rods $\langle \mathsf{X}, \circ, \succsim \rangle$, which can be compared for relative length, and which can be concatenated, is shown to be representable by a numerical structure, for example the real numbers under addition $\langle \mathbb{R}, +, \geq \rangle$. A homomorphism is a structure preserving mapping, so the representation theorem is established by an existence proof for a certain type of function, φ. Typically, there will not be a single such function, but a family of homomorphisms. To delineate the family of homomorphisms that yield appropriate representations is the purpose of the uniqueness theorem.

The uniqueness theorem, roughly, shows how unique the resulting representation is, that is, which changes to the mapping count as preserving the same structure. For example, because Celsius and Fahrenheit are interval scales, ratios formed using numerical values from these scales are not legitimate: we cannot say that it's twice as warm today as it was yesterday, even if it is true that today the temperature is 6 degrees Celsius and yesterday's temperature was 3 degrees Celsius. We cannot do so, because ratios are not invariant on interval scales, as can be seen from the fact that the equivalent numerical values in Fahrenheit do not form the same ratio: 37.4 degrees Fahrenheit and 42.8 degrees Fahrenheit do not stand in a 1:2 ratio. In the locution employed in measurement theory: ratio comparisons on interval scales are *meaningless*. Formally, a uniqueness theorem shows, which transformations of the initial homomorphism are permissible, that is, which $\varphi \rightarrow \varphi'$ are such that both φ and φ' are homomorphisms to the same numerical structure. The uniqueness theorem thereby tells us which structure is *invariant* in the mapping from the empirical relational structure to the numerical structure. Depending on the type of permissible transformation, the resulting representations are of different "strengths", along the lines of the hierarchy of scales originally introduced by Stevens (1946).[1]

[1] This hierarchy of scales is further developed and improved in the development of the representational theory of measurement. Especially results collected in volume III of *Foundations* indicate a more sophisticated understanding of uniqueness properties (Luce et al. [1990] 2007, chapter 20). In general, where Stevens'

Two further, interconnected points are worth mentioning beyond this initial characterisation of RTM. First, uniqueness theorems only capture permissible transformations of homomorphisms to *the same numerical* structure. Further variation in numerical representation arises from the choice of different numerical structures to represent the same empirical relational structure. In particular, the binary operation on an ordered extensive structure can be interpreted as addition —as in additive extensive structures—, but can also be interpreted as multiplication (Krantz et al. [1971] 2007, vol. 1, pp. 99f). Given the axiomatic characterisation of the relational structure, there is no formal reason not to do so. Second, in part in response to this first point, later more abstract work in the representationalist programme moves away from characterising uniqueness in terms of permissible transformations and moves instead towards characterising uniqueness more abstractly in terms of the automorphisms of structures representable at interval and ratio scales (Luce et al. [1990] 2007, section 20.2; Narens [2002] 2012, chapter 2). These points will become relevant in the discussion below.

3. *RTM as Epistemological Foundations of Measurement*

One reading of RTM's foundational ambition is to understand the project as a foundationalist epistemology for measurement.[2] Such epistemic foundations for measurement would demonstrate how the formal machinery of RTM, together with empirical observations, can be used to put measurement, and more specifically, numerical representations of phenomena, on a firm basis. It would offer a key element of an empiricist, foundationalist epistemology for science. Such firm, formal-empirical foundations would obviously be desirable as they would serve to justify measurement where they could be found to support numerical representations, while also forming the basis of criticism of numerical representations where no such foundations are available (Michell 1999). They would furthermore provide philosophical justification for the exalted status of measurement in science: if measurement results could be secured through a chain of

hierarchy seems to have been drawn from examples of measurement scales, *Foundations* aims to provide an overall mathematical theory of different measurement scales.

[2] This reading has also sometimes been called "evidential" (Tal 2021), in contrast to the "conceptual" reading I will be discussing in section 4. In addition to some of the authors of *Foundations*, the evidentialist project can be found in some traditional approaches to measurement (Campbell 1920; Carnap 1966) and has more recently been defended by Joel Michell (1999) as a form of measurement realism.

careful formal inferences from empirical observations to numerical representations, this would provide good reason for taking measurement to be a particularly secure form of empirical knowledge.

How might such epistemic foundations be provided by RTM? In the first instance, RTM shows —through its representation theorems— that a particular "empirical relational structure" may be represented by a particular numerical structure. Given a suitable axiomatisation of the empirical relational structure, RTM shows in a formally rigorous way that a numerical representation of that empirical structure is possible. Moreover, through the uniqueness theorem, RTM shows how strong this numerical representation is, which suggests what kinds of inferences we may draw from the numerical representation. So, once the axiomatic characterisation of the empirical structure in question is in place, RTM provides us with solid, formal tools for finding suitable numerical representations for the relational structure in question. But here is the one-million-dollar question: what justifies a particular axiomatization of the empirical relational structure?

Before we can address this question directly, we first need to clarify what is meant by an empirical relational structure. There are at least two candidates for empirical relational structures: on the one hand, one might have in mind a particular material collection of items, together with specific interventions and procedures. This is suggested by some of the measurement structures provided as examples in the *Foundations of Measurement*, for example a collection of rods together with the procedure of ordering them by length, and an intervention for concatenating them end-to-end, or by the example of weight measurements given above (Krantz et al. [1971] 2007, see, e.g., section 3.6., for physical interpretations of extensive measurement structures).

On the other hand, the "empirical relational structure" might also be interpreted as data-structure, and indeed, a data structure that has been adjusted for outliers and errors. Such cleaned-up data structures may already contain idealising assumptions and are generally abstractions from the raw data. Both interpretations of empirical relational structure are present in RTM, with the former suggested by the examples, while the latter is occasionally offered as a way of reading the empirical relational structure in question (e.g., "the choice of an empirical relational structure as an abstraction from the available data" (Krantz et al. [1971] 2007, vol. 1, p. 13)).

In order for RTM to play the kind of epistemic role envisioned by the epistemological foundationalist project, the first interpretation is the most relevant way of understanding "empirical relational structure". If "empirical relational structure" is understood as a cleaned-up data model, then questions about empirical foundations simply move to the step of constructing the cleaned-up data model. Yet RTM says little about the construction of such abstracted data models.[3] So, let's have a look at the understanding of "empirical relational structure" as a material system on which we may intervene.

Putting the measurement of such a structure on firm foundations requires that we axiomatize the structure, or that we show that the structure in question satisfies the axioms for one of the measurement structures described by RTM. This "formalisation" or "axiomatisation" is key to the foundations of measurement, as far as RTM is concerned (Krantz et al. [1971] 2007, vol. 1, p. 13). How do we choose a suitable axiom system for a given empirical relational structure? There are different types of axioms we need for the formalisation: necessary axioms and non-necessary or "structural" axioms (Krantz et al. [1971] 2007, vol. 1, pp. 21–23). An example of the former is the imposition that the binary relation on the empirical relational structure must be taken to be transitive; this requirement arises because the intended representing relation, $\leq$, is transitive. So here we have a case of imposing a feature of the numerical structure on the axiomatisation of the empirical relational structure. Non-necessary axioms come in a variety of flavours. Some of them are needed to rule out empirically uninteresting structures, by explicitly excluding trivial ways in which the axioms might be satisfied. Others are needed to ensure solvability for certain equalities and inequalities, for example by stipulating that the structure be sufficiently dense to contain an element c in between any two elements $a \lesssim b : a \lesssim c \lesssim b$.

This means that the process of finding a suitable axiom system will typically be bi-directional: on the one hand, we need to look at the numerical structure we wish to use to represent the empirical relational structure, to identify key features —such as transitivity for the binary relation— that we require an empirical relational structure to have in order for it to qualify as representable by the numerical structure. On the other hand, we need to look to the empirical structure to identify features we are interested in and to distinguish cases of structures trivially satisfying the axioms from those that non-trivially satisfy them.

To what extent can we interpret RTM as an attempt at providing epistemic, and indeed empiricist, foundations of measurement? Es-

[3] We will re-encounter this second reading in section 6.

pecially the early chapters of *Foundations of Measurement* suggest that RTM is in the spirit of empiricist foundations:

> What criteria, then, do we impose on an axiomatization for it to be satisfactory? One demand is for the axioms to have a direct and easily understood meaning in terms of empirical operations, so simple that either they are evidently empirically true on intuitive grounds or it is evident how to systematically test them. (Krantz et al. [1971] 2007, vol. 1, p. 25)

Here, the authors of *Foundations of Measurement* seem to subscribe to a project broadly in the spirit of earlier empiricist foundationalists, like Campbell (1920) or Carnap (1966). The theoretical terms of the axioms, including in particular, the relations and operations characterised by the axioms, are not to be understood merely as implicitly defined by the axioms, but require an empirical interpretation that is evident or intuitive, for example by identifying the binary operation on the structure with a particular empirical procedure. The idea very much seems to be that by looking at an empirical relational structure, we should have an observational or intuitive sense of how the axioms we are choosing to describe the system might apply to it, and in particular, how the relations and operations on the structure are to be understood. Similarly, whether axioms are appropriate is not just a matter of simplicity or elegance, but of descriptive adequacy: "The axioms purport to describe relations, perhaps idealized in some fashion, among certain potential observations, and adequacy of description is a more telling arbiter than beauty or simplicity" (Krantz et al. [1971] 2007, vol. 1, p. 27).

In taking "adequacy of description" as the primary criterion by which to judge the quality of an axiomatisation, the authors of *Foundations of Measurement* make it clear that the axioms are not to be understood as mere mathematical convenience; the formalisation is meant to capture something about the empirical relational structure. This seems to have two implications: first, the axiomatisation could get the empirical relational structure *wrong*, and second, there might be one axiomatisation that is better than all competitors on account of being the most descriptively adequate. Neither of these apparent implications is explicitly addressed by RTM, but they turn the empiricist foundationalist project in a distinctively realist direction. An alternative way of treating these axioms would be to allow for more pluralism in the evaluation criteria for axiom systems: simplicity could be traded off against strength, and either of them could be traded off against adequacy. If a more highly idealised axiom system, assuming, for example, error free transitivity or existence of relevant entities to satisfy various solvability axioms, serves to provide a simpler, more effective axiomatisation, this might be preferred over a more descriptively accurate one, which seeks to capture the various ways in which in the empirical structure fails to satisfy simpler or stronger axioms. Crucially, by elevating descriptive adequacy to the most important criterion for judging an axiom system, the authors of *Foundations* appear to reject such pluralism of criteria in favour of more monist view, which in turn invites a significant reduction in pluralism of axiom systems. Where from a pluralist view, choosing axioms for formalising a particular empirical relational structure would seem to be *pragmatic* —allowing for different preferences and competing axiomatisations, depending on local purposes and interests— the proposal here seems to be *epistemic*: there are better or worse ways of axiomatizing a given empirical relational structure, depending on how descriptively adequate they are. This commitment invites criticism.

First of all, there are several ways in which concrete material structures will simply fail to satisfy the axioms entirely: material structures will be finite, whereas the axioms typically assume infinite structures. Even for these finite structures, interventions and procedures like aligning rods end-to-end or placing metal cylinders on a beam balance are only feasible for a limited range of lengths and weights and under certain conditions: don't try using a beam balance in space. And even for cases where we can carry out the procedures in question, there is a finite degree of observational resolution: some differences will simply be too small to detect. This can lead to the introduction of error, whereby pairwise comparisons of a series of rods might lead us to judge that they are the same length, yet comparing the first and last members of such a series of pairwise comparisons reveals this pair to be unequal in length.[4]

The trouble for RTM as epistemic foundations of measurement is not merely that this means the fit of the axioms to any concrete material structure will at best be approximate or incomplete, but also that anyone attempting to make a case for representing a particular phenomenon as satisfying the axioms for a particular measurement structure will be faced with difficult and controversial choices: is

[4] These criticisms have been made in a variety of ways (Kyburg Jr. 1990; Savage and Ehrlich 1992), and the authors of *Foundations* were well aware of some of these concerns (e.g., Krantz et al. [1971] 2007, vol. 1, pp. 27–28).

the appearance of error in the pairwise comparisons a matter of observational resolution, or does it indicate a change in the standards we used to carry out our measurement procedures? Can it be fixed, by more precise measurement procedures, or is it a fundamental problem with the attribute in question?

A more recent challenge to RTM as epistemic foundations has been put forth by Eran Tal. Tal's objections target not merely the way in which RTM's treatment of empirical relational structures seems to require idealising assumptions, but the very idea of a foundationalist epistemology for measurement. In particular, Tal argues that RTM as a foundationalist epistemology for measurement falls prey to two myths: the "myth of well-ordered data", according to which "it is possible to infer the qualitative structure of objects and events from empirical data independently of any prior attribution of quantitative structure to those objects and events" and the "myth of qualitative foundations", according to which "the adequacy of numerical representations can only be justified by a homomorphism from an independently established qualitative structure to a quantitative one" (Tal 2021, p. 704). As Tal attributes only the first of these to RTM as presented in *Foundations*, I shall set aside the second myth in the discussion here.

The first of these myths, according to Tal, is prevalent in all evidential interpretations of RTM, and it is arguably a presupposition of the epistemic foundationalist project. If the aim is to justify that a particular attribute or phenomenon can be represented numerically, we must assume that we can characterise the phenomenon in such a way as to not already presuppose its numerical representability. So, in particular, in characterising a structure as, e.g., a closed extensive structure, this characterisation should not be informed or justified by an (implicit) assumption that a certain numerical representation of the structure is appropriate. Doing so would be circular, thereby undermining the foundationalist ambition of the project. Tal argues that such a non-quantitative characterisation is impossible. In particular, the presence of systematic error, especially systematic non-linear error means that the data themselves will violate transitivity or other axioms (Tal 2021, p. 719). Unlike random error, discussed above, correcting these errors requires imposing a quantitative structure on the attribute under measurement (Tal 2021, p. 718). In order to construct a mapping from the data structure to the numerical structure, we therefore have to make corrections to the data structure that already *presuppose* that the attribute in question is quantitative, thereby undermining the claim to finding out *whether* the attribute is quantitative in the first place. Without such presuppositions, the data would not yield a structure amenable to systematic numerical representation. At least one important ambition of the epistemic foundationalist project is thereby undermined: we cannot demonstrate, purely on the basis of qualitative data and formal theorems, that numerical representations (of a certain strength) are justified.

These concerns throw into doubt RTM's claim to offering epistemic foundations of measurement. RTM provides an epistemically secure route from axiomatized "empirical relational structures" to axiomatized "numerical relational structures", through its representation and uniqueness theorems. But when it comes to the justification for axiomatizing a particular phenomenon *as a particular relational structure*, RTM has little to offer. While we might point to the pragmatic success of axiomatizing a phenomenon a particular way as a reason to keep doing so, such a justification seems coherentist, not foundationalist. Chang, for example, interprets the quantification of a phenomenon like temperature as an iterative process, where earlier steps are revised in light of later improvements (Chang 2004, ch. 5). Crucially, what counts as a success will be shaped by our understanding of what we take the quantitative structure of the phenomenon to be (Chang 2004, pp. 212–217). This is in sharp contrast to the foundationalist project, which aims to anchor the numerical representation of the phenomenon firmly in purely qualitative terms, and to establish, once and for all, what the quantitative structure of the phenomenon is. The criticism of RTM as epistemic foundations is not that our use of measurement representations is unjustified, but that a foundationalist justification is impossible. If RTM fails to provide epistemic foundations of measurement, what alternative ways of understanding the foundational project are there?

4. *RTM as Formal Foundations*

In light of the difficulties of treating RTM as providing epistemic foundations for measurement, some RTM friendly philosophers have recently proposed a different, less ambitious reading of RTM. Instead of providing epistemic foundations of measurement, RTM's remit is more limited: it provides a library of mathematical theorems, useful as a tool in scientific concept formation (Heilmann 2015). This new interpretation of RTM departs from the epistemic interpretation in two steps: first, by viewing RTM as providing a library of mathematical theorems, it reduces RTM's foundational ambitions to purely formal foundations. Second, by dropping the idea that what is being mapped is an *empirical* relational structure, Heilmann's interpretation opens up the possibility of applying RTM to abstract concepts as well. As Heilmann sees it, the great advantage of this interpretation of RTM, aside from avoiding some of the criticism presented in the previous section, is that RTM can be explicitly used as a tool in concept formation, i.e., in constructing (or engineering) concepts with the aim of mathematically representing them. This is a concern not merely in science, but also in certain areas of philosophy. Similarly, Baccelli (2020) argues for a more abstract take on the role of axiomatic measurement theory: the aim is not to capture measurement procedures, but to describe idealised measurement structures, whether used in science or not.[5]

This new interpretation of RTM certainly seems to fit better with what RTM actually delivers: on the face of it, *Foundations of Measurement* is a mathematical textbook containing a large number of theorems for a variety of relational structures. Moreover, the more abstract developments of measurement theory in the later books of *Foundations of Measurement* point away from the more obviously empirical foundations attempted in volume I. Indeed, by volume III, the axioms are no longer classified into *necessary* and *non-necessary* (as in volume I, see discussion above), but instead are distinguished into: design axioms, technical axioms, empirical axioms, and idealised axioms (Luce et al. [1990] 2007, vol. 3, pp. 251–252; 28.1). Of these, only empirical axioms can be expected to be tested "under some standard interpretation of the primitive concepts" (Luce et al. [1990] 2007, vol. 3, p. 252). While these are still regarded as the most interesting kind of axioms, there is no longer an expectation that these interpretations will be empirically obvious; nor is there an expectation that descriptive adequacy trumps all other considerations. Finally, and perhaps most remarkably, this classification of axioms does not reflect the formal features of the axioms, but is context dependent:

> The important point is that the status of an axiom in terms of the four classes identified cannot be inferred from its mathematical form alone. Its status depends as well on the intended interpretations of the primitive concepts and, more loosely, on prior knowledge about the

[5] Tal suggests a further *heuristic* interpretation of RTM as a third alternative to the epistemic and formal (or "evidential" and "conceptual") foundations discussed here (Tal 2021, p. 733). But while his heuristic spin on the conceptual interpretation indicates how RTM might be useful in the empirical sciences, I don't think it offers any further sense in which RTM might be regarded as foundational.

> empirical domain and the kinds of experiments that are conducted in this domain in which the primitive concepts are ordinarily interpreted. (Luce et al. [1990] 2007, vol. 3, p. 253)

The later work in the *Foundations of Measurement* seems to fit well with Heilmann's more modest formal interpretation of the foundationalist project.

Despite this promising fit between the new interpretation and some of the later work in the representational theory of measurement, the restriction to purely formal foundations can seem like a consolation prize. After all, if the ambition had been to provide rigorous epistemic grounds for measurement, to end up merely with a formal framework that offers useful heuristics seems disappointing. Heilmann rightly insists that giving up on the stricture that the represented structure be *empirical* will greatly widen the applicability of RTM, and thereby increase its relevance to a range of fields, including philosophy, which are not in the first instance concerned with empirical data. Yet, the criticism that RTM is largely useless for actual measurement practice seems to gain force from this interpretation. While RTM might force researchers to distinguish clearly between assumption made for simplicity (idealising axioms), assumptions built into the design of the experiment (design axioms), assumptions made to enable the use of convenient mathematical tools (technical axioms), and finally empirical assumptions they are actually going to test (empirical axioms), the difficult work of deciding, how any particular assumption should be classified falls squarely onto researchers in a particular field. RTM provides no further guidance.

Perhaps the second advantage Heilmann claims for his new interpretation might help here. Instead of asking, how we might get from a qualitative structure to a numerical representation, Heilmann thinks RTM can also help us "backward engineer" the qualitative foundations of a given numerical representation (2015, p. 794). What he has in mind are cases where concepts like "happiness" are being numerically represented in various ways, but we find disagreement over the concept itself. RTM, Heilmann suggests, might help clarify what it would take for happiness to be numerically representable and further, we might then try to find the requisite empirical relations to "sustain both the concept of happiness and a theorem in RTM" (2015, p. 794). It is not entirely clear to me, why this form of backwards engineering requires Heilmann's new interpretation though, especially since the second, epistemic step of searching for empirical relations to bear out the theorems stipulated by RTM, seems to revert to the epistemic reading of RTM, as Heilmann himself acknowledges. It thereby seems at risk of the objections against such a reading discussed in section 3.

More importantly for the purposes of this paper, it seems less clear in what sense RTM provides *foundations of measurement* on this reading, as opposed to a useful collection of theorems and examples of mathematical structures for a wide range of possible applications. While many philosophers will appreciate a formal theory as an aid to conceptual clarity, the new interpretation seems to refrain from reading RTM as making claims about the nature of measurement. If RTM provides formal foundations of measurement, we would expect them to reveal something about measurement, not merely about the foundations of the particular concepts to which RTM may be applied. The reading of RTM as providing epistemic foundations of measurement provided a clear sense of why such foundational work would be philosophically significant: it would be a cornerstone for an empiricist epistemology of science. By contrast, the more modest proposal of RTM as a nifty piece of applied mathematics, while perhaps useful to some scientists and philosophers, lacks the promise of shedding light on foundational problems of measurement. While Heilmann is entirely correct that his reading of RTM is compatible with RTM as foundations of measurement, he makes it clear that the virtue of his reading is precisely to refrain from treating RTM as a full-fledged theory of measurement. Even if the new, modest interpretation offers a plausible reading of RTM, the question remains whether RTM can be understood as foundational at all.

5. *RTM as Semantic Foundations*

It is my contention that RTM is philosophically significant because it provides *semantic* foundations of measurement. RTM provides semantic foundations, insofar as its main concern is with the *representation* of measurement. To understand the philosophical significance of RTM, we need to have a careful look at what RTM's understanding of the representation of measurement tells us about the nature of measurement and about the nature of numerical representation.

The problem RTM addresses is how we can use numbers to represent empirical phenomena. This is a fundamental and philosophically interesting problem arising from measurement, although it is of course not the only philosophically interesting problem of measurement. The concern is best understood in the broader context of the problem of why mathematics is so effective in its application to physics and the empirical world more broadly (Wigner 1960; Benacerraf 1965). While not all applications of mathematics in science are cases of measurement, measurement provides a paradigmatic case of how mathematics, and numbers in particular, seem to apply to empirical phenomena.

The representational theory of measurement answers this challenge in two steps: first, by suggesting that the role of numbers is *representational* and second, by showing how numbers can play this representational role through their *structural*, rather than their intrinsic features. In these two steps, RTM has transformed the way we think about the role of numbers in measurement. Let's have a look at each step.

To say that the role of numbers in measurement is representational is to reject the idea that attributes like mass or length are inherently numerical. Numbers are *not* out there, as part of the empirical world we are trying to describe with our theories.[6] Instead, numbers are strictly part of the representational machinery we use to describe the physical world. It is important to note that in the *Foundations of Measurement*, this representationalism concerns the role of *numbers*, not merely numerals.[7] Numerals are of course representational features, insofar as they are symbols. The authors of *Foundations of Measurement*, by contrast, clearly take numbers and other mathematical objects to be playing a representational role. This is due to the second innovation of *Foundations*, which is the move to numerical *structures* as the representing entity and to *structure preserving maps* as the means by which a representation is achieved. Numbers do not represent individual magnitudes on their own, but by being part of a relational structure. The representation of the empirical phenomenon is achieved through a homomorphic mapping from a phenomenon —understood as an empirical relational structure— to a numerical relational structure. It is numbers, not numerals, which form the relevant structure. The representational theory of measurement, as presented in *Foundations* and in the subsequent literature

[6] This does not preclude Platonism about mathematics itself, i.e., the idea that mathematical objects are out there as independent abstract entities. But as such, these mathematical entities are not the target of measurement. Rather, the representational view opposes views of numbers in nature, like Michell 1999; Forrest and Armstrong 1987.

[7] As Berka (Berka and Riska 1983, chapter 2) has pointed out, the representationalist tradition contains a certain amount of ambiguity in the use of "numeral" and "number", going back at least to Campbell.

is a theory of structural representation: representation of structures, by structures.

The increasing importance of structure in science and in scientific representation has of course been noted before. A variety of *structuralisms*, both in mathematics and philosophy of science have been around since the mid-20th century (Benacerraf 1965; Maxwell 1971; Worrall 1989; Shapiro 1997; Ladyman 1998; French 2014). In mathematics, structuralism has often been offered as a form of mathematical realism that avoids commitment to mathematical objects (Shapiro 2000), while in the philosophy of science, structural realism has been offered a moderate form of scientific realism (Worrall 1989; Ladyman 1998). Both types of structuralism share the idea that *relations*, not objects are central to mathematics and physical science respectively.

We are apt to miss the structuralism embedded in RTM, precisely because *Foundations of Measurement* does not offer an explicit structuralist philosophy for either mathematics or science. Instead, a form of structuralism is implied by RTM's famous statement that measurement is "the construction of homomorphisms (scales) from empirical relational structures of interest into numerical relational structures that are useful" (Krantz et al. 1971, p. 9). This claim has often been criticised as an inadequate or incomplete definition of measurement, and rightly so. It seems too narrowly focussed on numerical representation as the defining feature of measurement —no mention is made of the actual experimental or observational processes needed to describe the empirical relational structure (Frigerio, Giordani, and Mari 2010). I propose instead, that we take this statement as an expression of RTM's structuralism.[8] The problem of measurement addressed in *Foundations* is the problem of numerical representation, and this problem can be solved by adopting a structural understanding of how numbers can represent empirical phenomena.[9]

What is the philosophical significance of moving to structural representation as the account of how numbers apply to empirical phe-

[8] Wolff (2019; 2020) offers a reading of RTM, which also emphasises the role of structures. But where Wolff (2020) investigates the metaphysical implications of RTM for our understanding of quantities, the present paper offers a reading of RTM as semantic foundations of numerical representations.

[9] A more explicit commitment to structuralism by one of the authors of *Foundations* can be found in Suppes' later book *Representation and Invariance of Scientific Structures*: "We cannot literally take a number in our hands and apply it to a physical object. What we can do is show that the structure of a set of phenomena under certain empirical operations is the same as the structure of some set of numbers under arithmetical operations and relations" (2002, p. 4).

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nomena? The first and most important consequence of the shift to structural representation is the articulation and partial solution of the problem of artifacts or conventional features in numerical representation. Such features are present in a numerical representation, but should not be taken to be significant or "meaningful". This problem was known prior to RTM: everyone was aware that choices of unit are conventional, for example. Representing mass in pounds or kilogram is not the expression of a substantive disagreement, but a difference in representation only. Nonetheless, measurement representations will contain a choice of unit, even if such a unit is acknowledged to be conventional.

What *Foundations* offers is a more systematic account of how such arbitrary features arise and, more importantly, how to correct for them. RTM treats such features as *surplus* structure.[10] Uniqueness theorems are built on the idea that in order to identify the surplus structure in a representation, we need to focus on the structure that is *invariant* across all the different representations. Structure that is not invariant across all the representations is surplus —an artifact of the particular representation. Such surplus structure is thought not to correspond to anything in the empirical structure represented, and accordingly does not legitimize inferences that depend on this non-invariant structure. By demanding that in addition to a representation theorem, a uniqueness theorem must also be proven, RTM puts the problem of surplus structure at the heart of the account of numerical representation. Uniqueness theorems are meant to identify the invariant structure, thereby protecting us from drawing illegitimate inferences based on representational artifacts.[11]

Uniqueness theorems provide only a partial solution to this problem, however, as they only help us to understand how mappings to the *same* numerical structure are related to one another. Uniqueness does not provide any help in selecting the numerical structure in the first place. The authors of RTM were well aware that not only are different mappings of a phenomenon to the same numerical structure

[10] The term "surplus structure" (Redhead 2003) is more commonly used in the philosophy of physics to discuss what, at least from the perspective of RTM, would appear to be a related phenomenon, namely the presence of symmetries in physical theories.

[11] There is a tight connection between the treatment of numerical representations as structural representations and the use of invariance to test for meaningfulness. RTM is here heir to the Erlanger Programme in geometry, which identified different geometries using different transformation groups. See Narens [2002] 2012; Wolff 2020 for further discussion of this relationship.

possible, but it is also often possible to construct a structure preserving map to a *different numerical* structure. For example, an ordered set of lengths under concatenation will typically be represented using an additive extensive structure, consisting of the real numbers under addition, but it is equally possible to represent this empirical relational structure using the real numbers under multiplication instead (Krantz et al. [1971] 2007, vol. 1, pp. 99–102).[12] So, in addition to containing arbitrary or conventional features, a particular numerical structure is typically not *uniquely* suited to represent a given phenomenon. Instead, a numerical structure has to be *chosen* as the representing structure. Uniqueness theorems, then, do not provide a complete answer to the problem of conventionality or arbitrariness of numerical representations, as not all arbitrariness arises from surplus structure.[13]

6. *How Successful is RTM as Semantic Foundations?*

The choice of numerical structure is a manifestation of a more fundamental objection to RTM's strategy of semantically grounding measurement: to describe a phenomenon as an empirical relational structure is already to impose a certain way of conceptualising and structuring the phenomenon. The structure itself is not empirically "given". The concern here, I take it, is not merely that the empirical phenomena typically *underdetermine*, which relational structure we should attribute to them, but rather, that the phenomena themselves are *structurable* in a variety of ways, and that there is no particular reason for thinking that one such structuring is preferable over others. To say that the axioms hold of a particular phenomenon is just to assert that the phenomenon has the requisite structure. As van Fraassen puts it: "The metaphysical realist's response depicts nature as itself a relational structure in precisely the same way that a mathematical object is a structure" (2008, p. 242). This concern goes beyond the epistemic worry whether we can find out, using empirical means only, what the relational structure of a phenomenon might be, and instead questions whether phenomena can be said

[12] Formally, the reason for this is that the binary operation characterised by the axioms for additive extensive structures can be interpreted either as addition or as multiplication (or indeed in a number of other ways), while preserving the axiomatic structure.

[13] Moreover, not all structures will have invariance properties that allow the application of invariance as tests for uniqueness. See Narens [2002] 2012 for discussion and for the development of alternative concepts of meaningfulness.

to have relational structures at all. As van Fraassen develops the objection, the problem is that, while it is clear what it means for mathematical objects to be relational structures —this can be understood set theoretically, it is not clear how we should understand the claim that concrete, material phenomena, such as some weights and their behaviour on beam balances form a relational structure. Sets, and relations on them, are *abstract* entities. To stipulate that an empirical phenomenon has the requisite mathematical structure is not to solve either the epistemic problem of finding out, what that structure is, nor to solve the more difficult semantic problem of connecting the phenomenon to its structural representation, let alone the metaphysical problem of saying what it means for a phenomenon to have a relational structure in the first place (and to have *this* relational structure, as opposed to some other relational structure).

This objection challenges the claim to semantic foundations by rejecting the idea that descriptive adequacy is even an eligible goal for a numerical representation, at least if the description is meant to adequately capture the structure of the phenomenon. That structure is *imposed*, not only in the sense that we are prone to looking for structure that will meet the axioms we would like to use in our representing structure, but because it involves treating the phenomenon as having a mathematical structure in the first place. This problem goes deeper than the worry about alternative numerical structures, because it questions the sense in which a structural representation can relate a concrete, empirical phenomenon to an abstract, mathematical structure. Van Fraassen seems to suspect that we are being misled into thinking the semantic problem of representing empirical phenomena using mathematics has been solved simply by *stipulating* that the phenomena have the requisite mathematical structure. But this does little to solve the original philosophical worry about the applicability of mathematics to empirical phenomena. If the structure the phenomena are meant to have is simply mathematical structure projected back on them, this is hardly a solution. If so, RTM as a semantic foundation of measurement would seem to have failed.

Is there a response on behalf of RTM as a semantic foundation of measurement? It seems to me there is. A first important step is to disentangle different aspects of the objection. First, there is the observation that empirical phenomena are structurable in a variety of way, an objection that finds support in the existence of alternative numerical representations. Second, there is the concern that mappings from concrete phenomena into abstract structures are not well defined or understood.

The first point needs to be conceded but does not seem devastating to the semantic project of RTM. It is a problem for a certain kind of (metaphysical) realist, who wishes to insist that there is a *unique* structure in the phenomena that our representations must aim to match.[14] RTM's semantic foundations need not adopt this kind of realism but could settle for a more pluralistic view of which representations are permissible. To retain a more modest realism, all that is required is that the representation tracks *some* structure in the phenomena, without laying claim to there being a unique structure to be captured. Indeed, this pluralistic understanding of the structure of a phenomenon fits well with the more pragmatic elements in RTM, according to which the representing structures have to be useful, with the implication that different uses might call for different representations. More importantly, RTM's contribution to clarifying this problem is clear: we understand both representational conventionality arising from equivalent mappings *and* representational conventionality arising from different choices of numerical structure better, thanks to RTM, even if not all representational conventionality is due to surplus structure. We can now say that one variety of conventionality arises as surplus structure, which determines the scale type, while the other variety of conventionality is due to a different choice of numerical structure.

The second objection is more problematic. There is a prima facie concession, made explicitly by Suppes, who acknowledges that the homomorphic mapping in fact holds between two abstract structures: the empirical model and the numerical structure.[15] If the homomorphic mapping constructed in the representation theorem holds between two abstract structures, however, it might seem that RTM does not even begin to offer semantic foundations. After all, we might have expected such semantic foundations to offer an account of how numbers are connected to the empirical phenomenon. Instead, it seems, we have an account of how one abstract structure is related to a different abstract structure. Insofar as there is a homomorphic mapping of one to the other, what is represented is not an empirical phenomenon, but another abstract structure. RTM seems to fail as

[14] Perhaps the most vocal and explicit contemporary defender of this form of realism is Ted Sider (2011).

[15] "[T]he concept of an empirical model used here is itself an abstraction from most of the empirical details of the actual empirical process of measurement. The function of the empirical model is to organize in a systematic way the *results* of the measurement procedures used" (Suppes 2002, p. 4, n. 1, emphasis in original).

semantic foundations on account of having lost traction with the empirical world.

To see how we might respond to this worry on RTM's behalf, it is worth considering the traditional empiricist solution to the problem of "tying" abstract representations to empirical phenomena: coordinative definitions. Unlike a linguistic definition, which defines one linguistic term in terms of other linguistic items, a coordinative definition is meant to connect empirical phenomena or concrete entities to abstract representations by stipulation. Most explicitly this can be taken to be the case when defining a unit through an artifact, like the former standard kilogram. The connection between the representation "1kg" and the world is explicitly stipulated. Critics of RTM as a semantic foundation of measurement find either that RTM fails to provide anything in the vicinity of coordinative definitions, or worse, that RTM might seem to sidestep the problem by means of homomorphic mappings, when such mappings are in fact insufficient, because they link abstract structures to one another, not empirical phenomena to numerical representations.

Interestingly, Suppes at least was not only aware of the idea of coordinative definition, but rejected it as inadequate, for some of the same reasons invoked by more recent discussions of coordination and operationalisation: it's more complicated than that. From his 1969 *Models of Data* (Suppes 1969) onwards, he made it clear that the relationship between theory and world is complicated and mediated by a whole hierarchy of models: "It is even a bowdlerization of the facts to say that coordinating definitions are given to establish the proper connections between models of the theory and models of the experimental results" (Suppes 2002, p. 7). What is needed, according to Suppes, is not a theory of coordinative definition, but instead a presentation of the statistical techniques used to interpret evidence, and close attention to the various interceding models needed to connect the experiential data to the high-level theory (if any).

This leaves us with a puzzle, though. If the homomorphisms described by RTM hold between abstract structures, and the connection between experimental data and high-level theory is so complicated as to require mediation by multiple interceding models, then how do our numerical representations become representations of particular empirical phenomena? Neither homomorphism nor coordinative definition seem to be up to the task. And consequently, if RTM cannot provide an account of this connection, in what sense might we think of RTM as semantic foundations?

My proposal is that RTM provides semantic foundations because it claims that what is being represented by numerical representations is structure. RTM shows us, what numerical representations, and measurement representations in particular, tell us about the phenomena: *they tell us something about the structure of phenomena, and nothing else*. Let's unpack this a bit. If a numerical representation is a representation of the structure of an empirical model, where an empirical model is understood as a systematic organisation of the measurement outcomes, then the numerical representation can only tell us about the structure of these outcomes, not any of their intrinsic, non-structural features. This is not to say that a phenomenon under measurement does not have any such non-structural features, but these will not be included in the numerical representation. This has important consequences, because the numerical representation will be used in reasoning about the phenomenon in question. The knowledge we obtain from measurement is hence structural, because our numerical representations of measurement convey structural features of the phenomena measured, and it is only (some of) these structural features that may enter our inferences. This is a bold understanding of what measurement tells us about the world, and one that is not especially clearly argued for in *Foundations*. Instead, a structuralist understanding of numerical representation seems to be presupposed, and the argument in its favour appears to be in terms of the benefits we reap: look what we can do if we understand numerical representations in terms of homomorphic mappings of different types of uniqueness!

RTM then provides semantic foundations in a limited, but at the same time revolutionary way. The foundations are limited, insofar as they do not show us how to justify a particular numerical representation of a phenomenon. This task is too much bound up with the epistemic difficulty of determining what the structure of a phenomenon might be, which suffers from severe underdetermination. Instead, RTM provides a semantics of numerical representation as structural representation, with sophisticated rules of engagement for this kind of representation. A numerical representation is meant to capture something about the phenomenon in question by being similar to the phenomenon, which is what enables us to reason indirectly about the represented phenomenon, using numbers. RTM maintains that this similarity is structural similarity. RTM investigates some of the usage conditions for numerical representations as structural representations, especially the conditions for "meaningfulness" of such representations. The semantic foundations provided by RTM, then, are not primarily about reference —how do we get the numerical representation to be about the phenomenon it represents, but rather about inference— how do we ensure our indirect reasoning about the phenomenon using a numerical representation is warranted.[16]

This interpretation of the semantic foundations provided by RTM also speaks to a third, related worry, namely that the existence of a homomorphism is not sufficient for a representation relationship. The problem is two-fold.[17] Firstly, similarity lacks directionality, whereas the representation relation has a direction. A represents B, but B does not represent A. Secondly, the mere existence of a structural similarity is not enough to establish any representational relationship at all, regardless of direction. Many entities are structurally similar in some way or another: mass and length can both be understood as additive extensive structures, a pendulum and a mass on a spring are both harmonic oscillators, and a palace and its reflection in a pond are mirror images of one another. Arguably, none of these are representations of one another.[18] Something more than structural similarity is required to establish the representational relationship.

Different suggestions are available for what the missing ingredient might be: agential intention, causation (Isaac 2013), self-locating belief (van Fraassen 2008), and more. My goal is not to adjudicate between these different options, but to point out that RTM does not settle the matter. The problem these additions are trying to solve is how a representation gets to be about what it represents. But this is not the problem RTM solved when we understand RTM as semantic foundations. Rather, RTM is aiming to solve a problem characteristic of structural representation, namely that the representing structure contains representational artifacts, which mustn't be used when inferring properties of the represented system from the representing system.

[16] This echoes the proposal that scientific models in general should be understood along inferentialist lines (see Suárez 2004). Of course, I'm not claiming here that RTM provides a full-blown inferentialist semantics for scientific models, or that there is a clear connection to the inferentialist tradition in the philosophy of language (Brandom 1998).

[17] The worry is articulated for similarity relations in general by Goodman (1976); for a discussion of problems with isomorphic relations between scientific models and the world, see Suárez 2003.

[18] On some views of natural meaning, the reflection in the pond does count as a representation of the palace, because in addition to being structurally similar, it is also caused by the palace (Isaac 2013). I take no stance on this issue here, as even on this view, some relation over and above structural similarity is needed, viz. causation.

So, RTM provides semantic foundations, because it provides an account of the species of representation numerical representations are, namely structural representations. Such representations are sufficiently different from familiar linguistic representations to warrant an explicit account of the meaningfulness conditions for such representations. The *Foundations of Measurement* offer invariance as the criterion for meaningfulness. While this is not exhaustive as a semantics for numerical representations, it provides a step change in our understanding of numerical representations and what we can infer from them. This is what makes RTM philosophically significant.

7. *Conclusion*

In this paper I've considered in what sense the representational theory of measurement provides foundations, and what we should take the philosophical significance of RTM to be. I concluded that RTM fails to provide epistemic foundations of measurement and that, while RTM does provide conceptual or formal foundations of measurement, such foundations do not account for the philosophical significance of RTM. I proposed instead to take RTM to deliver semantic foundations of numerical representation. These semantic foundations focus not on reference, but on *inference*. By treating numerical representations as structural representations, RTM brings into sharp focus the problem of surplus structure in such representations. The philosophical significance of RTM then lies in this semantic structuralism, which invites an epistemic structuralism, according to which all we can know through measurement is the structure of phenomena.[19,20]

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