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"A Tensorial Form of the Theory of Functions". An Engineering Application to: Polynomial Interpolation

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Abstract

From basicconcepts such as: tensor calculus (Flügge, 1972); functional analysis (Mikhlin, 1964) and solid mechanics (Soedel, 1972) the objective of yhis objetive is to show that besides the "n" covariant functions (of functional analysis), linearly independent and not necessarily orthogonal, there is another group of "n" contravariant functions that are biorthogonal to the former group. The presentation of these two families gives rise to a new formulation of functional analysis in skew coordinates. We will see that the concept of skew manifolds finds immediate applicability to the problem of interpolation of arbitrary functions via the use of the new concept of covariant and contravariant polynomials. The theory and the examples demonstrate that the problems of interpolation and Fourier analysis can be grouped into one single theory.

Keywords: *Interpolation, index notation, covariant and contravariant polynomials, general skew manifolds (Tensor calculus), tensorial theory of functions, convergence.*

Resumen

A partir de conceptos básicos de cálculo tensorial (Flügge, 1972), análisis funcional (Mikhlin, 1964) y de mecánica de sólidos (Soedel, 1972), el objetivo de este artículo es demostrar que además de las *"n"* funciones covariantes (de análisis funcional), linealmente independientes pero no necesariamente ortogonales, existe otro grupo de *"n"* funciones contravariantes que son biortogonales al grupo ante rior. La presentación de estas dos familias de funciones da origen a una nueva formulación de análisis funcional en coordenadas oblicuas. Veremos que el concepto de espacios coordenados oblicuos encuentra aplicación inmediata al problema de interpolación de funciones arbitrarias vía el uso del nuevo concepto de polinomios covariantes y contravariantes. La teoría y los ejemplos demuestran que los problemas de interpolación y análisis de Fourier se pueden agrupar y tratar dentro de una sola y única teoría.

Descriptores: Interpolación, notación índice, polinomios covariantes y contravariantes, espacios gener ales oblicuos (cálculo tensorial), teoría tensorial de funciones, convergencia.

Introduction

One of the most controversial topics in numerical analysis is the problem of interpolation and a great variety of approximate methods can be found. Ho-wever, when we examine "Why and in what sense are those methods accurate" we find a disenchanting panorama since there are no answers to those questions (Carnaham *et al*., 1969) and (Forsythe *et al*., 1977). When trying to approximate a given arbitrary function *f(x)* with some polynomial

$$f(x)=\sum_{n=0}^{N} a_n x^n ,$$

it is a common procedure to select $n + 1$ points and to obtain the a_n coefficients from the solution of the following $n + 1$ equations

$$\begin{aligned} a_0 + a_1 x_0^1 + a_2 x_0^2 + a_3 x_0^3 + \ldots + a_n x_0^n &= f(x_0) \\ a_0 + a_1 x_1^1 + a_2 x_1^2 + a_3 x_1^3 + \ldots + a_n x_1^n &= f(x_1) \qquad (1) \\ \ldots\ldots\ldots\ldots \\ a_0 + a_1 x_n^1 + a_2 x_n^2 + a_3 x_n^3 + \ldots + a_n x_n^n &= f(x_n) \end{aligned}$$

It is clear that the choice of the $n + 1$ points is not unique, and defining which group is the best is a tremendous task. There are a great number of possible sets of points to be selected. However, we can not decide conclusively from which group of points we can get our best approximation to *f(x)*. Quite easily we come across statements like (Forsythe *et al.*, 1977) "The criterion of reasonableness (of a given polynomial approximation to a function *f(x)*) may vary from problem to problem and may never be satisfactorily understood". When we deal with measured or tabulated values of a function *f(x)* that depends on *x*, one possible approach could be the method of divided differences of Newton. Unfortunately, the same doubts arise with respect to the approximation and the sense of convergence of the proposed interpolations.

In experimental analysis, it is usual to cull experimental values $f_i(x)$ and values of the experimental variable x_i. The problem is to find (Fraleigh and Beauregard, 1990) some function $f(x) = r_0 + r_1 x$ with certain values of r_0 and r_1 that fits accurately our experiments. However, no mention is made of the sense and rate of convergence of the function *f(x)* obtained. We only note that somehow our function approaches very closely our data points $f_i(x)$.

Maybe one of the most popular methods is the one proposed by Lagrange. It offers the possibility of getting one special polynomial that reproduces exactly each and every data. However the same doubt arises regarding exactness of our approximation. At this point it has to be noted that, one major drawback of other methods is the handling of sequences like, $(1, x, x^2, x^3, \ldots, x^n)$ not orthogonal among them by using the Gram-Schmidt orthogonalization procedure in an attempt to get simplicity. In view of this, it is not surprising that in many problems of interpolation we resort to orthogonal polynomials like those of Laguerre, Chebyshev or Legendre among many others. The reason for this choice is, apparently, a better convergence. However, no clear definitions of convergence are provided.

Searching for some clues to the convergence of some interpolating polynomial we find the following Faber's Theorem (Forsythe *et al.*, 1977):

"For any interpolating array there exists a continuous function g and an x in [a, b] such that Pn(g)(x) does not converge to g(x), as n $\rightarrow \infty$ ".

An example of this problem of divergence is Runge's Function presented in reference (Forsythe *et al.*, 1977).

Up to this point we have been speaking of interpolation with orthogonal (Legendre) and with nonorthogonal functions via different methods without mentioning that the problem of interpolation of data or functions can be gathered in the same mathematical scheme when we develop the concept of functional analysis with covariant and contravariant manifolds $\tilde{\phi}_n$ and $\tilde{\phi}^n$. This kind of manifold recently found and applied in the field of dynamics (Urrutia, 1992a and 1992b) sets up the basis for a generalized functional analysis with skew manifolds. We note that in some references (Urrutia, 1998) and (Bowen *et al.*, 1976) attention is focused on one manifold u^n and one dual manifold v^n which are biorthogonal and are associated to a nonsymmetric transformation matrix A. For a symmetric matrix both spaces are equal and no new information is given. In fact in a previous paper it has been seen that if the matrix transformation is symmetric we can still be able to calculate both manifolds which are identified now as un $u^n = \phi_n$ (covariant manifold) and $\gamma^n = \phi^n$ contravariant manifold). Besides, we will not be only concentrated in the problem of existence, already tackled in (Urrutia, 1992a and 1992b), but rather in the direct use of these mathematical tools in the solution and application of real problems.

Theory

Given a set of covariant functions $\tilde{\phi}_n$ linearly independent (not necessarily orthogonal) in a given domain Ω, there is another set $\tilde{\phi}^n$ of contravariant functions biorthogonal to the former ones. Therefore, given an arbitrary function $\tilde{F}$ in the same domain Ω with norm $|\tilde{F}|$, can be decomposed in the following manner

$$\tilde{F} = \sum_{n=1}^{\infty} f^n \tilde{\phi}_n \quad (2)$$

In covariant basis $\tilde{\phi}_n$ and contravariant components f^n (scalars) or in the form

$$\tilde{F} = \sum_{n=1}^{\infty} f_n \tilde{\phi}_n \quad (3)$$

in contravariant basis $\tilde{\phi}^n$ and covariant components f^n (scalars). Therefore if equations (2) and (3) are available we can calculate the norm of the function (or vector, Urrutia, 2003) $\tilde{F}$ in the following way

$$|F|^2 = \sum_{n=1}^{\infty} f^n f_n$$

$$|F| = \sqrt{\sum_{n=1}^{\infty} f^n f_n} \quad (4)$$

which for skew coordinate functions is the counterpart and constitutes a generalization of the Pythagorean theorem used in rectangular systems in the theory of vectors.

A particular case occurs, when the manifold $\tilde{\phi}_n$ is orthogonal or orthonormal. In this case all members of the covariant manifold $\tilde{\phi}_n$ are both linearly independent and orthogonal. The contravariant basis $\tilde{\phi}^n$ are collinear to the functions $\tilde{\phi}_n$ and therefore, $\tilde{\phi}^n$ is identical to $\tilde{\phi}_n$. In the same way $f^n = f_n$. Thus from equation (4) the norm of the function $\tilde{F}$ is equal to

$$F = \sqrt{\sum_{n=1}^{\infty} f^n f_n} = \sqrt{\sum_{n=1}^{\infty} f_n^2} \quad (5)$$

for orthogonal linear manifolds. For the general case of skew coordinates, if the covariant and contravariant approximations are complete and convergent we must respect the following two equations

$$F = \sqrt{\sum_{n=1}^{\infty} f^n f_n} \quad (6a)$$

$$F \geq \sqrt{\sum_{n=1}^{N} f^n f_n} \quad (6b)$$

Which are the Parseval and Bessel conditions respectively for skew coordinates.

Norms of Skew Vectors and Continuous Functions

Before embarking on further developments, we will define several operations used for discrete (vectors, Urrutia, 2003) and continuous functions in order to cover both cases in one presentation.

The scalar product of two vectors $\tilde{\phi}_n$ and $\tilde{\phi}_n$ (or $\tilde{\phi}^n$) and the energy norm of the same vectors with respect to the operator K_{mn} are defined by the following two equations

$$< \tilde{\phi}_n, \tilde{\phi}_m > = \tilde{\phi}_n^{T} \tilde{\phi}_m \quad (7)$$

$$< \tilde{\phi}_n K_{mn} \tilde{\phi}_m > = \sum_{n=1}^{N} \sum_{m=1}^{M} \tilde{\phi}_n^{T} K_{nm} \tilde{\phi}_m \quad (8)$$

where $\tilde{\phi}_n$ stands for a column vector, $\tilde{\phi}_n^{T}$ is a row vector which is the transpose of $\tilde{\phi}_n$ and $\boldsymbol{K}_{nm}$ is a transformation matrix.

The scalar product of two functions $\tilde{\phi}_n$ and $\tilde{\phi}_n$ (or $\tilde{\phi}^m$) and the energy norm of the same functions with respect to the operator K_{nm} are defined by the following two equations

$$<\tilde{\phi}_n, \tilde{\phi}_m> = \int \Omega \tilde{\phi}_n(x)\tilde{\phi}_m(x)dx \quad (9)$$

$$<\tilde{\phi}_n, K_{nm}\tilde{\phi}_m> = \sum_1^N \sum_1^M \int \Omega \tilde{\phi}_n(x) K_{nm} \tilde{\phi}_m(x)dx \quad (10)$$

Despite their different aspect, equations (7) to (10) stand for an integration process.

Covariant and Contravariant Basis for Continuous Manifolds

We define a manifold in a domain Ω by a set of contravariant functions $\tilde{\phi}^n$ linearly independent. A second group of covariant base functions $\tilde{\phi}_n$ is defined in the same domain Ω in such a way that the scalar product between these two kinds of coordinates leads us to the Kronecker symbol δ_n^n as follows

$$\int_\Omega \tilde{\phi}^m \tilde{\phi}_n d\Omega = <\tilde{\phi}_n, \tilde{\phi}^m> = \delta_n^m = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad (11)$$

An arbitrary function $\tilde{F}$ can be resolved in these two manifolds as follows

$$\tilde{F} = \sum_{n=1}^{\infty} c_n \tilde{\phi}^n \quad (12)$$

$$\tilde{F} = \sum_{n=1}^{\infty} c^n \tilde{\phi}_n \quad (13)$$

Where C^n and C_n stand for, the contravariant and the covariant components of $\tilde{F}$. Any continuous function can be decomposed in covariant and contravariant basis $\tilde{\phi}_n$ and $\tilde{\phi}^n$. So, it can be shown that when we attempt to resolve the covariant base function $\tilde{\phi}_n$ in covariant components the following result is obtained

$$\tilde{\phi}_n = \tilde{\phi}_{n1}\tilde{\phi}^1 + \tilde{\phi}_{n2}\tilde{\phi}^2 + \tilde{\phi}_{n3}\tilde{\phi}^3 + \ldots \quad (14)$$

In tensor notation

$$\tilde{\phi}_n = \tilde{\phi}_{nm}\tilde{\phi}^m \quad (15)$$

Recall that $\tilde{\phi}_1 \cdot \tilde{\phi}_n = \tilde{\phi}_n \cdot \tilde{\phi}_1$ and $\phi_{mn} = \phi_{nm}$. In the same fashion the following decomposition is possible

$$\tilde{\phi}^n = \phi^{nm}\tilde{\phi}_m \quad (16)$$

In the last two equations ϕ_{nm} and ϕ^{nm} are the covariant and contravariant metric tensors of tensor calculus. Usually, it is easy to choose an arbitrary and complete set of covariant base functions. The difficult part had been to find the contravariant base functions, to overcome this difficulty we continue as follows. By hypothesis we know that the Kronecker delta function is obtained when the following product is performed (Urrutia, 2003) (now an integral)

$$\tilde{\phi}_n \cdot \tilde{\phi}^m = \delta_n^m \quad (17)$$

Using the results (15) and (16) we find

$$<\phi_{ns}\tilde{\phi}^s, \phi^{mt}\tilde{\phi}_t> = \delta_n^m$$

$$\phi_{ns}\phi^{mt}\delta_t^s = \delta_n^m \quad (18)$$

$$\phi_{ns}\phi^{ms} = \delta_n^m$$

When we fix the value of m and we perform the summations over the repeated index s, the following set of m metric components ϕ^{m} is obtained

$$\phi_{11}\phi^{m1} + \phi_{12}\phi^{m2} + \phi_{13}\phi^{m3} + \phi_{14}\phi^{m4} + \ldots + = \delta_1^m$$

$$\phi_{21}\phi^{m1} + \phi_{22}\phi^{m2} + \phi_{23}\phi^{m3} + \phi_{24}\phi^{m4} + \ldots + = \delta_2^m \quad (19)$$

$$\phi_{31}\phi^{m1} + \phi_{32}\phi^{m2} + \phi_{33}\phi^{m3} + \phi_{34}\phi^{m4} + \ldots + = \delta_3^m$$

$$etc$$

To illustrate the use of equation (19) let us assume that the linear manifolds $\tilde{\phi}_m$ and $\tilde{\phi}^m$ have only three components. Then equation (19) will provide us with three systems of equations. If in equation (19) we set the value of $m = 1$ one set of equations is obtained as follows with ϕ_{mn} known

$$
\begin{aligned}
\phi_{11}\phi^{11} + \phi_{12}\phi^{12} + \phi_{13}\phi^{13} &= 1 \\
\phi_{21}\phi^{11} + \phi_{22}\phi^{12} + \phi_{23}\phi^{13} &= 0 \\
\phi_{31}\phi^{11} + \phi_{32}\phi^{12} + \phi_{33}\phi^{13} &= 0
\end{aligned}
\qquad (20)
$$

That in matrix form leads to

$$
\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{12} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}
\begin{bmatrix} \phi^{11} \\ \phi^{12} \\ \phi^{13} \end{bmatrix} =
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\qquad (21)
$$

In similar fashion

$$
\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{12} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}
\begin{bmatrix} \phi^{21} \\ \phi^{22} \\ \phi^{23} \end{bmatrix} =
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\qquad (22)
$$

And finally,

$$
\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{12} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}
\begin{bmatrix} \phi^{31} \\ \phi^{32} \\ \phi^{33} \end{bmatrix} =
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\qquad (23)
$$

From this the elements (ϕ^{mn}) of the contravariant metric tensor (3 $\times$ 3 tensor) are calculated. With the covariant and contravariant metrics ϕ_{mn} and ϕ^{mn} available we can calculate the contravariant base functions as follows

$$
\tilde{\phi}^{n} = \phi^{mn}\tilde{\phi}_{m} \qquad (24)
$$

We can now continue with any further analysis.

Example 1

Given a set of three skew covariant functions $\tilde{\phi}_0 = 1$, $\tilde{\phi}_2 = x^2$ and $\tilde{\phi}_4 = x^4$ find the corresponding set of contravariant functions in the domain $-1 \leq x \leq +1$. Odd powers (x, x^3, x^5, etc) do not intervene because in a later example the cos (x) function (an even function) will be analyzed.

First, we have to find the elements of the covariant metrics as follows

$$
< \tilde{\phi}_n, \tilde{\phi}_m >= \int_{-1}^{1} \tilde{\phi}_n \tilde{\phi}_m \, dx = \phi_{nm}
$$

$$
\therefore \phi_{00} = \int_{-1}^{1} (1)^2 dx = 2
$$

$$
\phi_{02} = \int_{-1}^{1} (1) x^2 dx = 2/3
$$

In similar fashion we find the rest of the elements to obtain

$$
\phi_{mn} = \begin{pmatrix} 2 & 2/3 & 2/5 \\ 2/3 & 2/5 & 2/7 \\ 2/5 & 2/7 & 2/9 \end{pmatrix}
$$

From equation (19) we find the following equation

$$
\begin{pmatrix} 2 & 2/3 & 2/5 \\ 2/3 & 2/5 & 2/7 \\ 2/5 & 2/7 & 2/9 \end{pmatrix}
\begin{bmatrix} \phi^{00} \\ \phi^{02} \\ \phi^{04} \end{bmatrix} =
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

From where, we obtain the first row of the metrics matrix ϕ^{mn}. With a similar procedure we get two more equations and the rest of the elements of ϕ^{mn}

$$
\phi^{mm} = \begin{pmatrix} 1.7578 & -8.2031 & 7.3828 \\ -8.2031 & 68.9063 & -73.8281 \\ 7.3828 & -73.8281 & 86.1328 \end{pmatrix}
$$

With the metric elements $\tilde{\phi}^{mn}$ we can get the contravariant basis $\tilde{\phi}^{n}$ from equation (16)

$$
\tilde{\phi}^{0} = 1.7578 - 8.2031(x^2) + 7.3828(x^4)
$$

$$
\tilde{\phi}^{0} = 1.7578 - 8.2031x^2 + 7.38284x^4
$$

Similarly,

$$\tilde{\phi}^2 = -8.2031 + 68.9063x^2 - 73.8281x^4$$

and $$\tilde{\phi}^4 = 7.3828 - 73.8281x^2 + 86.1328x^4$$

The reader can verify that the following equation holds true

$$< \tilde{\phi}_n, \tilde{\phi}^m > = \int_{-1}^{1} \tilde{\phi}_n(x)\tilde{\phi}^m(x)dx = \delta_n^m$$

Example 2, an Application to Interpolation

Find a polynomial approximation in three terms to the function $cos(x)$ in the domain [–1, 1] with the following form

$$\cos(x) = c^0 + c^2x^2 + c^4x^4$$

$$cos(x) = c^0\phi_0 + c^2\phi_2 + c^4\phi_4$$

Use the covariant functions $\tilde{\phi}_n$ and the contravariant functions $\tilde{\phi}^n$ from example 1. According to reference (Carnaham *et al*., 1969) that uses Chebyshev polynomials the solution to this problem is

$$cos(x) = 0.99995795 - 0.49924045x^2 + 0.03962674x^4$$

Solution

When we dot multiply equation (23) by the coordinate function $\tilde{\phi}^0$ we get the following

$$< \tilde{\phi}^0, cos(x) > = < \tilde{\phi}^0, (c^0\tilde{\phi}_0 + c^2\tilde{\phi}_2 + c^4\tilde{\phi}_4) >$$

If we remember that $< \tilde{\phi}^n, \tilde{\phi}_m > = \delta_m^n$, it is clear that the coefficient c^0 is obtained from the following equation, written now in form of an integral

$$\int_{-1}^{1} \tilde{\phi}^0 \cos(x)dx = c^0$$

With $\tilde{\phi}^0 = 1.7578 - 8.2031x^2 + 7.3828x^4$.

When the integral is evaluated we see that c^0 = 0.999958197. If we now perform the scalar product of equation (24) $\tilde{\phi}^2$ we will get the following

$$< \tilde{\phi}^2, \cos(x) > = < \tilde{\phi}^2, (c^0\tilde{\phi}_0 + c^2\tilde{\phi}_4 + c^4\tilde{\phi}_4) >$$

From where it is clear that the coefficient c^2 is equal to

$$\int_{-1}^{1} \tilde{\phi}^2 \cos(x)dx = c^2$$

Where $\tilde{\phi}^2 = -8.2031 + 68.9063x^2 - 73.8281x^4$. When the integral is performed we see that c^2 = –.4999309946. In similar fashion we find c^4 = 0.039793817. Therefore, we have that within the interval $-1 \leq x \leq 1$ the best approximation to the function $cos(x)$ is the following

$$cos(x) = 0.999958197 - 0.4993099946x^2$$

$$+0.039793817x^4$$

In the basis $\tilde{\phi}_0 = 1$, $\tilde{\phi}_2 = x^2$ and $\tilde{\phi}_4 = x_4$. However, this approximation to $cos(x)$ is not unique as we can resort to the contravariant functions $\tilde{\phi}^0$, $\tilde{\phi}^2$ and $\tilde{\phi}^4$ from the first example. To make this fact clearer, we require the following approximation

$$cos(x) = c_0\tilde{\phi}^0 + c_2\tilde{\phi}^2 + c_4\tilde{\phi}^4$$

This is now dot multiplied by $\tilde{\phi}_0 = 1$ as follows

$$< \tilde{\phi}_0, cos(x) > = < \tilde{\phi}_0, (c_0\tilde{\phi}^0 + c_2\tilde{\phi}^2 + c_4\tilde{\phi}^4) >$$

From where the following result is obtained

$$\int_{-1}^{1} \tilde{\phi}_0 \cos(x)dx = c_0$$

When the integral is done we see that c_0 = 1.682941973. When $\tilde{\phi}_0$ is replaced by $\tilde{\phi}_2$ and by $\tilde{\phi}_4$ we obtain c_2 = 0.478267241 and the last coefficient

c_4 = 0.266153329. Therefore, the function cos (x) can be equally represented by

$$cos(x) = 1.682941973\,\tilde{\phi}^0 + 0.478267241\,\tilde{\phi}^2 + 0.266153329\,\tilde{\phi}^4 \qquad (26)$$

With $\tilde{\phi}^0, \tilde{\phi}^2$ y $\tilde{\phi}^4$ given by the following functions

$$\tilde{\phi}^0 = 1.7578 - 8.2031\,X^2 + 7.3828\,4\,X^4$$

$$\tilde{\phi}^2 = \_8.2031 + 68.9063x^2 - 73.8281x^4$$

$$\tilde{\phi}^4 = 7.3828 - 73.8281x^2 + 86.1328\,x^4$$

Equations (25) and (26) somehow fall very close to the solution (24) given in reference (Carnaham *et al*., 1969). At this point we note that from the three possible approximations (24) to (26), the solutions (24) and (25) that use the same covariant basis $\tilde{\phi}_n$ are comparable. The problem now is to decide which of the solutions (24) and (25) is the best and in what sense. Any approach with given c^n and c_n must satisfy equations (6a) and (6b) of Parseval and Bessel for skew manifolds. In this connection, Table 1 presents the coefficients of the three approximations (24) to (26) to the function cos (x). In columns 2, 3 and 4 are located the coefficients calculated according to the methods of Chebyshev and those of the present paper. When formula (6b) is applied using the coefficients of columns two and four we obtain the squared norm $|cos(x)|^2$ = 1.45464763 and we get the squared root of this value we in turn obtain the norm $cos(x)$ = 1.20608774. When the coefficients of columns three and four are equally multiplied we find that the norm of our function is $|cos(x)$ = 1.206088186. When we find the differences of these two norms with respect to the exact value $|cos\ (x)|$ = 1.206088187 (calculated at the bottom of table 1) is 0.00000045 and 0.000000001 respectively, for the Chebyshev and the covariant approximations in the sense of norm. From this we conclude that the error of the covariant representation is 450 times smaller that the Chebyshev approximation.

As we can observe neither the Chebyshev nor the Contravariant approximations overshoot the exact norm $|cos\,(x)|$ = 1.206088187. Therefore we can now confirm that both solutions satisfy the Bessel's inequality (6b). Up to this point we have accomplished several goals. First, we have obtained the best approximation to cos (x), in covariant basis, second, we have found a new approximation the contravariant that allows us to recover the simplicity of the Pythagorean theorem, with equation (5), for the handling of the concepts of NORM and CONVERGENCE in skew manifolds. In addition we knew (Carnaham *et al*., 1969) that the Chebyshev approximation had an error smaller than 4.234x10 $^{-5}$ and now we have a new approximation the covariant with an error 450 times smaller and with a rate of convergence that satisfies the convergence laws of Parseval and Bessel. This in turn allows us to focus our attention on polynomials with powers higher than four and to appreciate other problems of numerical analysis.

Table 1

	Chebyshev4	Contravariant	Covariant
a^0	0.99995795	0.999970781	1.68294197
a^2	–0.49924045	–0.499384548	0.478267252
a^4	0.03962674	0.038408595	0.266153368
$\|cos(x)\|$	1.20608774	1.206088186	
error	0.00000045	0.000000001	

Higher Order Polynomial approximations to $cos(x)$ for $-1 \le x \le 1$

According to what we have seen in this paper, in principle, we can obtain a covariant and a contravariant polynomials that tend to $cos\ (x)$ in all points in the domain, i.e. we can obtain

$$cos(x) = c^0 \tilde{\phi}_0 + c^2 \tilde{\phi}_2 + c^4 \tilde{\phi}_4 + \ldots + c^n \tilde{\phi}_n$$

$$cos(x) = c_0 \tilde{\phi}^0 + c_2 \tilde{\phi}^2 + c_4 \tilde{\phi}^4 + \ldots + c_n \tilde{\phi}^n$$

and the norm of $|cos\,(x)|$ would be equal to $c^0 c_0 + c^2 c_2 + c^4 c_4 + \ldots + c^n c_n$ when $n \to \infty$. However, as we increase the order of the matrices ϕ_{mn} and ϕ^{mn} we note that the matrix ϕ_{mn} has very small elements of the order of 2/(2(i + j)–3) that tend to zero when i and j tend to infinite. The variables i and j stand for the i–th row and the j–th column. This problem will lead us to the handling of very ill–conditioned matrices of the kind of the famous matrices of Hilbert with elements of the type 1/(i + j), see reference (Fraleigh and Beauregard, 1990). As it is indicated in (Fraleigh and Beauregard, 1990), for matrices of order greater than 10×10 today's computers accuracy give rise to contravariant matrices (when they are calculated) ϕ^{mn} with extremely large numbers that will lead us to divergent results.

When we add the results of polynomials up to 10th order to the results of the polynomial of fourth order we obtain the coefficients shown in table 2. At this point some doubts arise with respect to the values to which the coefficients a^n tend when $n \to \infty$. We immediately note that a^0 is contained between 0.999970781 and 1.000000538, a^2 changes between –0.499384548 and –0.500019533, a^4 between 0.039808595 and 0.41778820, a^6 between –0.001342159 and –0.001585556 but now we see that the coefficient of the tenth polynomial does not converge any more and it even changes its sign.

Besides, the alternating sign of the coefficients of the polynomial of order fourth to eight is lost in the tenth order polynomial and this warns us that from this point on –for some reason– we start having numerical instability. From reference (Forsythe *et al.*, 1977) we might conclude that this divergence may be the result of the Faber's Theorem, shown in the introduction. However we can not accept it because we know that the following expansion exists

$$cos(x) = 1.0 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + etc$$

and whose coefficients exactly fall between the limiting values in which the coefficients of polynomials of fourth to eighth degree. The tenth degree polynomial starts to diverge from expansion (27) in view of the ill conditioning of the matrix ϕ_{mn} as it can be seen in equation (21). Working with double or higher precision we recover some exactness but soon we confront divergent approximations for higher values of n again. In table 3 we present the exact first eleven significant con- travariant coefficients obtained from equation (27), that our intuition suggests must be the coefficients that we should obtain in table 2 if we will increase the precision of our calculations. Following a similar procedure to the one used to calculate the contravariant polynomial (26) the covariant coefficients c_n were calculated and are presented in the third column of table 3. If the coefficients a^n and a_n of table 3 are certainly the contravariant and the covariant coefficients of $cos\ x$ between $-1 \le x \le 1$ then if we calculate the norm of this function using equation (4) we must satisfy Bessel's inequa- lity (6b) when $n \to \infty$. In this sense it is readily observed that in the fourth column of table 3 we present the accumulated norm of $cos\ (x)$ when we use equation 4. When $n = 10$ the squared norm is $|cos(x)|^2 = 1.454648715$ (smaller than 1.454648716) and it is not affected any more for the inclusion of the rest of the elements. From this we conclude that the polynomial (27) converges to $cos\ (x)$ everywhere in the domain $-1 \le x \le 1$ and converges to the norm of $cos\ (x)$ according to the Bessel's inequality (6b). In order to observe one more effect of the divergence of the different approximations to $cos\ (x)$

we obtained the norms of contravariant coefficients of table 2 and the covariant coefficients of the third column of table 3. The different approximations to the norm of *cos (x)* are shown in the last row of table 2.

As it can be seen, the norm of the polynomial of fourth order is 1.454648713, the polynomial of sixth degree has a norm of 1.454648692 (actually it starts to diverge) and up to this point there is no major objection. However, the last two columns show norms that are greater than the exact value of 1.454648716 and this is a clear violation of the Bessel's inequality (6b) and a proof of divergence.

Table 2

CONTRAVARIANT COEFFICIENTS OF POWERS 4, 6, 8 AND 10				
a^n	4	6	8	10
a^0	0.999970781	0.999999835	1.000000538	0.999997793
a^2	–0.499384548	–0.499994769	–0.50001953	–0.49987840
a^4	0.039808595	0.041638979	0.041778820	0.040454756
a^6	------	–0.001342159	–0.00158556	0.002279407
a^8	------	------	0.000129896	–0.00450388
a^{10}	------	------	------	0.002038310
NORM	1.454648713	1.454648692	1.454648824	1.454650073

Table 3

n	a^n eq (28)	Covariant coeffic. a_n	Norm of *cos* (x) cumulative sum $a^n a_n$
0	+1.00	+1.682941970	1.682941970
2	–0.50	0.478267252	1.443808344
4	+1/4!	0.266153368	1.454898068
6	–1/6!	0.181968530	1.454645334
8	+1/8!	0.137541095	1.454648745
10	–1/10!	0.110289862	1.454648715
12	+1/12!	0.091937628	1.454648715
14	–1/14!	0.078765706	1.454648715
16	+1/16!	0.068865056	1.454648715

Conclusions

From example 1 it is concluded that given a sequence of covariant functions (complete) $\tilde{\phi}_n$ there exists another set of contravariant functions $\tilde{\phi}^n$ which is biorthogonal to the former one and that satisfies the Kronecker Delta function $<\tilde{\phi}^m, \tilde{\phi}_n> = \delta^m_n$. From example 2 we saw that any polynomial approximation to any function $f(x)$ can now be tackled by using the concept of manifold theory in skew coordinates. We must be only careful with the convergence analysis that is directly related to the precision of the computing device available. As it was observed, the theorem of Faber that denies the existence of a polynomial $P_n(x)$ that approaches f(x), everywhere, as $n \to \infty$ is not valid. The problem of divergence shown in reference (Forsythe *et al*., 1977) is due to the lack of precision rather than to questions related with the existence or non existence of a polynomial $P_n(x)$ that approaches $f(x)$ as $n\infty$. The problem of interpolation can now be seen as analysis in skew manifolds where equations (6a) and (6b) of Parseval and Bessel can be used to guarantee convergence of our approximating polynomials. To avoid duplication of work the interested reader should review references (Urrutia, 1992a and 1992b), to get a deeper insight in the mechanical and physical meaning of the manifold theory presented in this paper.

Future Work

As a follow up to the findings of references (Urrutia, 1992a, 1992b and 1998), and of the present paper we will use the same theory now focused on the solution of nonlinear differential equations. As we will see, using covariant and contravariant manifolds will allows us to obtain an easy and novel method of solution for this kind of nonlinear problems.

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