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Método de Perturbación y Aproximación de Laplace-Padé como una herramienta novedosa para encontrar soluciones aproximadas en el problema de Troesch

Perturbation Method and Laplace-Padé Approximation as a Novel Tool to Find Approximate Solutions for Troesch's Problem

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Resumen

En este artículo el Método de Perturbación (PM) es empleado para obtener una solución aproximada para el problema de Troesch. Además describiremos el uso de la Transformada de Laplace y la Aproximación de Padé para trabajar con las series truncadas obtenidas por el Método de Perturbación, y así obtener soluciones aproximadas compactas. Finalmente se propone una tabla comparativa entre la solución propuesta y otras soluciones reportadas en la literatura: Método de Descomposición de Adomian, Método de Perturbación Homotópica, Método de Análisis Homotópico y la solución numérica exacta. Los resultados muestran que nuestra solución es la más exacta (Error Relativo Absoluto Promedio $1.705648354x10^{-8}$).

Palabras clave: Ecuación de Troesch, Ecuación Diferencial no lineal, Método de Perturbación, Transformada de Laplace, Aproximación de Padé

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Abstract

In this article, Perturbation Method (PM) is employed to obtain an approximate solution for Troesch equation. In addition, we will describe the use of Laplace transform and Padé transformation to deal with the truncated series obtained by the PM method, in order to obtain handy approximations. Finally a table of comparison, between proposed solution, and other solutions reported in the literature: Adomian's Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and exact numerical solution, shows that our solution is the most accurate one (Average Absolute Relative Error $1.705648354x10^{-8}$).

Keywords: Troesch Equation, Nonlinear Differential Equation, Perturbation Method, Laplace transform, Padé transformation

1. Introduction

Troesch equation is relevant in physics because it models the confinement of a plasma column by radiation pressure. Therefore, it is important to search for accurate solutions for this equation. Unfortunately, it is difficult to solve nonlinear differential equations, like many others that appear in the physical sciences.

The perturbation method (PM) is a well established method; it is among the pioneer techniques to approach various kinds of nonlinear problems. This procedure was originated by S. D. Poisson and extended by J. H. Poincare. Although the method appeared in the early 19th century, the application of a perturbation procedure to solve nonlinear differential equations was performed later on that century. The most significant efforts were focused on celestial mechanics, fluid mechanics, and aerodynamics [1, 2, 54, 55].

In general, it is assumed that the differential equation to be solved can be expressed as the sum of two parts, one linear and the other nonlinear. The nonlinear part is considered as a small perturbation through a small parameter (the perturbation parameter). The assumption that the nonlinear part is small compared to the linear is considered as a disadvantage of the method. There are other modern alternatives to find approximate solutions to the differential equations that describe some nonlinear problems such as those based on: Variational approaches [5-7, 29], Tanh method [8], Exp-function [9, 10], Adomian's decomposition method [11-16,40], Parameter expansion [17], Homotopy perturbation method [3,4,18-28,31-36,39,45,46,48-53,56,57] and Homotopy analysis method [30,47], among many others.

Although the PM method provides in general, better results for small perturbation parameters $\varepsilon << 1$, we will see that our approximation has good accuracy, even for big values of the perturbation parameter. Finally, we will couple the PM and Padé methods, in order to express the results of perturbation method in a handy way.

The paper is organized as follows. In Section 2, we introduce the basic idea of the PM method. Section 3 will provide a brief introduction to the Padé approximation. For Section 4, we provide an application of the PM method. Section 5 shows an approximate solution to the Troesch equation by using Laplace-Padé approximation. Section 6 discusses the main results obtained. Finally, a brief conclusion is given in Section 7.

2. Basic idea of Perturbation Method.

Let the differential equation of one dimensional nonlinear system be in the form

$$L(x)+\varepsilon N(x)=0. \quad (1)$$

Where we assume that x is a function of one variable $x=x(t)$, $L(x)$ is a linear operator which, in general, contains derivatives in terms of t, $N(x)$ is a nonlinear operator, and ε is a small parameter.

Considering the nonlinear term in (1) to be a small perturbation and assuming that the solution for (1) can be written as a power series in the small parameter ε.

$$x(t)=x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+... \quad (2)$$

Substituting (2) into (1) and equating terms having identical powers of ε, we obtain a number of differential equations that can be integrated, recursively, to find the values for the functions: $x_0(t)$, $x_1(t)$, $x_2(t)\ldots$

3. Padé Aproximation.

A Rational approximation to $f(x)$ on $\left[a,b\right]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M, respectively. We use the notation $R_{N,M}(x)$ to denote this quotient. The $R_{N,M}(x)$ Padé approximations to a function are given by [41,43]

$$R_{NM}=\frac{P_N(x)}{Q_M(x)}, \qquad a\le x\le b\ . \quad (3)$$

The method of Padé requires that $f(x)$ and its derivatives be continuous at $x=0$. The polynomials used in (3) are

$$P_N(x)=p_0+p_1x+p_2x^2+...+p_N(x)x^N, \quad (4)$$

$$Q_N(x)=q_0+q_1x+q_2x^2+...+q_N(x)x^N. \quad (5)$$

The polynomials in (4) and (5) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x=0$ and their derivatives up to $N+M$ agree at $x=0$. In this case $Q_0(x)=1$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N+M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher than $Q_M(x)$.

Notice that the constant coefficient of $Q_M(x)$ is $q_0 - 1$. This is permissible, because it can be noted that 0 and $R_{N,M}(x)$ do not change when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N+M+1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin expansion.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots \tag{6}$$

And from the difference $f(x)Q_M(x) - P_N(x) = Z(x)$

$$\left[\sum_{i=0}^{\infty} a_i x^i\right]\left[\sum_{i=0}^{M} q_i x^i\right] - \left[\sum_{i=0}^{N} p_i x^i\right] = \left[\sum_{i=N+M+1}^{\infty} c_i x^i\right]. \tag{7}$$

The lower index $j = N+M+1$ in the summation on the right side of (7) is chosen because the first $N+M$ derivatives of $f(x)$ and $R_{N,M}(x)$ should agree at $x=0$.

When the left side of (7) is multiplied and the coefficients of the powers of x^i are set equal to zero for $k = 0,1,2,\ldots,N+M$, the result is a system of $N+M+1$ linear equations.

$$\begin{aligned}
&a_0 - p_0 = 0,\\
&q_1 a_0 + a_1 - p_1 = 0,\\
&q_2 a_0 + q_1 a_1 + a_2 - p_2 = 0,\\
&q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 = 0,\\
&q_M a_{N-M} + |q_{M-1} a_{N-M-1}| a_N - p_N = 0,
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \ldots + q_1 a_N + a_{N+2} = 0,\\
&q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + \ldots + q_1 a_{N+1} + a_{N+3} = 0,\\
&\ldots\\
&q_M a_N + q_M a_{N+1} + \ldots + q_1 a_{N+M+1} + a_{N+M} = 0.
\end{aligned} \tag{9}$$

Notice that in each equation the sum of the subscripts on the factors of each product is the same. This sum increases consecutively from 0 to $N+M$. The M equations in (9) involve only the unknowns: $q_1, q_2, \ldots, q_M$ and must be solved first. Then the equations in (8) are used successively to find $p_1, p_2, \ldots, p_N$ [41].

4. Approximate solution of Troesch Equation.

The equation to solve is

$$y'' - \varepsilon \sinh(\varepsilon y) = 0, \qquad 0 \le x \le 1, \quad y(0) = 0, \quad y(1) = 1, \tag{10}$$

where ε is known as Troesch's parameter.

It is possible to find a handy solution for (10) by applying PM. Identifying terms:

$$L(y) = y''(x), \tag{11}$$

$$N(y) = -\sinh(\varepsilon y), \tag{12}$$

and ε with the PM parameter.

Since the parameter is embedded into the nonlinear operator $N(y)$, we express the right hand side of (12), in terms of Taylor series expansion as it is shown

$$y'' = \varepsilon^2 y + \frac{\varepsilon^4 y^3}{6} + \frac{\varepsilon^6 y^5}{120} + \frac{\varepsilon^8 y^7}{5040} + \ldots \tag{13}$$

Assuming a solution for (13) in the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \varepsilon^4 y_4(x) + \ldots, \qquad \text{(see (2))} \tag{14}$$

and equating the terms with identical powers of ε, it can be solved for $y_0(x)$, $y_1(x)$, $y_2(x)$, …, and so on. Later it will be seen that, a very good result is obtained, by keeping up to eighth order approximation.

$$\varepsilon^0) \qquad y_0'' = 0, \tag{15}$$

$$\varepsilon^1) \qquad y_1'' = 0, \tag{16}$$

$$\varepsilon^2) \qquad y_2'' = y_0, \tag{17}$$

$$\varepsilon^3) \qquad y_3'' = y_1, \tag{18}$$

$\varepsilon^4)$ $$y_4'' = y_2 + \frac{y_0^3}{6}, \tag{19}$$

$\varepsilon^5)$ $$y_5'' = y_3 + \frac{y_0^2 y_1}{2}, \tag{20}$$

$\varepsilon^6)$ $$y_6'' = y_4 + \frac{\left[3y_0^2 y_1 + 3y_0^2 y_2\right]}{6} + \frac{y_0^5}{120}, \tag{21}$$

$\varepsilon^7)$ $$y_7'' = y_5 + \frac{\left[3y_3 y_0^2 + y_1^3 + 6y_0 y_1 y_2\right]}{6} + \frac{y_0^4 y_1}{24}, \tag{22}$$

$\varepsilon^8)$ $$y_8'' = y_6 + \frac{\left[3y_0^2 y_4 + 3y_2 y_1^2 + 3y_0 y_2^2 + 6y_0 y_1 y_3\right]}{6} + \frac{\left[5y_0^4 y_2 + 9y_0^3 y_1^2 + y_0^4 y_1^2\right]}{120} + \frac{y_0^7}{5040}, \tag{23}$$

$$\vdots$$

In order to fulfill the boundary conditions from (10), it follows that $y_0(0)=0$, $y_0(1)=1$, $y_1(0)=0$, $y_1(1)=0$, $y_2(0)=0$, $y_2(1)=0$, $y_3(0)=0$, $y_3(1)=0$, $y_4(0)=0$, $y_4(1)=0$, and so on. Thus, the results obtained are

$$y_0(x) = x, \tag{24}$$

$$y_1(x) = 0, \tag{25}$$

$$y_2(x) = \frac{x^3}{6} - \frac{x}{6}, \tag{26}$$

$$y_3(x) = 0, \tag{27}$$

$$y_4(x) = \frac{x^5}{60} - \frac{x^3}{36} + \frac{x}{90}, \tag{28}$$

$$y_5(x) = 0, \tag{29}$$

$$y_6(x) = \frac{13x^7}{5040} - \frac{x^5}{180} + \frac{x^3}{540} + \frac{51x}{45360}, \tag{30}$$

$$y_7(x) = 0, \tag{31}$$

$$y_8(x) = \frac{55459x^9}{125000000} - \frac{13x^7}{10080} + \frac{23x^5}{21600} + \frac{51x^3}{272160} - \frac{202821x}{500000000}, \tag{32}$$

$$\vdots$$

By substituting (24)-(32) into (14), we obtain an approximate solution to (10), as it is shown

$$y(x)=\frac{55459x^9\varepsilon^8}{125000000}+\left(\frac{13\varepsilon^6}{5040}-\frac{13\varepsilon^8}{10080}\right)x^7+\left(\frac{\varepsilon^4}{60}-\frac{\varepsilon^6}{180}+\frac{23\varepsilon^8}{21600}\right)x^5+\left(\frac{\varepsilon^2}{6}-\frac{\varepsilon^4}{36}+\frac{\varepsilon^6}{540}+\frac{51\varepsilon^8}{272160}\right)x^3+$$

$$+\left(1-\frac{\varepsilon^2}{6}+\frac{\varepsilon^4}{90}+\frac{51\varepsilon^6}{45360}-\frac{202821\varepsilon^8}{500000000}\right)x\,. \qquad (33)$$

We consider as a case of study, the following values of the Troesch's parameter: $\varepsilon=0.5$, $\varepsilon=1$, $\varepsilon=1.5$, $\varepsilon=2$, so that

$$y(x)=8.665468x10^{-7}x^9+\frac{91}{2580480}x^7+\frac{5303}{5529600}x^5+\frac{16704627}{418037760}x^3+\frac{2784143}{2903040}x\,,\ (\varepsilon=0.5) \qquad (34)$$

$$y(x)=\frac{55459x^9}{125000000}+\frac{13}{10080}x^7+\frac{487037}{40000000}x^5+\frac{1101001}{7812500}x^3+\frac{105746803}{125000000}x\,, \qquad (\varepsilon=1) \qquad (35)$$

$$y(x)=\frac{11370828x^9}{1000000000}-\frac{9477}{2580480}x^7+\frac{267543}{5529600}x^5+\frac{107130195}{418037760}x^3+\frac{198496453}{290304000}x\,, \qquad (\varepsilon=1.5) \qquad (36)$$

$$y(x)=\frac{221836x^9}{1953125}-\frac{52}{315}x^7+\frac{124}{675}x^5+\frac{808}{2835}x^3+\frac{479224431}{1000000000}x\,. \qquad (\varepsilon=2) \qquad (37)$$

5. An approximate solution by using Laplace-Padé transformation and PM Method.

In this section we will describe the use of Laplace transform and Padé transformation [40] to deal with the truncated series (34), (35), (36) and (37) obtained by PM, in order to obtain handy approximate solutions to equation (10), keeping the same domain of the original problem [56].

First, Laplace transformation is applied for example, to series (34) and then $1/x$ is written in place of s in the equation obtained. Then Padé approximant [4/4] is applied and 1/s is written in place of x. Finally, by using the inverse Laplace transformation, we obtain the modified approximate solution.

$$y(x)=\frac{4}{573}Sinh\left(\frac{85x}{54}\right)+\frac{869}{434}Sinh\left(\frac{116x}{245}\right). \quad (38)$$

Then, applying the same procedure to the series (35), (36) and (37), we obtain the following alternative expressions.

$$y(x)=\frac{1}{178}Sinh\left(\frac{73x}{27}\right)+\frac{15}{17}Sinh\left(\frac{16x}{17}\right), \quad (39)$$

$$y(x)=\frac{16}{37}Sinh\left(\frac{47x}{30}\right)+\frac{1}{659}Sin\left(\frac{49x}{12}\right), \quad (40)$$

$$y(x)=\frac{18}{85}Sinh\left(\frac{70x}{31}\right)+\frac{1}{8641}Sin\left(\frac{106x}{11}\right). \quad (41)$$

6. Discussion

The fact that the PM depends on a parameter which is assumed small, suggests that the method is limited. In this work, the PM method has been applied to the problem of finding an approximate solution for the nonlinear differential equation which describes the Troesch's problem. Table 1 shows the comparison between exact solution given in [42], and approximations (34), (38), ADM [44], HPM [45], HPM [46] and HAM [47] for the case $\varepsilon = 0.5$. It is clear that (34), has the best accuracy and also the lowest Average Absolute Relative Error (A.A.R.E) $1.705648354x10^{-8}$, followed by HAM [47], with accuracy $2.51374x10^{-6}$, despite of the fact that HPM, ADM and HAM methods are considered more general and difficult to use. Table 2 shows that for $\varepsilon = 1$, our approximations (35) and (39) were the best. In particular PM possesses the lowest A.A.R.E $2.379908276x10^{-5}$, although $\varepsilon = 1$ cannot be considered as small.

The PM method provides in general, better results for small perturbation parameters $\varepsilon << 1$ (see (1)) and when are included the most number of terms from (2). To be precise, ε is a parameter of smallness, that measures how greater is the contribution of linear term $L(x)$ than the one of $N(x)$ in (1). From the approximations (34), (35), (36), (37), as well as of Figure 1, it is clear that the term proportional to x ($y_0(x)=x$), is the contribution to the approximation of the linear operator (see (15) and (24)) besides, they are the dominant terms in the approximate solutions, even for the big values of: $\varepsilon = 0.5$, $\varepsilon = 1$, $\varepsilon = 1.5$ and $\varepsilon = 2$. This happens because Troesch' s problem is defined in [0,1] (see (10)); in that interval $x > x^3, x^5, x^7, x^9$, for $x \in (0,1)$. Also, the coefficients of powers: x^3, x^5, x^7, x^9 of the aforementioned equations (34)-(37) are small, compared with that of x.

Figure 1 shows the comparison between approximate solutions (34) and (35) for $\varepsilon = 0.5$ and $\varepsilon = 1$ respectively, and the exact solutions given in [42]. Besides, the same Figure compares (36) $(\varepsilon = 1.5)$ and (37) $(\varepsilon = 2)$ with the four order Runge Kutta numerical solution of (10), for the same values of ε. It can be noticed that, figures are very similar in all cases, of which is clear the accuracy of results (34)-(37) as approximated solutions for (10).

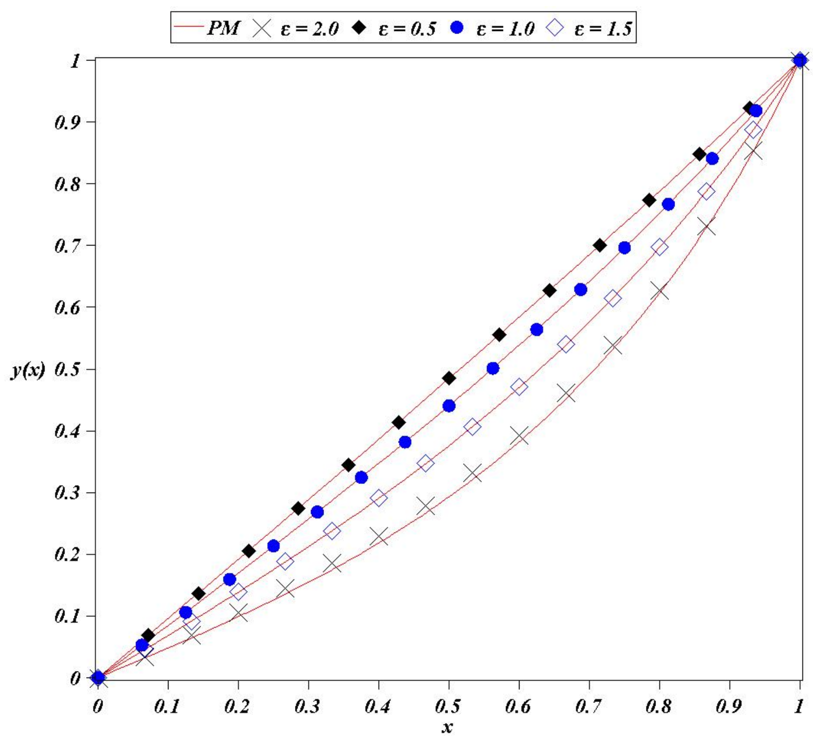

Fig 1: Comparison of proposed solution (34) (solid line) for $\varepsilon = 0.5$, $\varepsilon = 1$, $\varepsilon = 1.5$ and $\varepsilon = 2$; with $\varepsilon = 0.5$ (solid square), $\varepsilon = 1$ (solid circles) reported in [42]; $\varepsilon = 1.5$ (empty square) and $\varepsilon = 2$ (diagonal-cross) calculated by using four order Runge Kutta.

We employed Laplace transform and Padé transformation to obtain the approximate solutions of equation (10), given by (38)-(41). Although some precision is lost compared with PM approximations, expressions (38)-(41) are handier and computationally more efficient than (34)-(37). In fact from tables 1 and 2, we conclude that our PM Padé expressions are also competitive. For $\varepsilon = 0.5$, (38) has an acceptable accuracy, since it's A.A.R.E $2.720152702x10^{-5}$, is better than the ones of ADM [44] and HPM [45], while for $\varepsilon = 1$, (39) is even better, when it is compared with the other approximations, having the second best A.A.R.E $5.51616232x10^{-4}$ as it had been already mentioned.

Unlike other methods, our approximate solution (33) does not depend of any adjustment parameter, for which, it is in principle, a general expression for Troesch' s problem.

It is important to remark, that further research may be focused on the development of a sensitivity analysis for the solutions emanating from the perturbation method (PM), since it is possible that small perturbations on the coefficients change the accuracy of the approximate solutions.

x	Exact[8]	PM	PM-Pade	ADM[44]	HPM[45]	HPM[46]	HAM[47]
0.1	0.0959443493	0.0959443459	0.0959417697	0.0959383534	0.0959395656	0.095948026	0.0959446190
0.2	0.1921287477	0.1921287413	0.1921235763	0.1921180592	0.1921193244	0.192135797	0.1921292845
0.3	0.2887944009	0.2887943925	0.2887866136	0.2887803297	0.2887806940	0.288804238	0.2887952148
0.4	0.3861848464	0.3861848373	0.3861744075	0.3861687095	0.3861675428	0.386196642	0.3861859313
0.5	0.4845471647	0.4845471566	0.4845340266	0.4845302901	0.4845274183	0.4845599	0.4845485110
0.6	0.5841332484	0.5841332428	0.5841173497	0.5841169798	0.5841127822	0.584145785	0.5841348222
0.7	0.6852011483	0.6852011458	0.6851824083	0.6851868451	0.6851822495	0.685212297	0.6852028604
0.8	0.7880165227	0.7880165229	0.7879948277	0.7880055691	0.7880018367	0.788025104	0.7880181729
0.9	0.8928542161	0.8928542178	0.8928293923	0.8928480234	0.8928462193	0.892859085	0.8928553997
A.A.R.E		$1.705648354x10^{-8}$	$2.720152702x10^{-5}$	$3.47802x10^{-5}$	$3.57932x10^{-5}$	$2.44418x10^{-5}$	$2.51374x10^{-6}$

Table 1: Comparison between (34), exact solution [42], and other reported approximate solutions, using $\varepsilon = 0.5$.

x	Exact[8]	PM	PM-Pade	ADM[44]	HPM[45]	HPM[46]	HAM[47]
0.1	0.0846612565	0.0846573641	0.0847051491	0.084248760	0.0843817004	0.084934415	0.0846732692
0.2	0.1701713582	0.1701639663	0.1702606304	0.169430700	0.1696207644	0.170697546	0.1701954538
0.3	0.2573939080	0.2573838801	0.2575316283	0.256414500	0.2565929224	0.258133224	0.2574302342
0.4	0.3472228551	0.3472115676	0.3474137007	0.346085720	0.3462107378	0.348116627	0.3472715981
0.5	0.4405998351	0.4405890267	0.4408496855	0.439401985	0.4394422743	0.44157274	0.4406610140
0.6	0.5385343980	0.5385257349	0.5388488010	0.537365700	0.5373300622	0.539498234	0.5386072529
0.7	0.6421286091	0.6421230701	0.6425088996	0.641083800	0.6410104651	0.642987984	0.7526899495
0.8	0.7526080939	0.7526055378	0.7530430509	0.751788000	0.7517335467	0.753267551	0.7526899495
0.9	0.8713625196	0.8713619354	0.8718119297	0.870908700	0.8708835371	0.871733059	0.8714249118
A.A.R.E		$2.379908276x10^{-5}$	$5.51616232x10^{-4}$	.002714577	.002320107	002044737	0.019244326

Table 2: Comparison between (35), exact solution [42], and other reported approximate solutions, using $\varepsilon = 1$.

7. Conclusion

This work showed that some nonlinear problems can be adequately approximated by using the PM method, even for large values of the perturbation parameter; as it was done for the Troesch' s problem described by (10). The fact that the term proportional to x, is the dominant one in approximations: (34), (35), (36), (37), even for the big values of ε , contributed to the success of the method for this case and could be useful to apply it in similar cases, instead of using other sophisticated and difficult methods. Finally we showed that, it is possible to use a novel technique that coupled the PM method and the Padé–Laplace transformation to obtain, handy approximate solutions of equation (10), given by (38)-(41). In all cases, the numerical and graphical results show that the proposed solutions have good accuracy.

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