Ranking of Exponential Vague Sets With an Application to Decision Making Problems

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ABSTRACT
The main aim of this paper is to propose a new approach for the ranking of exponential vague sets. The concepts of exponential vague sets and arithmetic operations between two exponential vague sets are introduced. The shortcomings of some existing ranking approaches for the ranking of generalized fuzzy sets and intuitionistic fuzzy numbers are pointed out. The proposed method consider not only the rank but also the decision maker optimistic attitude and it is shown that proposed ranking approach is more intuitive and reasonable as compared to existing ranking approaches. Also the proposed ranking function satisfies the reasonable properties for the ordering of fuzzy quantity. For practical use, proposed ranking approach is applied to decision making problem.

Keywords: Fuzzy sets, vague sets, exponential vague sets, intuitionistic fuzzy numbers, ranking functions, decision making problems.

RESUMEN
El objetivo principal de este trabajo es proponer un nuevo enfoque para la clasificación de los conjuntos inciertos exponenciales. Se introducen los conceptos de conjuntos inciertos exponenciales y operaciones aritméticas entre dos conjuntos inciertos exponenciales. Se señalan las deficiencias de algunos enfoques de clasificación existentes para la clasificación de los conjuntos difusos generalizados y de los números difusos intuicionistas. El método propuesto toma en cuenta no sólo el rango, sino también el enfoque optimista para toma de decisiones y se muestra que el enfoque de clasificación propuesto es más intuitivo y razonable en comparación con los enfoques de clasificación existentes. Asimismo, la función de clasificación propuesta satisface las propiedades razonables para el ordenamiento de la cantidad difusa. Para usos prácticos, el enfoque de clasificación propuesto se aplica al problema de toma de decisiones.

1. Introduction

The theory of fuzzy sets was first introduced by Zadeh [37] in 1965. Since then, the theory of fuzzy sets is applied in many fields such as pattern recognition, control theory, management sciences and picture processing, etc. In the field of fuzzy mathematics many mathematical theory such as fuzzy optimization, fuzzy topology, fuzzy logic, fuzzy analysis and fuzzy algebra etc. are obtained [3, 10, 14, 22, 23, 29, 32, 35]. In many applications of fuzzy set theory to decision making, we are faced with the problem of selecting one from a collection of possible solution, and in general we want to know which one is the best. This selection process may require that we rank or order fuzzy numbers. In order to rank fuzzy numbers, one fuzzy number needs to be evaluated and compared to others but this may not be easy. As known, the real numbers in can be linearly ordered by, however, fuzzy numbers cannot be done in such a way. Since fuzzy numbers are represented by possibility distributions, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than the other.

To the task of comparing fuzzy numbers, many authors proposed fuzzy ranking methods [5, 6, 8, 9, 12, 15, 16, 17, 18, 20, 25, 30, 31, 34]. But among all the methods, most of them consider only one point of view on comparing fuzzy quantities in spite of the different demand of the decision maker, so some improved methods have been brought forward which lead to produce different rankings for the same problem. Until now, there have not one unify method to this problem. Fuzzy set theory [37] has been shown to be useful tool to handle the situations, in which the data is imprecise, by attributing a degree to which a certain object belongs to a set. In real life, a person may assume that an object belongs to a set, but it is possible that he is not sure about it. In other words, there may be hesitation or uncertainty thatwhether an object belongs to a set or not. In fuzzy set theory, there is no means to incorporate such type of hesitation or uncertainty. A possible solution is to use intuitionistic fuzzy set [2] and vague set [11]. Bustince and Burillo [4] pointed out that the notion of vague set is the same as that of intuitionistic fuzzy set. Lu and Ng [21] proved that vague sets is more natural than using an intuitionistic fuzzy set. Several authors [19, 24, 27, 28] have proposed different methods for the ranking of intuitionistic fuzzy sets but to the best of our knowledge till now there no method in the literature for the ranking of vague sets.

The main aim of this paper is to propose a new approach for the ranking of exponential vague sets. The concepts of exponential vague sets and arithmetic operations between two exponential vague sets are introduced. The shortcomings of some existing ranking approaches [8, 19] for the ranking of generalized fuzzy sets and intuitionistic fuzzy numbers are pointed out. Also it is shown that proposed ranking approach is more intuitive and reasonable as compared to existing ranking approaches Rest of the paper is organized as follows: In Section 2, some basic definitions related to generalized fuzzy sets, intuitionistic fuzzy sets, vague sets and arithmetic operations between vague sets are presented. In Section 3, a brief review of the existing approach [8] for the ranking of generalized trapezoidal fuzzy numbers and the existing approach [19] for the ranking of intuitionistic fuzzy numbers are presented. In Section 4, the shortcomings of existing approaches [8, 19] are discussed. In Section 5, a new approach is proposed for the ranking of exponential vague sets. In Section 6, it is proved that the proposed ranking function satisfies the reasonable properties for the ordering of fuzzy quantities and results are compared with some existing approached. In Section 7, an application of proposed ranking method to decision making is presented. Section 8 draws the conclusions.

2. Preliminaries

In this section some basic definitions related to generalized fuzzy sets, intuitionistic fuzzy sets, vague sets and arithmetic operations between exponential vague sets are presented.

2.1 Generalized Fuzzy Sets

Definition 1. [6] A fuzzy set A, defined on the universal set of real numbers R, is said to be a generalized fuzzy number if its membership function has the following characteristics:

1. $\mu_A : R \to [0, w]$ is continuous.

2. $\mu_A(x) = 0$, for all $x \in (-\infty, a] \cup [d, \infty)$.

3. $\mu_A(x)$ is strictly on $[a, b]$ and strictly decreasing on $[c, d]$.

4. $\mu_A(x) = w$, for all $x \in [b, c]$, where $0 < w \le 1$.

Definition 2. [6] A generalized fuzzy number, denoted as $A = (a, b.c, d; w)$ is said to be a generalized trapezoidal fuzzy number if its membership function is given by

$$\mu_A(x) = \begin{cases} \dfrac{w(x-a)}{(b-a)}, & a \le x < b, \\ w & , b \le x \le c, \\ \dfrac{w(x-d)}{(c-d)}, & c < x \le d, \\ 0 & , \text{otherwise} \end{cases}$$

2.2 Intuitionisitic Fuzzy Sets

Definition 3. [19] An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ on the universal set X is characterized by a truth membership function $\mu_A(x), \mu_A(x) : X \to [0,1]$ and a false membership $\nu_A(x), \nu_A(x) : X \to [0,1]$. The values $\mu_A(x)$ and $\nu_A(x)$ represents the degree of membership and degree of non-membership of x and always satisfies the condition $\mu_A(x) + \nu_A \le 1$. The value $1 - \mu_A(x) + \nu_A$ represents the degree of hesitation of $x \in X$.

Definition 4. [19] An intuitionistic fuzzy set A, defined on the universal set of real numbers R, said to be triangular intuitionistic fuzzy number, denoted as $A=(a,b,c;w;u)$, if degree of membership and degree of non-membership are given by:

$$\mu_A(x)=\begin{cases}\dfrac{w(x-a)}{(b-a)}, & a\le x<b,\\ w & ,x=b,\\ \dfrac{w(c-x)}{(c-b)}, & b<x\le c,\\ 0 & ,\text{otherwise}\end{cases}$$

and

$$\nu_A(x)=\begin{cases}\dfrac{(b-x+u(x-a))}{(b-a)}, & a\le x<b,\\ u & ,x=b,\\ \dfrac{(x-b+u(c-x)}{(c-d)}, & b<x\le c,\\ 1 & ,\text{otherwise}\end{cases}$$

respectively. The values w and u represent the supremum of the degree of membership and the infimum of the degree of non-membership membership, respectively.

2.3 Vague sets and exponential vague sets

Definition 5. [11] A vague set $A=\{(x,\mu_A(x),1-\nu_A(x))\,|\,x\in X\}$ on the universal set X is characterized by a truth membership function $\mu_A(x),\mu_A(x):X\to[0,1]$ and a false membership $\nu_A(x),\nu_A(x):X\to[0,1]$. The values $\mu_A(x)$ and $\nu_A(x)$ represents the degree of membership and degree of non-membership of x and always satisfies the condition $\mu_A(x)+\nu_A\le 1$. The value $1-\mu_A(x)+\nu_A$ represents the degree of hesitation of $x\in X$.

Definition 6. [11] A vague set A, defined on the universal set of real numbers R, denoted as $A=(a,b,c,d;\lambda,\rho)$, where $a\le b\le c\le d$ and $\lambda\le\rho$, is said to be a triangular vague set if degree of membership, $\mu_A(x)$, and complement of the degree of non-membership, $1-\nu_A(x)$, are given by

$$\mu_A(x)=\begin{cases}\dfrac{\lambda(x-a)}{(b-a)}, & a\le x<b,\\ \lambda & ,x=b,\\ \dfrac{\lambda(x-c)}{(b-c)}, & b<x\le c,\\ 0 & ,\text{otherwise}\end{cases}$$

and

$$1-\nu_A(x)=\begin{cases}\dfrac{\rho(x-a)}{(b-a)}, & a\le x<b,\\ \rho & ,x=b,\\ \dfrac{\rho(x-c)}{(b-c)}, & b<x\le c,\\ 0 & ,\text{otherwise}\end{cases}$$

Definition 7. A vague set A, defined on the universal set of real numbers R, denoted as $A=(a,b,c,d;\lambda,\rho)$, where $a\le b\le c\le d$ and $\lambda\le\rho$, is said to be a exponential vague set if degree of membership, $\mu_A(x)$, and complement of the degree of non-membership, $1-\nu_A(x)$, are given by

$$\mu_A(x)=\begin{cases}\lambda\exp\left\{-\dfrac{(b-x)}{(b-a)}\right\}, & a\le x<b,\\ \lambda, & b\le x\le c,\\ \lambda\exp\left\{-\dfrac{(x-c)}{(d-c)}\right\}, & c<x\le d,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\mu_A(x)=\begin{cases}\rho\exp\left\{-\dfrac{(b-x)}{(b-a)}\right\}, & a\le x<b,\\ \rho, & b\le x\le c,\\ \rho\exp\left\{-\dfrac{(x-c)}{(d-c)}\right\}, & c<x\le d,\\ 0, & \text{otherwise}\end{cases}$$

2.4 Arithmetic operations between exponential vague sets

Let $A=(a_1,b_1,c_1,d_1;\lambda_1,\rho_1)$ and $B=(a_2,b_2,c_2,d_2;\lambda_2,\rho_2)$ be two exponential vague sets then

(i) $A\oplus B$
$=(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2;\min(\lambda_1,\lambda_2),\min(\rho_1,\rho_2))$

(ii) $A\Theta B$
$=(a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2;\min(\lambda_1,\lambda_2),\min(\rho_1,\rho_2))$

(iii) $\alpha A=\begin{cases}(\alpha a_1,\alpha b_1,\alpha c_1,\alpha d_1;\lambda_1,\rho_1),\alpha\geq 0\\(\alpha d_1,\alpha c_1,\alpha b_1,\alpha a_1;\lambda_1,\rho_1),\alpha<0\end{cases}$

3. Overview of existing ranking approaches

In this section, the existing ranking approach [8, 19] are briefly discussed.

3.1 Chen and Sanguansat ranking approach

Let $A=(a_1,b_1,c_1,d_1;w_A)$ and $B=(a_2,b_2,c_2,d_2;w_B)$ be two generalized fuzzy numbers, then use the following steps to compare A and B .

Step 1. Standardize each generalized fuzzy number A and B into A^* and B^* as follows:

$$A=\left(\frac{a_1}{k},\frac{b_1}{k},\frac{c_1}{k},\frac{d_1}{k};w_A\right)=(a_1^*,b_1^*,c_1^*,d_1^*;w_A) \qquad (1)$$

Where, $k=\max(\lceil|a_1|\rceil,\lceil|b_1|\rceil,\lceil|c_1|\rceil,\lceil|d_1|\rceil)$, $|a_1|,|b_1|,|c_1|,|d_1|$ denoted the absolute value of a_1,b_1,c_1,d_1 respectively, and $\lceil|a_1|\rceil,\lceil|b_1|\rceil,\lceil|c_1|\rceil,\lceil|d_1|\rceil$ denoted the upper bound of $|a_1|,|b_1|,|c_1|,|d_1|$ respectively.

$$B=\left(\frac{a_2}{k},\frac{b_2}{k},\frac{c_2}{k},\frac{d_2}{k};w_b\right)=(a_2^*,b_2^*,c_2^*,d_2^*;w_B) \qquad (2)$$

Where, $k=\max(\lceil|a_2|\rceil,\lceil|b_2|\rceil,\lceil|c_2|\rceil,\lceil|d_2|\rceil)$, $|a_2|,|b_2|,|c_2|,|d_2|$ denoted the absolute value of a_2,b_2,c_2,d_2 respectively, and $\lceil|a_2|\rceil,\lceil|b_2|\rceil,\lceil|c_2|\rceil,\lceil|d_2|\rceil$ denoted the upper bound of $|a_2|,|b_2|,|c_2|,|d_2|$ respectively.

Step 2. Calculate the areas $AreaA_L^-$ and $AreaA_R^-$, respectively, which denote the areas from the membership function curves of μ_A^L and μ_A^R respectively, to the membership function curve of the generalized fuzzy number $(-1,-1,-1,-1;w_A)$ respectively, where

$$\mu_A^L=w_A\frac{(x-a_1^*)}{(b_1^*-a_1^*)},a_1^*\leq x<b_1^* \qquad (3)$$

$$\mu_A^R=w_A\frac{(x-d_1^*)}{(c_1^*-d_1^*)},c_1^*<x\leq d_1^* \qquad (4)$$

and

$$AreaA_L^-=w_A\frac{(a_1^*+1)+(b_1^*+1)}{2} \qquad (5)$$

$$AreaA_R^-=w_A\frac{(c_1^*+1)+(d_1^*+1)}{2} \qquad (6)$$

Then, calculate the areas $AreaA_L^+$ and $AreaA_R^+$, respectively, which denote the areas from the membership function curves of μ_A^L and μ_A^R defined in Eqs. (3) and, (4), respectively, to the membership function curve of the generalized fuzzy number $(-1,-1,-1,-1;w_A)$, where

$$AreaA_L^+=w_A\frac{(1-a_1^*)+(1-b_1^*)}{2} \qquad (7)$$

$$AreaA_R^+=w_A\frac{(1-c_1^*)+(1-d_1^*)}{2} \qquad (8)$$

Similarly, for generalized fuzzy set B

$$AreaB_L^-=w_A\frac{(a_2^*+1)+(b_2^*+1)}{2} \qquad (9)$$

$$AreaB_R^-=w_A\frac{(c_2^*+1)+(d_2^*+1)}{2} \qquad (10)$$

$$AreaB_L^+ = w_A \frac{(1-a_2^*)+(1-b_2^*)}{2} \quad (11)$$

$$AreaB_R^+ = w_A \frac{(1-c_2^*)+(1-d_2^*)}{2} \quad (12)$$

Step 3. Calculate the values

$$XI_{A^*} = AreaA_L^- + AreaA_R^- \quad (13)$$

$$XD_{A^*} = AreaA_L^+ + AreaA_R^+ \quad (14)$$

and

$$XI_{B^*} = AreaB_L^- + AreaB_R^- \quad (15)$$

$$XD_{B^*} = AreaB_L^+ + AreaB_R^+ \quad (16)$$

Step 4. Calculate the ranking score $Score(A^*)$ and $Score(B^*)$ of each generalized trapezoidal fuzzy number A^* and B^* as follows:

$$Score(A^*) = \frac{XI_{A^*} - XD_{A^*}}{XI_{A^*} + XD_{A^*} + (1-w_A)} \quad (17)$$

and

$$Score(B^*) = \frac{XI_{B^*} - XD_{B^*}}{XI_{B^*} + XD_{B^*} + (1-w_B)} \quad (18)$$

The following three cases may arise

(i) If $Score(A^*) > Score(B^*)$, then $A \succ B$.

(ii) If $Score(A^*) < Score(B^*)$, then $A \prec B$.

(iii) If $Score(A^*) = Score(B^*)$, then $A \sim B$

3.2 Li ranking approach

Let $A = (a_1, b_1, c_1; w_1, u_1)$ and $B = (a_2, b_2, c_2; w_2, u_2)$ be two triangular intuitionistic fuzzy numbers then use the following steps to compare A and B:

Step 1. Calculate

$$V(A,\alpha) = V_\mu(A) + \alpha(V_\nu(A) - V_\mu(A)),$$

$A(A,\alpha) = A_\nu(A) - \alpha(A_\nu(A) - A_\mu(A))$, where

$$V_\mu(A) = \frac{w_1(a_1 + 4b_1 + c_1)}{6}$$

$V_\nu(A) = \dfrac{(1-u_1)(a_1 + 4b_1 + c_1)}{6}$, $A_\mu(A) = \dfrac{w_1(c_1 - a_1)}{3}$

$A_\mu(A) = \dfrac{(1-u_1)(c_1 - a_1)}{3}$ $\quad \alpha \in [0,1]$

and

$$V(B,\alpha) = V_\mu(B) + \alpha(V_\nu(B) - V_\mu(B))$$

$A(B,\alpha) = A_\nu(B) - \alpha(A_\nu(B) - A_\mu(B))$, where

$V_\mu(B) = \dfrac{w_2(a_2 + 4b_2 + c_2)}{6}$,

$V_\nu(B) = \dfrac{(1-u_2)(a_2 + 4b_2 + c_2)}{6}$, $A_\mu(B) = \dfrac{w_2(c_2 - a_2)}{3}$

$A_\mu(B) = \dfrac{(1-u_2)(c_2 - a_2)}{3}$, $\alpha \in [0,1]$

Step 2. Calculate

$\Re(A,\alpha) = \dfrac{V(A,\alpha)}{1 + A(A,\alpha)}$ and $\Re(b,\alpha) = \dfrac{V(B,\alpha)}{1 + A(B,\alpha)}$

The following three cases may arise

(i) If $\Re(A,\alpha) < \Re(B,\alpha)$, then $A \prec B$.

(ii) If $\Re(A,\alpha) > \Re(B,\alpha)$, then $A \succ B$.

(iii) If $\Re(A,\alpha) = \Re(B,\alpha)$, then $A \sim B$

4. Shortcomings of existing ranking approaches

In this section, the shortcomings of the existing ranking approaches [8, 19] are Pointed out.

4.1 Shortcomings of Chen and Sanguansat ranking approach

Here we pointed out the shortcomings of Chen and Sanguansat ranking approach [8], on the basis of reasonable properties of fuzzy quantities [34] and on the basis of height of fuzzy numbers.

4.1.1 On the basis of reasonable properties for fuzzy quantities.

With the help of some examples it shown that the ranking function, proposed by Chen and Sanguansat [8] does not satisfy the reasonable property, $A \succ B \Rightarrow A\Theta B \succ B\Theta B$, for the ordering of fuzzy quantities i.e., according to Chen and Sanguansat approach [8] $A \succ B \not\Rightarrow A\Theta B \succ B\Theta B$, which is a contradiction according to Wang and Lee [34].

Example 1. Let $A = (0.1,0.3,0.3,0.5;0.8)$, and $B = (0.1,0.3,0.3,0.5;1)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8] $B \succ A$ but $B\Theta A \prec A\Theta A$ i.e., $B \succ A \not\Rightarrow B\Theta A \succ A\Theta A.$

Example 2. Let $A = (0.1,0.3,0.3,0.5;1)$, and $B = (0.3,0.5,0.5,0.7;1)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8] $B \succ A$ but $B\Theta A \prec A\Theta A$ i.e., $B \succ A \not\Rightarrow B\Theta A \succ A\Theta A.$

Example 3. Let $A = (-0.5,-0.3,-0.3,-0.1;1)$, and $B = (0.1,0.3,0.3,0.5;1)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8] $B \succ A$ but $B\Theta A \prec A\Theta A$ i.e., $B \succ A \not\Rightarrow B\Theta A \succ A\Theta A.$

4.1.2 On the basis of height of fuzzy numbers

In some cases Chen and Sanguansat approach [8] indicates that the ranking of fuzzy numbers depends upon height of fuzzy numbers while in several cases it indicates that the ranking of fuzzy number does not depend upon the height of fuzzy numbers, which is not reasonable.

Let $A = (a_1, a_2, a_3, a_4; w_1)$ and $B = (a_1, a_2, a_3, a_4; w_2)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8]

Case (i) If $(a_1 + a_2 + a_3 + a_4) \neq 0$ then

$$\begin{cases} A \prec B, \text{ if } w_1 < w_2, \\ A \succ B, \text{ if } w_1 > w_2, \\ A \sim B, \text{ if } w_1 = w_2. \end{cases}$$

Case (ii) If $(a_1 + a_2 + a_3 + a_4) = 0$ then $A \sim B$ for all values of w_1 and w_2.

Example 4. Let $A = (1,1,1,1; w_1)$ and $B = (1,1,1,1; w_2)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Sanguansat approach [8]

$A \prec B$ if $w_1 < w_2$, $A \succ B$ if $w_1 > w_2$, and $A \sim B$ if $w_1 = w_2$.

Example 5. Let $A = (-0.4,-0.2,-0.1,0.7; w_1)$ and $B = (-0.4,-0.2,-0.1,0.7; w_2)$, be two generalized trapezoidal fuzzy numbers then $A \sim B$ for all values of w_1 and w_2.

According to Chen and Sanguansat approach [8], in first case ranking of fuzzy numbers depends upon height and in second case ranking does not depend upon the height which is not reasonable.

4.2 Shortcomings of Li ranking approach

Li [19] pointed out the shortcomings of all the existing approaches for the ranking of intuitionistic fuzzy numbers and proposed a new approach for the ranking of intuitionistic fuzzy numbers. In this section, the shortcomings of existing approach [19] are pointed out.

(i) The existing results, presented in Theorem 1 to Theorem 6 [19], are correct only if both ~ A and ~B are intuitionistic fuzzy numbers with same degree of membership and the same degree of non-membership but in the real life problems, we need to compare the intuitionistic fuzzy numbers with different degree of membership and degree of non-membership.

If we have two intuitionistic fuzzy numbers with unequal degree of membership and degree of non-membership than the existing approach [19] can not be used, also in that case the ranking function does not satisfy the reasonable property,

$A \succ B \Rightarrow A\Theta B \succ B\Theta B$ for the fuzzy quantities [34] i.e., $\Re(A,\alpha) > \Re(B,\alpha) \not\Rightarrow \Re(A\Theta B,\alpha) > \Re(B\Theta B,\alpha)$.

Example 6. Let $A = (1,3,5;0.3,0.2)$ and $A = (4,8,9;0.4,0.1)$ be two triangular intuitionistic fuzzy numbers. According to existing ranking approach [19] the values of $\Re(A,\frac{1}{2})$ and $\Re(B,\frac{1}{2})$ are 0.6347 and 0.822 respectively so $A \prec B$ but the values of $\Re(A\Theta B)$ and $\Re(B\Theta B)$ are -10.2 and 0 respectively which is a contradiction i.e., $\Re(A\Theta B,\frac{1}{2}) \nsucc \Re(B\Theta B,\frac{1}{2})$

Hence, the results, obtained by using the existing ranking approach [19], are valid for intuitionistic fuzzy numbers with equal degree of membership and degree of non-membership.

(ii) In some cases, the existing approach [19] indicates that the ranking of intuitionistic fuzzy numbers depends upon degree of membership and degree of nonmembership of intuitionistic fuzzy numbers while in several cases the ranking does not upon degree of membership and degree of non-membership of intuitionistic fuzzy numbers.

Let $A = (a_1,a_2,a_3;w_1,u_1)$ and $B = (a_1,a_2,a_3;w_2,u_2)$ be two intuitionistic fuzzy numbers then according to existing approach [19].

Case (i) If $(a_1 + 4a_2 + a_3) \neq 0$ then

$$\begin{cases} A \prec B, \text{ if } \Re(\mathrm{A},\alpha) < \Re(\mathrm{B},\alpha), \\ A \succ B, \text{ if } \Re(\mathrm{A},\alpha) > \Re(\mathrm{B},\alpha), \\ A \sim B, \text{ if } \Re(\mathrm{A},\alpha) = \Re(\mathrm{B},\alpha). \end{cases}$$

Case (ii) If $(a_1 + 4a_2 + a_3) = 0$ then $A \sim B$ for all values of w_1, u_1, w_2 and u_2.

Example 7. Let $A = (-0.2,0,0.2;w_1,u_1)$ and $B = (-0.2,0,0.2;w_2,u_2)$ be two intuitionitsic fuzzy numbers then, according to existing approach [19] $A \sim B$ for all values of w_1, u_1, w_2 and u_2.

Example 8. Let $A = (-8,1,4;w_1,u_1)$ and $B = (-8,1,4;w_2,u_2)$ be two intuitionitsic fuzzy numbers then, according to existing approach [19] $A \sim B$ for all values of w_1, u_1, w_2 and u_2.

Example 9. Let $A = (1,1,1;w_1,u_1)$ and $B = (1,1,1;w_2,u_2)$ be two intuitionitsic fuzzy numbers then, according to existing approach [19]

- $A \prec B$ if $\alpha(1-u_1) + (1-\alpha)w_1 < \alpha(1-u_2) + (1-\alpha)w_2$
- $A \succ B$ if $\alpha(1-u_1) + (1-\alpha)w_1 > \alpha(1-u_2) + (1-\alpha)w_2$,
- $A \sim B$ if $\alpha(1-u_1) + (1-\alpha)w_1 = \alpha(1-u_2) + (1-\alpha)w_2$,

According to existing approach [19], case (i) shows that ranking of fuzzy numbers depends upon degree of membership and degree of non-membership of intuitionistic fuzzy numbers while case (ii) shows ranking does not depend upon degree of membership and degree of non-membership of intuitionistic fuzzy numbers, which is again not reasonable.

5. Proposed approach

Let $A = (a_1,b_1,c_1,d_1;\lambda_1,\rho_1)$ and $B = (a_2,b_2,c_2,d_2;\lambda_2,\rho_2)$ be two exponential vague sets, where $a_1 \le b_1 \le c_1 \le d_1, \lambda_1 \le \rho_1$ and $a_2 \le b_2 \le c_2 \le d_2, \lambda_2 \le \rho_2$ then use the following steps to compare A and B:

Step 1. Find $(\lambda,\rho) = (\min(\lambda_1,\lambda_2), \min(\rho_1,\rho_2))$.

Step 2. Find $\Re(A_\lambda) = \int_0^\lambda \{\alpha(b_1 + (b_1 - a_1)\log(\frac{x}{\lambda}))$

$$+ (1-\alpha)(c_1 - (d_1 - c_1)\log(\frac{x}{\lambda}))\}$$
$$\Rightarrow \Re(A_\lambda) = \alpha a_1 \lambda + (1-\alpha) d_1 \lambda$$

and

$\Re(A_{\rho-\lambda})=\int_{\lambda}^{\rho}\{\alpha L_{\rho-\lambda}^{-1}(x)+(1-\alpha)R_{\rho-\lambda}^{-1}(x)\}dx$, where

$L_{\rho-\lambda}^{-1}(x)=b_1-(b_1-a_1)\log(\frac{x}{\lambda})\log(\frac{x}{\rho}),$

$R_{\rho-\lambda}^{-1}(x)=c_1+(d_1-c_1)\log(\frac{x}{\lambda})\log(\frac{x}{\rho}),$

$$\Rightarrow \Re(A_{\rho-\lambda})=\alpha\{(b_1(\rho-\lambda)-(b_1-a_1)\log\frac{\lambda}{\rho}(-\lambda\log(\frac{\lambda}{\rho})-(\rho-\lambda))\}+(1-\alpha)\{(c_1(\rho-\lambda)-(d_1-c_1)\log\frac{\lambda}{\rho}(-\lambda\log(\frac{\lambda}{\rho})-(\rho-\lambda))\}$$

Now, $\Re(A)=\Re(A_\lambda)+\Re(A_{\rho-\lambda})$

$$=\alpha\{a_1\lambda+(b_1(\rho-\lambda)-(b_1-a_1)\log\frac{\lambda}{\rho}(-\lambda\log(\frac{\lambda}{\rho})-(\rho-\lambda))\}+(1-\alpha)\{d_1\lambda+(c_1(\rho-\lambda)-(d_1-c_1)\log\frac{\lambda}{\rho}(-\lambda\log(\frac{\lambda}{\rho})-(\rho-\lambda))\}$$, where $\alpha\in[0,1]$

Similarly $\Re(B)=\Re(B_\lambda)+\Re(B_{\rho-\lambda})$

$$=\alpha\{a_2\lambda+(b_2(\rho-\lambda)-(b_2-a_2)\log\frac{\lambda}{\rho}(-\lambda\log(\frac{\lambda}{\rho})-(\rho-\lambda))\}+(1-\alpha)\{d_2\lambda+(c_2(\rho-\lambda)-(d_2-c_2)\log\frac{\lambda}{\rho}(-\lambda\log(\frac{\lambda}{\rho})-(\rho-\lambda))\}$$, where $\alpha\in[0,1]$

Step 3. Check $\Re(A)>\Re(B)$ or $\Re(A)<\Re(B)$ or $\Re(A)=\Re(B)$

Case (i) if $\Re(A)>\Re(B)$ then $A\succ B\forall\alpha$.

Case (ii) if $\Re(A)<\Re(B)$ then $A\prec B\forall\alpha$.

Case (i) if $\Re(A)=\Re(B)$ then $A\sim B\forall\alpha$

It can be easily shown that the proposed ranking formula for the ordering of vague sets is generalization of existing ranking approaches [1, 7, 20].

Corollary 1. If $0\le\lambda=\rho\le1$, then (17) reduces to $\Re(A)=\frac{\lambda(a_1+d_1)}{2}$, which is the ranking formula for generalized exponential fuzzy numbers [7].

Corollary 2. If $0\le\lambda=\rho\le1$ and left and right membership functions of A as linear functions, then (17) reduces to $\Re(A)=\frac{1}{2}(\lambda(a_1+d_1)+\frac{\lambda(b_1-a_1+c_1-d_1)}{3})=\frac{\lambda(a_1+b_1+c_1+d_1)}{4}$, which is the ranking formula for generalized trapezoidal fuzzy numbers [20].

Corollary 3. If $0\le\lambda=\rho\le1$, $b_1=c_1$ and left and right membership functions of A as linear functions, then (17) reduces to $\Re(A)=\frac{1}{2}(\lambda(a_1+d_1)+\frac{\lambda(b_1-a_1+b_1-d_1)}{3})=\frac{\lambda(a_1+2b_1+d_1)}{4}$, which is the ranking formula for generalized triangular fuzzy numbers [20].

Corollary 4. If $0\le\lambda=\rho\le1$ and left and right membership functions of A as linear functions, then (17) reduces $\Re(A)=\frac{1}{2}((a_1+d_1)+\frac{(b_1-a_1+c_1-d_1)}{3})=\frac{(a_1+b_1+c_1+d_1)}{4}$, which is the ranking formula for normal trapezoidal fuzzy numbers [1, 20].

Corollary 5. If $0\le\lambda=\rho\le1$, $b_1=c_1$ and left and right membership functions of A as linear functions, then (17) reduces to $\Re(A)=\frac{1}{2}((a_1+d_1)+\frac{(b_1-a_1+b_1-d_1)}{3})=\frac{(a_1+2b_1+d_1)}{4}$, which is the ranking formula for generalized triangular fuzzy numbers [1,20].

Corollary 6. If left and right membership functions of A as linear functions, then (17) reduces to $\Re(A)=\alpha\{\lambda\frac{(a_1+b_1)}{2}+\frac{(\rho-\lambda)}{2}(b_1+a_1+\frac{\lambda}{\rho}(b_1-a_1))\}$

$$+(1-\alpha)\{\lambda\frac{(d_1+c_1)}{2}+\frac{(\rho-\lambda)}{2}(d_1+c_1+\frac{\lambda}{\rho}(d_1-c_1))\},$$

which is the ranking formula for trapezoidal vague sets.

Corollary 7. If left and right membership functions of A as linear functions and $b=c$, then (17) reduces to

$$\Re(A)=\alpha\{\lambda\frac{(a_1+b_1)}{2}+\frac{(\rho-\lambda)}{2}(b_1+a_1+\frac{\lambda}{\rho}(b_1-a_1))\}$$
$$+(1-\alpha)\{\lambda\frac{(d_1+c_1)}{2}+\frac{(\rho-\lambda)}{2}(d_1+b_1+\frac{\lambda}{\rho}(d_1-b_1))\},$$

which is the ranking formula for triangular vague sets.

6. Results and discussion

In this section, it is proved that the proposed ranking function satisfies the reasonable properties for the ordering of fuzzy quantities. Also some examples are taken to show, shortcoming that exits in existing methods are not occurred in proposed approach.

Results of the proposed approach are compared with the existing ranking approaches.

6.1 Reasonable properties: Validation of proposed ranking approach

Wang and Kerre [33] proposed some axioms as a reasonable properties of ordering fuzzy quantities for the ranking approach $\Re$. These properties are:

P1. For an arbitrary subset S of F and $A\in S$, $A \underset{\sim}{\succ} A$ by $\Re$ on S where, F is a set of exponential vague sets.

P2. For an arbitrary subset S of F and $A,B\in S$, $A \underset{\sim}{\succ} B$ and $B \underset{\sim}{\succ} A$ by $\Re$ on S, then $A\sim B$ by $\Re$ on S.

P3. For an arbitrary subset S of F and $A,B,C\in S$, $A \underset{\sim}{\succ} B$ and $B \underset{\sim}{\succ} C$ by $\Re$ on S, then $A \underset{\sim}{\succ} C$ by $\Re$ on.

P4. If $A\cap B=0$ and A is on the right of B then $A \underset{\sim}{\succ} B$.

P5. Let S and S' be two arbitrary finite sets of exponential vague sets in which proposed ranking function can be applied, and $A,B\in S\cap S'$, we obtain the ranking order $A \underset{\sim}{\succ} B$ by $\Re$ in S iff $A \underset{\sim}{\succ} B$ by $\Re$ in S'.

P6. If $A \underset{\sim}{\succ} B$ by $\Re$, then $A+C \underset{\sim}{\succ} B+C$ by $\Re$ when $C\neq 0$.

P7. Let A,B,AC,BC be the elements of S and $C\geq 0$. $A \underset{\sim}{\succ} B$ by $\Re \Rightarrow AC \underset{\sim}{\succ} BC$ by $\Re$.

Now, we prove that proposed ranking function satisfies the some of the reasonable properties.

Proposition 1. For an arbitrary subset S of F and $A\in S$, $A \underset{\sim}{\succ} A$ by $\Re$ on S where, F is a set of exponential vague sets.

Proof. Since $A\in S \Rightarrow A,A\in S$ and $\Re(A)\geq\Re(A)$ $\Rightarrow A \underset{\sim}{\succ} A$ by proposed ranking function.

Proposition 2. For an arbitrary subset S of F and $A,B\in S$, $A \underset{\sim}{\succ} B$ and $B \underset{\sim}{\succ} A$ by $\Re$ on S, then $A\sim B$ by $\Re$ on S.

Proof. By proposed ranking function, $A \underset{\sim}{\succ} B$ means $\Re(A)\geq\Re(B)$ and $B \underset{\sim}{\succ} A$ means $\Re(A)\leq\Re(B)$, and hence $\Re(A)=\Re(B)$ $\Rightarrow A\sim B$.

Proposition 3. For an arbitrary subset S of F and $A,B,C\in S$, $A \underset{\sim}{\succ} B$ and $B \underset{\sim}{\succ} C$ by $\Re$ on S, then $A \underset{\sim}{\succ} C$ by $\Re$ on.

Proof. $A \underset{\sim}{\succ} B \Rightarrow \Re(A)\geq\Re(B)$ and $B \underset{\sim}{\succ} C$ $\Rightarrow \Re(B)\geq\Re(C)$. Since Ranking of exponential vague sets is a crisp number i.e., the ranking function map each exponential vague set into a real line, Therefore $\Re(A)\geq\Re(C)$ hence $A \underset{\sim}{\succ} C$.

Proposition 4. If $A \cap B = 0$ and A is on the right of B then $A \underset{\sim}{\succ} B$.

Proof. If $A \cap B = 0$ and A is on the right of B, then $\Re(A) \geq \Re(B)$ ie., $A \underset{\sim}{\succ} B$.

Proposition 5. Let S and S' be two arbitrary finite sets of exponential vague sets in which proposed ranking function can be applied, and $A, B \in S \cap S'$, we obtain the ranking order $A \underset{\sim}{\succ} B$ by $\Re$ in S iff $A \underset{\sim}{\succ} B$ by $\Re$ in S'.

Proof. Given that $A \underset{\sim}{\succ} B$ in $S' \Leftrightarrow \Re(A) \geq \Re(B)$ in S'

$\Leftrightarrow \Re(A) \geq \Re(B)$ in $S \cap S'$ ($\because A, B \in S \cap S'$)

$\Leftrightarrow \Re(A) \geq \Re(B)$ in $S' (\because S \cap S' \subset S) \Rightarrow A \underset{\sim}{\succ} B$ in S.

Proposition 6. If $A \underset{\sim}{\succ} B$ by $\Re$, then $A + C \underset{\sim}{\succ} B + C$ by $\Re$ when $\Re(C) \neq 0$ with degree of membership and non membership of C is less than or equal to the degree of membership of both, A and B.

Proof. If $A \underset{\sim}{\succ} B \Rightarrow \Re(A) \geq \Re(B) \Rightarrow \Re(A) + \Re(C) \geq \Re(B) + \Re(C) \Rightarrow \Re(A + C) + \Re(B + C)$ ($\because$ degree of membership and non membership of C is less than or equal to the degree of membership of both, A and B) $\Rightarrow A + C \underset{\sim}{\succ} B + C$.

We take the examples from Section 4, and solved them with proposed ranking approach, to show that shortcomings are now removed.

Example 10. Let $A = (0.1, 0.3, 0.3, 0.5; 0.8)$, and $B = (0.1, 0.3, 0.3, 0.5; 1)$ be two generalized trapezoidal fuzzy numbers then $A \Theta B = (-0.4, 0, 0, 0.4; 0.8)$ and $B \Theta B = (-0.4, 0, 0, 0.4; 1)$

Step 1. $\min(0.8, 1) = 0.8$

Step 2. $\Re(A \Theta B) = 0.4(0.4 - 0.8\alpha)$ and $\Re(B \Theta B) = 0.4(0.4 - 0.8\alpha)$. Since, $\Re(A \Theta B) = \Re(B \Theta B)\ \forall \alpha$, so $A \Theta B \sim B \Theta B$.

Example 11. Let $A = (0.1, 0.3, 0.3, 0.5; 1)$, and $B = (0.3, 0.5, 0.5, 0.7; 1)$ be two generalized trapezoidal fuzzy numbers then $A \Theta B = (-0.6, -0.2, -0.2, 0.2; 1)$ and $B \Theta B = (-0.4, 0, 0, 0.4; 1)$

Step 1. $\min(1, 1) = 1$

Step 2. $\Re(A \Theta B) = \frac{1}{2}(-0.8\alpha)$ and

$\Re(B \Theta B) = \frac{1}{2}(0.4 - 0.8\alpha)$.

For a pessimistic decision maker, with $\alpha = 0$, $\Re(A \Theta B) = 0$ and $\Re(B \Theta B) = 0.2$. Since $\Re(A \Theta B) < \Re(B \Theta B)$, so $(A \Theta B) \prec (B \Theta B)$.

For optimistic decision maker, with $\alpha = 1$, $\Re(A \Theta B) = -0.4$ and $\Re(B \Theta B) = -0.2$. Since $\Re(A \Theta B) < \Re(B \Theta B)$, so $(A \Theta B) \prec (B \Theta B)$.

For moderate decision maker with $\alpha = 0.5$, $\Re(A \Theta B) = -0.2$ and $\Re(B \Theta B) = 0.$ Since $\Re(A \Theta B) < \Re(B \Theta B)$, so $(A \Theta B) \prec (B \Theta B)$.

Example 12. Let $A = (-0.5, -0.3, -0.3, -0.1; 1)$, and $B = (0.1, 0.3, 0.3, 0.5; 1)$ be two generalized trapezoidal fuzzy numbers then $A \Theta B = (-0.6, -0.2, -0.2, 0.2; 1)$ and $B \Theta B = (-0.4, 0, 0, 0.4; 1)$

Step 1. $\min(1, 1) = 1$

Step 2. $\Re(A \Theta B) = \frac{1}{2}(-0.8 + 2.4\alpha)$ and

$\Re(B \Theta B) = \frac{1}{2}(0.4 - 0.8\alpha)$.

For a pessimistic decision maker, with $\alpha = 0$, $\Re(A \Theta B) = -0.4$ and $\Re(B \Theta B) = 0.2$. Since $\Re(A \Theta B) < \Re(B \Theta B)$, so $(A \Theta B) \prec (B \Theta B)$.

For optimistic decision maker, with $\alpha = 1$, $\Re(A \Theta B) = -0.266$ and $\Re(B \Theta B) = -0.2$. Since $\Re(A \Theta B) > \Re(B \Theta B)$, so $(A \Theta B) \succ (B \Theta B)$.

For moderate decision maker with $\alpha = 0.5$, $\Re(A\Theta B) = 0.2$ and $\Re(B\Theta B) = 0.$ Since $\Re(A\Theta B) > \Re(B\Theta B)$, so $(A\Theta B) \succ (B\Theta B)$.

Example 13. Let $A = (1,1,1,1; w_1)$ and $B = (1,1,1,1; w_2)$ be two generalized trapezoidal fuzzy numbers then

Step 1. $\min(w_1, w_2) = w$ (say)

Step 2. $\Re(A\Theta B) = \frac{w}{2}(2\alpha + (1-\alpha)2) = w$ and

$\Re(A\Theta B) = \frac{w}{2}(2\alpha + (1-\alpha)2) = w$

$\Re(B\Theta B) = \frac{w}{2}(2\alpha + (1-\alpha)2) = w$.

Since, $\Re(A) = \Re(B) \ \ \forall \alpha$, so $A \sim B$.

Example 14. Let $A = (-0.4,-0.2,-0.1,0.7; w_1)$ and $B = (-0.4,-0.2,-0.1,0.7; w_2)$, be two generalized trapezoidal fuzzy numbers

Step 1. $\min(w_1, w_2) = w$ (say)

Step 2. $\Re(A\Theta B) = \frac{w}{2}(-0.6\alpha + (1-\alpha)0.6) = 0.3w$ and

$\Re(B\Theta B) = \frac{w}{2}(-0.6\alpha + (1-\alpha)0.6) = 0.3w$ .Since, $\Re(A) = \Re(B) \ \ \forall \alpha$, so $A \sim B$.

Example 15. Let $A = (1,3,5;0.3,0.8)$, and $B = (4,8,9;0.4,0.9)$ be two generalized triangular vague sets then $A\Theta B = (-8,-5,1;0.3,0.8)$ and $B\Theta B = (-5,0,5;0.4,0.9)$.

Step 1. $\min(0.3,0.4) = 0.3$ and $\min(0.8,0.9) = 0.8$.

Step 2. $\Re(A\Theta B) = \frac{1}{2}(-1.037 - 3.881\alpha)$ and

$\Re(B\Theta B) = \frac{1}{2}(2.47 - 0.94\alpha)$.

For a pessimistic decision maker, with $\alpha = 0$, $\Re(A\Theta B) = -1.0375$ and $\Re(B\Theta B) = 2.47$. Since $\Re(A\Theta B) < \Re(B\Theta B)$, so $(A\Theta B) \prec (B\Theta B)$.

For optimistic decision maker, with $\alpha = 1$, $\Re(A\Theta B) = -4.919$ and $\Re(B\Theta B) = 1.53$. Since $\Re(A\Theta B) < \Re(B\Theta B)$, so $(A\Theta B) \prec (B\Theta B)$.

For moderate decision maker with $\alpha = 0.5$, $\Re(A\Theta B) = -2.978$ and $\Re(B\Theta B) = 2$ Since $\Re(A\Theta B) < \Re(B\Theta B)$, so $(A\Theta B) \prec (B\Theta B)$.

Example 16. Let $A = (-2,0,2; \lambda_1, \rho_1)$, and $B = (-2,0,2; \lambda_2, \rho_2)$ be two triangular vague numbers then $A\Theta B = (-0.4,0,0.4; \lambda_3, \rho_3)$ where $\lambda_3 = \min(\lambda_1, \lambda_2)$ and $B\Theta B = (-0.4,0,0.4; \lambda_2, \rho_2)$.

Step 1. $\min(\lambda_3, \lambda_2) = \lambda$ and $\min(\rho_3, \rho_2) = \rho$

Step 2. Since, $\Re(A\Theta B) = \Re(B\Theta B) \ \ \forall \alpha$, so $A\Theta B \sim B\Theta B$.

Example 17. Let $A = (-8,1,4; \lambda_1, \rho_1)$, and $B = (-8,1,4; \lambda_2, \rho_2)$ be be two triangular vague numbers then $A\Theta B = (-12,0,12; \lambda_3, \rho_3)$ where $\lambda_3 = \min(\lambda_1, \lambda_2)$ and $B\Theta B = (-12,0,12; \lambda_2, \rho_2)$.

Step 1. $\min(\lambda_3, \lambda_2) = \lambda$ and $\min(\rho_3, \rho_2) = \rho$

Step 2. Since, $\Re(A\Theta B) = \Re(B\Theta B) \ \ \forall \alpha$, so $A\Theta B \sim B\Theta B$.

Example 18. Let $A = (1,1,1; \lambda_1, \rho_1)$, and $B = (1,1,1; \lambda_2, \rho_2)$ be be two triangular vague numbers then $A\Theta B = (0,0,0; \lambda_3, \rho_3)$ where $\lambda_3 = \min(\lambda_1, \lambda_2)$ and $B\Theta B = (0,0,0,; \lambda_2, \rho_2)$.

Step 1. $\min(\lambda_3, \lambda_2) = \lambda$ and $\min(\rho_3, \rho_2) = \rho$

Step 2. Since, $\Re(A\Theta B) = \Re(B\Theta B) \ \ \forall \alpha$, so $A\Theta B \sim B\Theta B$.

6.2 Comparative study

Different sets of fuzzy sets and vague sets are taken to compare the results of proposed and existing ranking approaches.

Set 1.

Let $A=(0.1,0.3,0.3,0.5;0.8)$ and $B=(0.1,0.3,0.3,0.5;1)$ be two trapezoidal fuzzy sets, take $\alpha=\frac{1}{2}$

Step 1. $\min(0.8,1)=0.8$

Step 2. $\Re(A)=0.24$ and $\Re(B)=0.24$.
Since $\Re(A)=\Re(B)$, so $A\sim B$.

Set 2.

Let $A=(0.1,0.3,0.3,0.5;1)$ and $B=(0.3,0.5,0.5,0.7;1)$ be two trapezoidal fuzzy sets, take $\alpha=\frac{1}{2}$

Step 1. $\min(1,1)=1$

Step 2. $\Re(A)=0.3$ and $\Re(B)=0.5$.
Since $\Re(A)<\Re(B)$, so $A\prec B$.

Set 3.

Let $A=(0.1,0.3,0.3,0.5;0.8)$ and $B=(-0.5,-0.3,-0.3,0.1;1)$ be two trapezoidal fuzzy sets, take $\alpha=\frac{1}{2}$

Step 1. $\min(1,1)=1$

Step 2. $\Re(A)=0.3$ and $\Re(B)=-0.3$.
Since $\Re(A)>\Re(B)$, so $A\succ B$.

Set 4.

Let $A=(5,7,9,11;0.4)$ and $B=(2,4,8,9;0.2)$ be two generalized exponential trapezoidal fuzzy sets, take $\alpha=\frac{1}{2}$

Step 1. $\min(0.4,0.2)=0.2$

Step 2. $\Re(A)=1.6$ and $\Re(B)=1.1$.
Since $\Re(A)>\Re(B)$, so $A\succ B$.

Set 5.

Let $A=(3,6,7,14;0.2,0.7)$ and $A=(1,4,6,9;0.3,0.4)$ be two exponential vague fuzzy sets, take $\alpha=\frac{1}{2}$

Step 1. $\min(0.2,0.3)=0.2$ and $\min(0.7,0.4)=0.4$

Step 2. $\Re(A)=1.52$ and $\Re(B)=1.13$.
Since $\Re(A)>\Re(B)$, so $A\succ B$.

Methods	Set 1	Set 2	Set 3	Set 4	Set5
Yager [36]	$A\sim B$	$A\sim B$	$A\sim B$	N.A	N.A
Liou and Wang [20]	$A\sim B$	$A\succ B$	$A\succ B$	N.A	N.A
Cheng [9]	$A\prec B$	$A\prec B$	$A\sim B$	N.A	N.A
Murakami et al. [26]	$A\prec B$	$A\prec B$	$A\succ B$	N.A	N.A
Chen and Li [7]	$A\sim B$	$A\prec B$	$A\succ B$	$A\succ B$	N.A
Chen and Chen [6]	$A\prec B$	$A\prec B$	$A\succ B$	N.A	N.A
Chen and Sanguansat [8]	$A\prec B$	$A\prec B$	$A\succ B$	N.A	N.A
Li [19]	$A\prec B$	$A\prec B$	$A\succ B$	N.A	N.A
Nehi [28]	N.A	$A\prec B$	$A\succ B$	N.A	N.A
Kumar et al. [15, 16, 17]	$A\sim B$	$A\prec B$	$A\succ B$	N.A	N.A
Proposed approach	$A\sim B$	$A\prec B$	$A\succ B$	$A\succ B$	$A\succ B$

Table 1. Comparison of proposed ranking approach with existing ranking approach where N.A denotes the not applicable.

7. Multicriteria decision-making problem based on the proposed ranking approach

Let $A=\{A_1,A_2,...,A_n\}$ be a set of alternatives and let $C=\{C_1,C_2,...,C_n\}$ be set of criteria. The values of an alternative on criteria $C_j(j=1,2,...,n)$ are exponential vague sets S_{ij}, which indicates the degree that the alternative A_i satisfies or does not satisfies the criterion C_j given by decision makers or experts according to linguistic values of exponential sets for linguistic terms.

The weights of criterion $C_j(j=1,2,...,n)$ are represented by exponential sets. The ranking weight value w_j for exponential vague set is obtained by Eq. (17). The normalized weights are obtained using the following equation:

$$w_j=\frac{\Re(w_j)}{\sum_{j=1}^{n}\Re(w_j)} \tag{19}$$

Therefore, the weighted ranking value for an alternative $A_i(i=1,2,...,m)$ is given by:

$$\Re_{w_j}(A_i)=\sum_{j=1}^{n}w_j\Re(S_{ij}) \tag{20}$$

Thus, the calculated weighted ranking value for an alternative is used to rank alternatives and then to select the best one in all the alternatives.

The above method can be summarized as follows:

(i) Calculate the ranking weight value w_j for criterion $C_j(j=1,2,...,n)$ by using Eq. (17) and (18).

(ii) Calculate the weighted ranking value for alternative $A_i(i=1,2,...,m)$ by using Eq. (17) and (19).

(iii) Rank the alternative and select the best one in accordance with weighted ranking vales $\Re_{w_j}(A_i)$.

7.1 Illustrative example

A numerical example has been taken to show the applicability of ranking function in multicriteria decision making problem.

Suppose there is a panel with three alternative to invest the money: (i) A_1 is car company, (ii) A_2 is food company, (iii) A_3 is computer company.

The investment must take a decision according to the following three criteria: (i) C_1 is the riskanalysis, (ii) C_2 is the growth analysis, (iii) C_3 is the environmental impact analysis.

The three possible alternative are to be evaluated under the above three criteria by corresponding to linguistic values of exponential sets for linguistic terms, as shown in Table 2.

Linguistic terms	Linguistic values of exponential vague sets
Absolute value	(0.0,0.0,0.0,0.0;0.0,0.0)
Low	(0.0,0.1,0.2,0.3;0.1,0.2)
Fairly low	(0.0,0.2,0.3,0.4;0.2,0.5)
Medium	(0.3,0.4,0.5,0.6;0.4,0.5)
Fairly high	(0.5,0.6,0.7,0.8;0.5,0.7)
High	(0.7,0.8,0.9,1.0;0.8,0.9)
Absolutely high	(1.0,1.0,1.0,1.0;1.0,1.0)

Table 2. Linguistic vales of exponential vague sets for linguistic terms.

Suppose we called three experts ($k=3$) to make the decision. They give the linguistic values of exponential vague sets.

The values of alternatives and criteria weights based on the decision makers or expert's knowledge are shown in Table 3.

Use the following steps to find the best alternative.

(i) Using Eqs. (17) and (18), we get the ranking weight value wj for criterion $C_j (j = 1,2,3)$, $w_1 = 0.383$, $w_2 = 0.271$, $w_3 = 0.345$.

(ii) Using Eqs. (17) and (19), we get the weighted ranking value for alternative $A_i (i = 1,2,3)$, $\Re_{w_1}(A_1) = 0.901$, $\Re_{w_2}(A_2) = 0.778$, $\Re_{w_3}(A_3) = 0.937$.

(iii) Rank the alternative as follows:

$$A_1 \prec A_2 \prec A_3$$

Thus according to above results the most desirable alternative is A_3.

8. Conclusions

In this paper, we present a new approach for the ranking of exponential vague sets. New representation and arithmetic operations between two exponential vague sets has been introduced. Some shortcomings of existing ranking approaches [8, 19] are pointed out. The proposed method consider not only the rank but also the decision maker optimistic attitude also with the help of some comparative examples it is shown that proposed ranking approach is more intuitive than the existing ranking approaches. Also the proposed ranking function satisfies the reasonable properties for the ordering of fuzzy quantity. Further the proposed ranking approach can effectively rank of various types of fuzzy sets and vague sets (normal, generalized, triangular, trapezoidal and exponential), which is another advantage of proposed ranking approach over the other exiting ranking approaches. For practical use, proposed ranking approach is applied to decision making problem.

	k	C_1	C_2	C_3
A_1	1 2 3	(0.1,0.2,0.3,0.4;0.2,0.3) (0.3,0.5,0.6,0.8;0.5,0.7) (0.3,0.5,0.6,0.8;0.5,0.7)	(0.2,0.3,0.5,0.7;0.3,0.6) (0.2,0.3,0.5,0.7;0.3,0.6) (0.2,0.3,0.5,0.7;0.3,0.6)	(0.3,0.4,0.6,0.7;0.8,0.9) (0.3,0.4,0.6,0.7;0.8,0.9) (0.3,0.5,0.6,0.8;0.5,0.7)
A_2	1 2 3	(0.3,0.4,0.6,0.7;0.8,0.9) (0.3,0.4,0.6,0.7;0.8,0.9) (0.1,0.2,0.3,0.4;0.2,0.3)	(0.1,0.2,0.3,0.4;0.2,0.3) (0.2,0.3,0.5,0.7;0.3,0.6) (0.1,0.4,0.6,0.7;0.5,0.6)	(0.1,0.4,0.6,0.7;0.5,0.6) (0.1,0.4,0.6,0.7;0.5,0.6) (0.1,0.4,0.6,0.7;0.5,0.6)
A_3	1 2 3	(0.2,0.3,0.5,0.7;0.3,0.6) (0.1,0.4,0.6,0.7;0.5,0.6) (0.3,0.5,0.6,0.8;0.5,0.7)	(0.1,0.4,0.6,0.7;0.5,0.6) (0.3,0.5,0.6,0.8;0.5,0.7) (0.1,0.4,0.6,0.7;0.5,0.6)	(0.2,0.3,0.5,0.7;0.3,0.6) (0.1,0.4,0.6,0.7;0.5,0.6) (0.1,0.4,0.6,0.7;0.5,0.6)
weight	1 2 3	(0.1,0.4,0.6,0.7;0.5,0.6) (0.3,0.4,0.6,0.7;0.8,0.9) (0.3,0.5,0.6,0.8;0.5,0.7)	(0.1,0.4,0.6,0.7;0.5,0.6) (0.1,0.4,0.6,0.7;0.5,0.6) (0.3,0.4,0.6,0.7;0.8,0.9)	(0.2,0.3,0.5,0.7;0.3,0.6) (0.1,0.4,0.6,0.7;0.5,0.6) (0.3,0.4,0.6,0.7;0.8,0.9)

Table 3. Linguistic vales of exponential vague sets for linguistic terms.

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References

[1] S. Abbasbandy and T. Hajjari, "A new approach for ranking of trapezoidal fuzzy numbers", Computers and Mathematics with Applications, 57, 413-419, 2009.

[2] K.T. Atanassov, "Intuitionistic fuzzy sets", Fuzzy Sets and Systems, 20, 87- 96, 1986.

[3] N. Bhatia and A. Kumar, "Sensitivity analysis for interval-valued fully fuzzy linear programming problems", Journal of Applied Research and Technology, 10, 871-884, 2012.

[4] H. Bustince and P. Burillo, "Vague sets are intuitionistic fuzzy sets", Fuzzy Sets and Systems, 79, 403-405, 1996.

[5] S. J. Chen and S. M. Chen, "Fuzzy risk analysis based on the ranking of generalized trapezoidal fuzzy numbers", Applied Intelligence, 26, 1-11, 2007.

[6] S. M. Chen and J. H. Chen, "Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads", Expert Systems with Applications, 36, 6833-6842, 2009.

[7] S.H. Chen and G.C. Li, "Representation, ranking, and distance of fuzzy number with exponential membership function using grade mean integration method", Tamsui Oxford journal of Mathematical Sciences, 16, 123-131, 2000.

[8] S.M. Chen and K. Sanguansat, "Analyzing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers", Expert Systems with Applications, 38, 2163-2171, 2011.

[9] C.H. Cheng, "A new approach for ranking fuzzy numbers by distance method", Fuzzy Sets and Systems, 95, 307-317, 1998.

[10] E. Elbas, "Fuzzy logic-based scenario recognition from video sequences", Journal of Applied Research and Technology, 11, 702-707, 2013.

[11] W.L. Gau and D.L. Buehrer, "Vague sets", IEEE Transactions on Systems, Man and Cybernetics, 23, 610-614, 1993.

[12] R. Jain, "Decision-making in the presence of fuzzy variables", IEEE Transactions on Systems Man and Cybernetics, 6, 698-703, 1976.

[13] A. Kaufmann and M.M. Gupta, "Fuzzy mathematical models in engineering and managment science", Elseiver Science Publishers. Amsterdam, Netherlands 1985.

[14] A. Kumar and M. Kaur, "A Ranking Approach for Intuitionistic Fuzzy Numbers and its Application", Journal of Applied Research and Technology, 11, 381-396, 2013.

[15] A. Kumar, P. Singh, P. Kaur and A. Kaur, "A new approach for ranking of L–R type generalized fuzzy numbers", Expert Systems with Applications, 38, 10906-10910, 2011.

[16] A. Kumar, P. Singh, A. Kaur and P. Kaur, "A new approach for ranking nonnormal p-norm trapezoidal fuzzy numbers", Computers and Mathematics with Applications, 61, 881-887, 2011.

[17] A. Kumar, P. Singh, A. Kaur and P. Kaur, "RM approach for ranking of L-R type Generalized Fuzzy Numbers", Soft Computing, 15, 1373-1381, 2011.

[18] H.C. Kwang and J.H. Lee, "A method for ranking fuzzy numbers and its application to decision making", IEEE Transactions on Fuzzy Systems, 7, 677-685, 1999.

[19] D.F. Li, "A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems", Computers and Mathematics with Applications, 60, 1557-1570, 2010.

[20] S. Liou and M. J. Wang, "Ranking fuzzy numbers with integral value", Fuzzy Sets and Systems, 50, 247-255, 1992.

[21] A. Lu and W. Ng, "Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better?" Lecture Notes in Computer Science, 3716, 401-416, 2005.

[22] S. P. Martagon, G. R. Salgado, G. F. Becerra, E. T. Cuautle, L.G. Fraga, I.G. Gomez and M.A.D. Villasenor, "selection of mosfet sizes by fuzzy sets intersection in

the feasible solutions space", Journal of Applied Research and Technology, 10, 472-483, 2012.

[23] J. A. Marquez , R. Cortes , H. R. Siller, V. Coello and D. Escamilla, "Fuzzy logic scheme for tip-sample distance control for a low cost near _eld optical microscope", Journal of Applied Research and Technology, 11, 886-894, 2013.

[24] H.B. Mitchell, "Ranking intuitionistic fuzzy numbers", International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 12, 377-386, 2004.

[25] M. Modarres and S. Sadi-Nezhad, "Ranking fuzzy numbers by preference ratio", Fuzzy Sets and Systems, 118, 429-436, 2001.

[26] S. Murakami, S. Maeda and S. Imamura, "Fuzzy decision analysis on the development of centralized regional energy control system", In Proceedings of the IFAC symposium on fuzzy information, knowledge representation and decision analysis, 363-368, 1983.

[27] V.L.G. Nayagam, G. Venkateshwari and G. Sivaraman, "Ranking of intuitionistic fuzzy numbers", IEEE International Conference on Fuzzy Systems, 1971-1974, 2008.

[28] H.M. Nehi, "A new ranking method for intuitionistic fuzzy numbers", International Journal of Fuzzy Systems, 12, 80-86, 2010.

[29] V. Rafe and M. H. Goodarzi, "A novel web-based human advisor fuzzy expert system", Journal of Applied Research and Technology, 11, 161-168, 2013.

[30] N. Ramli and D. Mohamad, "A comparative analysis of centroid methods in ranking fuzzy numbers", European Journal of Scientific Research, 28, 492-501, 2009.

[31] P. Singh, "A new approach for the ranking of fuzzy sets with different heights", Journal of Applied Research and Technology, 10, 941-949, 2012.

[32] J. Tian and Z.F. Yan, Fuzzy analytic hierarchy process for risk assessment to general assembling of satellite, Journal of Applied Research and Technology, 11, 568-577, 2013.

[33] X. Wang and E.E. Kerre, "Reasonable properties for the ordering of fuzzy quantities (I)", Fuzzy Sets and Systems, 118, 375-385, 2001.

[34] Y.J. Wang and H.S. Lee, "The revised method of ranking fuzzy numbers with an area between the centroid and original points", Computers and Mathematics with Applications, 55, 2033-2042, 2008.

[35] C. F. Wu , C. J. Lin ,H. Y. Lin and H. Chung, "Adjacent lane detection and lateral vehicle distance measurement using vision-based neuro-fuzzy approaches", Journal of Applied Research and Technology, 11, 251-258, 2013.

[36] R.R. Yager, "Ranking fuzzy subsets over the unit interval", In Proceeding of the 17th IEEE international conference on decision and control, 1435-1437, San Diego, CA, 2013.

[37] L.A. Zadeh, "Fuzzy sets", Information and Control, 8, 338-353 1965.