Appendix

Appendix A. Dynamic model of manipulator robot.

a. Inertia matrix

$$M(q) = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \tag{A.1}$$

Where:

$$\begin{aligned}
M_{11} = {} & (m_2 lc_2^2 c_2^2 + m_3 l_2^2 c_2^2 + 2m_3 l_2 l_{c3} c_{23} + m_3 l_{c3}^2 c_{23}^2 + m_4(l_2^2 c_2^2 + l_3 c_{23}^2 + l_{c4}^2 c_{234}^2 + 2l_2 l_3 c_2 c_{23} + \\
& 2l_3 l_{c4} c_{23} c_{234} + 2l_2 l_{c4} c_4 c_2 c_{234}) + I_{yy_1} + s_2^2 I_{xx_1} + c_2^2 I_{yy_2} + I_{xx_3}\left(c_3^2 s_2^2 + s_3^2 c_2^2 - \tfrac{1}{2}\mathrm{s}(2q_2)\,\mathrm{s}(2q_3)\right) + \\
& I_{yy_3}\left(s_3^2 s_2^2 + c_{31}^2 c_2^2 - \tfrac{1}{2}\mathrm{s}(2q_2)\,\mathrm{s}(2q_3)\right) + \left(c_4^2 I_{xx_4} + s_4^2 I_{yy_4}\right)\left(s_2^2 c_3^2 + \tfrac{1}{2}\mathrm{s}(2q_2) + \mathrm{s}(2q_3) + s_3^2 c_2^2\right) + \\
& \left(I_{xx_4} - I_{yy_4}\right)\left(\tfrac{1}{2}\mathrm{c}(2q_2)\,\mathrm{s}(2q_3)\,s(2q_4) + \tfrac{1}{2}\mathrm{c}(2q_3)\,\mathrm{s}(2q_2)\,\mathrm{s}(2q_1)\right) + \left(s_4^2 I_{xx_4} - c_4^2 I_{yy_4}\right)\left(s_2^2 s_3^2 - \right. \\
& \left. \tfrac{1}{2}\mathrm{s}(2q_2)\,\mathrm{s}(2q_3) + c_2^2 c_3^2\right) \\
M_{12} = {} & M_{13} = M_{14} = M_{31} = M_{41} = 0 \\
M_{22} = {} & m_2\, lc_2^2 + m_3\, l_2^2 + 2m_3 l_2 lc_3 c_3 + m_3\, lc_3^2 + m_4(l_2^2 + l_3^2 + lc_4^2 + l_2 l_3 c_3 + 2\, l_3 lc_4 c_4 + \\
& 2\, l_2 lc_4 c_{34}) + I_{zz_2} + I_{zz_3} \\
M_{23} = {} & m_3\, l_2 lc_3\, c_3 + m_3\, lc_3^2 + m_4(l_3^2 + lc_4^2 + l_2\, l_3\, c_3 + 2\, l_3\, lc_4\, c_4 + l_2\, lc_4\, c_{34}) + I_{zz_3} \\
M_{24} = {} & m_4(lc_4^2 + l_3 lc_4 c_4 + l_2 lc_4 c_{34}) \\
M_{32} = {} & m_3 l_2 lc_3 c_3 + m_3 lc_3^2 + m_4(l_3^2 + lc_4^2 + 2l_2 l_3 c_3 + 2\, l_3 lc_4 c_4 + l_3 lc_4 c_{34}) + I_{zz_3} \\
M_{33} = {} & m_3 lc_3^2 + m_4(l_3^2 + lc_4^2 + 2l_3 lc_4 c_4) + I_{zz_3} \\
M_{34} = {} & m_4(lc_4^2 + l_3 lc_4 c_4) \\
M_{42} = {} & m_4[lc_4^2 + l_3 lc_4 c_4 + l_2 lc_4 c_{34}] \\
M_{43} = {} & m_4[lc_4^2 + l_3 lc_4 c_4] \\
M_{44} = {} & m_4[lc_4^2]
\end{aligned}$$

b. Coriolis matrix

$$C(q, \dot{q}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \tag{A.2}$$

Where:

$$\begin{aligned}
C_{11} = {} & 2m_2\, lc_2^2\, c_2\, s_2\, \dot{q}_2 + [\left(I_{xx_2} - I_{yy_2}\right)s(2q_2) + \left(I_{xx_3} - I_{yy_3}\right)s(2q_2)\, c(2q_3) \\
& + \left(I_{xx_3} - I_{yy_3}\right)s(2q_3)\, c(2q_2) - (2m_4)(l_2^2\, c_2 s_2 + l_2 l_3\, c_{23}\, s_2 + l_2 lc_4 c_{234} s_2)]\dot{q}_2 \\
C_{12} = {} & [(-2)(m_3 l_2^2 c_2 s_2 + m_3 l_2 lc_3 c_2 s_{23} + m_3 l_2 lc_3 c_{23} s_2 + m_3 lc_3^2 c_{23} s_{23}) - 2m_4(l_3^2 c_{23} s_{23} + lc_4 c_{234} s_{234} \\
& + l_2 l_3 c_2 s_{23} + l_3 lc_4 c_{23} s_{234} + l_3 lc_4 c_{234} s_{23} + l_2 lc_4 c_2 s_{234}) + (c_4^2 I_{xx_4} \\
& + s_4^2 I_{yy_4})(\, c_3^2\, s(2q_2) - s_3^2\, s(2q_2) + s(2q_3)\, c(2q_2)) + (s_4 I_{xx_4}\} \\
& + c_4^2 I_{yy_4})(\, s_3^2\, s(2q_2) - c_3^2\, s(2q_2) - s(2q_3)\, c(2q_2)\,) \\
& + \left(I_{yy_4} - I_{xx_4}\right)\left(s(2q_3)s(2q_4)s(2q_2)\right) + (I_{xx_4} \\
& - I_{yy_4})\, (c(2q_3)\, s(2q_1)\, c(2q_2))]\dot{q}_1
\end{aligned}$$

$$
\begin{aligned}
C_{13} = \; & [2\, m_3 l_2 lc_3 c_2 s_{23} + 2\, m_3 lc_3^2 c_{23} s_{23} + (I_{xx_3} - I_{yy_3})\, c(2q_2)\, s(2q_3) + (I_{xx_3} - I_{yy_3}) s(2q_2) c(2q_3) \\
& - 2m_4 (l_3^2 c_{23} s_{23} + lc_4 c_{234} s_{234} + l_2 l_3 c_2 s_{23} + l_3 lc_4 c_{23} s_{234} + l_3 lc_4 c_{234} s_{23} \\
& + l_2 lc_4 c_2 s_{234}) + (c_4^2\, I_{xx_4} + s_4^2\, I_{yy_4})(-s_2^2\, s(2q_3) + c_2^2\, s(2q_3) + s(2q_2) c(2q_3)) \\
& + (s_4^2\, I_{xx_4} + c_4^2\, I_{yy_4})(s_2^2\, s(2q_3) - c_3\, s(2q_3) - s(2q_2) c(2q_3)) \\
& + (I_{xx_4} - I_{yy_4})(\, c(2q_2) s(2q_4) c(2q_3) - s(2q_2) s(2q_4) s(2q_3))] \dot{q}_1 \\
C_{14} = \; & \dot{q}_1 [-2m_4 (lc_4^2 c_{234}\, s_{234} + l_3 lc_4 c_{23} s_{234} + l_2 lc_4 c_2 s_{234}) \\
& + (I_{yy_4} - I_{xx_4}) \left(s_2^2\, c_2^2\, s(2q_4) + s_3^2 c_2^2\, s(2q_4) + \frac{1}{2}\, s(2q_2) s(2q_3) s(2q_4) \right) \\
& + (I_{xx_4} - I_{yy_4})\, (s_3^2\, s_2^2\, s(2q_4) + c_2^2 c_3^2\, s(2q_4) + c(2q_2)\, s(2q_3)\, c(2q_4) \\
& + c(2q_3)\, s(2q_2)\, c(2q_4))] \\
C_{21} = \; & \dot{q}_1 [m_2 lc_2^2 c_2\, s_2 \dot{q}_2 + m_3 l_2^2 c_2 s_2 \dot{q}_2 + m_3 l_2 lc_3 c_2 s_{23} + m_3 l_2 lc_3 c_{23} s_2 + m_3 lc_3^2\, c_{23} s_{23} \\
& + m_4 (\, l_2^2 c_2 s_2 + l_3^2 c_{23} s_{23} + lc_4 c_{234} s_{234} + l_2 l_3 c_2 s_{23} + l_2 l_3 c_{23} s_2 \\
& + l_3 lc_4 c_{23} s_{234} + l_3 lc_4 c_{234} s_{23} + l_2 lc_4 c_2 s_{234} + l_2 lc_4 c_{234} s_2) - \frac{1}{2} [(I_{xx_2} \\
& - I_{yy_2})\, s(2q_2) + (I_{xx_3} - I_{yy_3}) c(2q_3) s(2q_2) + (I_{xx_3} - I_{yy_3})\, s(2q_3) c(2q_2) \\
& + (I_{xx_4} - I_{yy_4})\, (c(2q_3) s(2q_2)\, c(2q_4) + c(2q_4)\, s(2q_3)\, c(2q_2) \\
& + s(2q_4)\, (c(2q_3)\, c(2q_2) - s(2q_3)\, s(2q_2)))]] \\
C_{22} = \; & -2m_3 l_2 lc_3 s_3\, \dot{q}_3 - 2m_4 [\dot{q}_3\, (l_2 l_3 s_3 + l_2 lc_4 s_{34}) + \dot{q}_4\, (l_3 lc_4 s_4 + l_2 lc_4 s_{34}\,) \\
C_{23} = \; & \dot{q}_3\, (-m\_3 l\_2 lc\_3\, s\_3) - m\_4 [\dot{q}_3 (l_2 l_3 s_3 + l_2 lc_4 s_{34}) + \dot{q}_4\, (2\, l_3 lc_4 s_4 + l_2 lc_4 s_{34}) \\
C_{24} = \; & -m_4 [l_2 lc_4 s_{34}\, \dot{q}_3 + \dot{q}_4\, (l_3 lc_4 s_4 + l_2 lc_4 s_{34})] \\
C_{31} = \; & \dot{q}_1 [m_3 l_2 lc_3 c_2 s_{23} + m_3 lc_3^2 c_{23} s_{23}\; m_4\, (l_3^2 c_{23} s_{23} + lc_4 c_{234} s_{234} + l_2 l_3 c_2 s_{23} \\
& + l_3 lc_4 c_{23} s_{234} + l_3 lc_4 c_{234} s_{23} + l_2 lc_4 c_2 s_{234}) - \frac{1}{2} [(I_{yy_3} + I_{xx_3})\, c(2q_2)\, s(2q_3) \\
& + (I_{xx_3} - I_{yy_3})\, s(2q_2)\, c(2q_3)\; (I_{xx_4} - I_{yy_4})\, (c(2q_2)\, s(2q_3)\, c(2q_4) \\
& + c(2q_4)\, s(2q_2)\, c(2q_3) + s(2q_4)\, (\, c(2q_2)\, c(2q_3) - s(2q_2)\, s(2q_3)))]] \\
C_{32} = \; & \dot{q}_2 [m_3 l_2 lc_3 s_3 + m_4 (l_2 l_3 s_3 + l_2 lc_4 s_{34})] \\
C_{33} = \; & -2m_4\, l_3 lc_4 s_4 \dot{q}_4 \\
C_{34} = \; & -2m_4\, [\dot{q}_2\, (2\, l_3 lc_4 s_4) + \dot{q}_4\, (l_3 lc_4 s_4)] \\
C_{41} = \; & -\frac{1}{2}\, \dot{q}_1\, [(c(2q_3)\, c(2q_2)\, s(2q_4) - s(2q_2)\, s(2q_3)\, s(2q_4) + c(2q_4)\, (c(2q_2)\, s(2q_3) \\
& + c(2q_3)\, s(2q_2)))\, (I_{xx_4} - I_{yy_4})] \\
C_{42} = \; & m_4\, [\dot{q}_2\, (l_3 lc_4 s_4 + l_2 lc_4 s_{34})] \\
C_{43} = \; & + m_4\, [\dot{q}_2\, (\, 2\, l_3 lc_4 s_4\,) + l_3 lc_4 s_4\, \dot{q}_3] \\
C_{44} = \; & 0
\end{aligned}
$$

c. Gravity vector

$$
g(q) = \begin{bmatrix} 0 \\ lc_2 m_2 c_2 - l_2 m_3 c_2 - lc_3 m_3 c_{23} + m_4\, (l\_2\, c\_2 + l_3 c_{23} + lc_4 c_{234}) \\ lc_3 m_3 c_{23} + m_4\, (l_3\, c_{23} + lc_4\, c_{234}) \\ m_4 lc_4 c_{234} \end{bmatrix} \tag{A.3}
$$

Nomenclature:

$$
\begin{aligned}
s_a^y = sin^y(a) \qquad & c_a^y = cos^y(a) \\
s_{ab}^y = sin^y(a+b) \qquad & c_{ab}^y = cos^y(a+b) \\
s_{abc}^y = sin^y(a+b+c) \qquad & c_{abc}^y = cos^y(a+b+c)
\end{aligned}
$$

$$I_a = \begin{bmatrix} I_{xx_a} & 0 & 0 \\ 0 & I_{yy_a} & 0 \\ 0 & 0 & I_{zz_a} \end{bmatrix}, \text{ with } a = 1, 2, 3, 4.$$