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Solvability and Primal-dual Partitions of the Space of Continuous Linear Semi-infinite Optimization Problems

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Abstract. Different partitions of the parameter space of all linear semi-infinite programming problems with a fixed compact set of indices and continuous right and left hand side coefficients have been considered in this paper. The optimization problems are classified in a different manner, e.g., consistent and inconsistent, solvable (with bounded optimal value and nonempty optimal set), unsolvable (with bounded optimal value and empty optimal set) and unbounded (with infinite optimal value). The classification we propose generates a partition of the parameter space, called second general primal-dual partition. We characterize each cell of the partition by means of necessary and sufficient, and in some cases only necessary or sufficient conditions, assuring that the pair of problems (primal and dual), belongs to that cell. In addition, we show non emptiness of each cell of the partition and with plenty of examples we demonstrate that some of the conditions are only necessary or sufficient. Finally, we investigate various questions of stability of the presented partition.

Keywords. Linear semi-infinite programming, parameter space of continuous problems, primal-dual partition, stability properties.

1 Introduction

For a given infinite compact Hausdorff topological space T and $n \in \mathbb{N}$, we associate with each triple $\pi = (a, b, c) \in \Pi := C(T)^n \times C(T) \times \mathbb{R}^n$, where $C(T)$ is the Banach space of all continuous functions over the compact T, a continuous linear semi-infinite programming problem:

$$\begin{array}{lll} P: & \min\limits_{x\in\mathbb{R}^n} & c'x, \\ & s.t. & a_t'x \geq b_t,\ t \in T, \end{array}$$

and its corresponding Haar's dual problem:

$$\begin{array}{lll} D: & \max\limits_{\lambda\in\mathbb{R}_+^{(T)}} & \sum\limits_{t\in T} \lambda_t b_t \\ & s.t. & \sum\limits_{t\varepsilon T} \lambda_t a_t = c, \end{array}$$

where $\mathbb{R}_+^{(T)}$ is the set of nonnegative functions λ, and $\lambda : T \to \mathbb{R}_+$ such that $\lambda_t \neq 0$ for at most a finite number of indices belonging to T.

Among the recent applications of continuous linear semi-infinite programming (LSIP), let us mention that the primal problem P arises in functional approximation [5, 6], finance [12], Bayesian statistics [14], and the design of telecommunications networks [4, 13, 16], whereas the dual D has been used in robust Bayesian analysis [3] and optimization under uncertainty [1].

We denote the feasible (optimal) set of P and D by F (F^*) and Λ (Λ^*), respectively. In the space of parameters Π we consider the topology of the uniform convergence generated by the following extended distance: for given $\pi^i = (a^i, b^i, c^i) \in \Pi$, $i = 1, 2$, the distance between π^1 and π^2 is:

$$d(\pi^1, \pi^2) :=$$

$$= \max\left\{ ||c^1 - c^2||_\infty\ ,\ \max_{t\in T} \left|\left| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t^2 \\ b_t^2 \end{pmatrix} \right|\right|_\infty \right\}.$$

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By Π_C^P, Π_{IC}^P, Π_B^P and Π_{UB}^P, (Π_C^D, Π_{IC}^D, Π_B^D and Π_{UB}^D) we denote the sets of parameters providing primal (dual) consistent, inconsistent, bounded, (with finite optimal value), and unbounded, (with unbounded optimal value) problem, respectively.

In ordinary linear programming (LP), if the primal problem is solvable then the dual problem is also solvable and the optimal values of both problems coincide. In LSIP these properties fail in general. The continuity property of $\pi = (a, b, c)$ ensures nice theoretical properties (e.g., in the duality context) and has computational implications (for instance, continuity guarantees the convergence of LSIP discretization algorithms). In particular, Goberna, Lopez, Todorov, Ochoa and Vera de Serio, among others, have investigated conditions under which the primal-dual pair in LSIP satisfies some of the above mentioned properties (see [7, 10, 11, 15]).

In [11], Goberna and Todorov have considered the consistency of the primal and dual problems, and have presented a characterization of the sets of the primal partition $\{\Pi_{IC}^P,\ \Pi_B^P,\ \Pi_{UB}^P\}$, the sets of the dual partition $\{\Pi_{IC}^D,\ \Pi_B^D,\ \Pi_{UB}^D\}$ and the sets (or states) of the primal-dual partition, formed by the intersections of the corresponding states of the primal and dual partitions. The stability properties of the different states have been studied, as well.

In [10], Goberna and Todorov divided the set of parameters with bounded primal (dual) problem Π_B^P (Π_B^D) into sets of parameters that have solvable primal (dual), problem with bounded optimal set Π_S^P (Π_S^D) and a set of parameters that have unsolvable primal (dual), problem or unbounded optimal set Π_N^P (Π_N^D). This generates what we call a first general primal partition $\{\Pi_{IC}^P, \Pi_S^P, \Pi_N^P, \Pi_{UB}^P\}$, a first general dual partition $\{\Pi_{IC}^D, \Pi_S^D, \Pi_N^D, \Pi_{UB}^D\}$, and a first general primal-dual partition. In the same article, Goberna and Todorov characterized, by means of necessary and sufficient conditions, when a given parameter belongs to a certain state of the above partitions.

They have also studied several topological and stability properties of each cell of the partitions. Later on, Ochoa and Vera de Serio reconsidered the characterizations presented in [11], and investigated the stability of the states in the general case [15], i.e., without continuity properties of the functions involved in LSIP problems.

In this paper we present a more natural primal-dual partition in continuous LSIP than the one given in [10]. The new partition is generated by dividing the set of bounded primal (dual) problems Π_B^P (Π_B^D) in two: the set of parameters that have solvable primal (dual) problem Π_s^P (Π_s^D), and the set of parameters that have unsolvable primal (dual) problem Π_n^P (Π_n^D). Formally, we consider the partitions $\{\Pi_{IC}^P, \Pi_s^P, \Pi_n^P, \Pi_{UB}^P\}$ ($\{\Pi_{IC}^D, \Pi_s^D, \Pi_n^D, \Pi_{UB}^D\}$) of Π from the primal (dual) perspective, and the primal-dual $\{\Pi_s^P \cap \Pi_s^D, \Pi_n^P \cap \Pi_s^D, \Pi_s^P \cap \Pi_n^D, \Pi_n^P \cap \Pi_n^D\}$ of the set of parameters with primal and dual bounded problems.

We have obtained some necessary and some sufficient conditions showing that a certain element of the space of parameters belongs to certain subset generated by the second general primal-dual partition. Intersecting the nonempty pairs of the new general primal and dual partitions we obtain the second general primal-dual partition. By means of suitable examples we demonstrate that each subset of the second general primal-dual partition is nonempty. Only in a few cases we have succeeded to find necessary and sufficient conditions. If it is not the case, we provide counterexamples showing that given conditions are only necessary or sufficient. Finally, we investigate several topological and stability properties of the cells in the second general primal-dual partition.

This paper is organized as follows. In Section 2, we introduce the necessary notations, recall the basic results on LSIP which are frequently used throughout this article, and summarize the conditions presented in [10] and [11] that characterize the states of the primal-dual partition and the first general primal-dual partition. Section 2.1 provides new conditions which are either necessary of sufficient guaranteeing that a given parameter belongs to a certain element of the second general primal-dual partition. Finally, we study some topological properties of the states of the second general primal-dual partition in Section 2.1.1. More precisely, we prove that the interior of certain cells is nonempty and provide some density results.

The results of this paper are partially announced without proofs in [2].

2 Preliminaries

Let us introduce the necessary notations for this paper. The symbol 0_n denotes the null-vector in $\mathbb{R}^n$, the j-th element of the canonical basis of $\mathbb{R}^n$ is e_j. Given a nonempty set $X \subset \mathbb{R}^n$, $\operatorname{conv} X$ and $\operatorname{cone} X$ are the convex and the canonical hull of X, respectively ($\operatorname{cone} \varnothing = \{0_n\}$). If X is a convex set, $\dim X$ ($\dim \varnothing = -1$) denotes its dimension. From the topological side, if X is a subset of any topological space, $\operatorname{int} X$, $\operatorname{cl} X$ and $\operatorname{bd} X$ represent the interior, the closure and the boundary of X, respectively.

We recall some concepts and basic results on LSIP, we shall use (all the proofs and references can be found in [9] and [17]). We associate with each triple $\pi = (a, b, c)$ the first and second moment cones of π :

$$M = \text{cone}\,\{a_t : t \in T\},$$

and

$$N = \text{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix} : t \in T\right\},$$

as well as its characteristic cone

$$K = \text{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix} : t \in T\,;\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}.$$

The Existence Theorem establishes that P is consistent if and only if $(0_n, 1)' \notin \operatorname{cl} N$. In such a case, the non-homogeneous Farkas Lemma establishes that the inequality $c'x \geq d$ holds for all $x \in F$, if and only if $(c, d) \in \operatorname{cl} K$. For the dual problem, D is consistent if and only if $c \in M$.

When various triples are simultaneously considered, they and their associated feasible, optimal, etc. sets will be distinguished by means of superscripts or subscripts: π^i, P_i, D_i, F_i, F_i^*, Λ_i, Λ_i^*, M_i, N_i, K_i.

We denote by $v^P(\pi)$ and $v^D(\pi)$ the optimal value of P and D, defining as usual:

$$v^P(\pi) = +\infty \text{ and } v^D(\pi) = -\infty$$

respectively, when the corresponding problem is inconsistent.

If we consider different primal-dual states of the LSIP problems, since P and D can be either inconsistent (IC), bounded (B), or unbounded (UB), crossing both criteria, we get nine possible duality states, which are reduced to six by the Weak Duality Theorem: $v^D(\pi) \leq v^P(\pi)$. The primal-dual partition is presented in Table 1 (according to the Duality Theorem, the duality states 5 and 6 are impossible in LP [8], Proposition 4.2):

Table 1. Primal-dual partition

$D \setminus P$	IC	B	UB
IC	Π_4	Π_5	Π_2
B	Π_6	Π_1	
UB	Π_3		

where $\Pi_1 = \Pi_B^P \cap \Pi_B^D$, $\Pi_2 = \Pi_{UB}^P \cap \Pi_{IC}^D$, $\Pi_3 = \Pi_{IC}^P \cap \Pi_{UB}^D$, $\Pi_4 = \Pi_{IC}^P \cap \Pi_{IC}^D$, $\Pi_5 = \Pi_B^P \cap \Pi_{IC}^D$, and $\Pi_6 = \Pi_{IC}^P \cap \Pi_B^D$.

The next Lemma describes the characterization of the duality states Π_i, $i = 1, ..., 6$ in terms of M, N, and K. This characterization appears in [11].

Lemma 2.1. *The following assertions hold:*
(i) $\pi \in \Pi_1$ *if and only if* $(0_n, 1)' \notin \operatorname{cl} N$ *and* $c \in M$.
(ii) $\pi \in \Pi_2$ *if and only if* $(0_n, 1)' \notin \operatorname{cl} N$ *and* $(\{c\} \times \mathbb{R}) \cap \operatorname{cl} N = \varnothing$.
(iii) $\pi \in \Pi_3$ *if and only if* $\{c\} \times \mathbb{R} \subseteq K$.
(iv) $\pi \in \Pi_4$ *if and only if* $(0_n, 1)' \in \operatorname{cl} N$ *and* $c \notin M$.
(v) $\pi \in \Pi_5$ *if and only if* $c \notin M$, $(0_n, 1)' \notin \operatorname{cl} N$ *and* $(\{c\} \times \mathbb{R}) \cap \operatorname{cl} N \neq \varnothing$.
(vi) $\pi \in \Pi_6$ *if and only if* $(0_n, 1)' \in \operatorname{cl} N$, $c \in M$ *and* $\{c\} \times \mathbb{R} \nsubseteq K$.

Corollary 2.2. *[[9], Corollary 9.3.1] Given a consistent problem P in LSIP, the next statements are equivalent.*
(i) F^* *is nonempty and bounded;*
(ii) $c \in \operatorname{int} M$.

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The next results are valid in continuous LSIP, where the Slater Condition plays a crucial role. Recall that $\pi = (a, b, c)$ satisfies the Slater Condition if there exists $\overline{x} \in \mathbb{R}^n$ such that $a_t' \overline{x} > b_t$, for all $t \in T$. π satisfies the Slater Condition if and only if $\pi \in \operatorname{int} \Pi_C^P$ (This result can be found in [9]).

Theorem 2.3. *[[9], Theorem 9.8] If P is a consistent LSIP problem with consistent dual problem D, then the next statements are equivalent.*
(i) Λ^* is nonempty and bounded;
(ii) π satisfies the Slater Condition.

We will use the next characterization of Π_S^P and Π_S^D .

Lemma 2.4. *[[10], Lemma 2.2]*
(i) $\pi \in \Pi_S^P$ if and only if $\binom{0_n}{1} \notin \operatorname{cl} N$ and $c \in \operatorname{int} M$.
(ii) $\pi \in \Pi_S^D$ if and only if $c \in M$ and π satisfies the Slater Condition.

If we consider the parameter space Π, P and D can be either consistent, inconsistent, bounded or unbounded. Now, if in addition to the boundedness we consider the solvability, the bounded problems can be either solvable with optimal set nonempty and bounded (S) or unsolvable (N). In the latter case we include the problems that have optimal set unbounded. With this classification we obtain the first general primal partition $\{\Pi_{IC}^P, \Pi_S^P, \Pi_N^P, \Pi_{UB}^P\}$ and the first general dual partition $\{\Pi_{IC}^D, \Pi_S^D, \Pi_N^D, \Pi_{UB}^D\}$ of the optimization problems space. Crossing both partitions, we get the first general primal-dual partition, which is presented in Table 2:

Table 2. First general primal-dual partition

$D \setminus P$	IC	B: S	B: N	UB
IC	Π_4	Π_5		Π_2
B: S		Π_1^1	Π_1^3	
B: N	Π_6	Π_1^2	Π_1^4	
UB	Π_3			

where $\Pi_1^1 = \Pi_S^P \cap \Pi_S^D$, $\Pi_1^2 = \Pi_S^P \cap \Pi_N^D$, $\Pi_1^3 = \Pi_N^P \cap \Pi_S^D$ and $\Pi_1^4 = \Pi_N^P \cap \Pi_N^D$.

In [10], Goberna and Todorov showed that $\Pi_S^P \cap \Pi_{IC}^D$ and $\Pi_{IC}^P \cap \Pi_S^D$ are empty sets, for this reason the corresponding boxes do not appear numbered. The next Theorem confirms that Π_1^i, $i = 1, ..., 4$ are nonempty.

Theorem 2.5. *[[10], Theorem 3.1]* $\Pi_1^i \neq \varnothing$, $i = 1,\ ...,\ 4$.

The states $\Pi_1^i \neq \varnothing$, $i = 1,\ ...,\ 4$, in continuous LSIP, are characterized by Goberna and Todorov in the next Theorem.

Theorem 2.6. *[[10], Theorem 3.3]*
(i) $\pi \in \Pi_1^1$ if and only if $c \in \operatorname{int} M$ and π satisfies the Slater Condition.
(ii) $\pi \in \Pi_1^2$ if and only if $\binom{0_n}{1} \notin \operatorname{cl} N$, $c \in \operatorname{int} M$ and π does not satisfy the Slater Condition.
(iii) $\pi \in \Pi_1^3$ if and only if $c \in M \setminus \operatorname{int} M$ and π satisfies the Slater Condition.
(iv) $\pi \in \Pi_1^4$ if and only if $\binom{0_n}{1} \notin \operatorname{cl} N$, $c \in M \setminus \operatorname{int} M$ and π does not satisfy the Slater Condition.

2.1 Second Refined Primal-dual Partition

In this section we present a refinement of Table 1, different to the refinement of Goberna and Todorov, presented in Table 2. To do this, we separate the parameter set with bounded primal problem Π_B^P, into two parameter sets. The first one, with solvable primal problem Π_s^P and the other with unsolvable primal problem Π_n^P. The same classification is made with respect to the dual problem. Having in mind the new notations, we get the second general primal partition $\{\Pi_{IC}^P, \Pi_s^P, \Pi_n^P, \Pi_{UB}^P\}$ and the second general dual partition $\{\Pi_{IC}^D, \Pi_s^D, \Pi_n^D, \Pi_{UB}^D\}$ of the parameter space. Crossing both partitions we obtain the

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second general primal-dual partition. The possible duality states in continuous linear optimization are enumerated in Table 3:

Table 3. Possible duality states in continuous linear optimization

$D \setminus P$	IC	B: s	B: n	UB
IC	Π_4	$\widehat{\Pi}_5^1$	$\widehat{\Pi}_5^2$	Π_2
B: s	$\widehat{\Pi}_6^1$	$\widehat{\Pi}_1^1$	$\widehat{\Pi}_1^3$	
B: n	$\widehat{\Pi}_6^2$	$\widehat{\Pi}_1^2$	$\widehat{\Pi}_1^4$	
UB	Π_3			

where $\widehat{\Pi}_1^1 = \Pi_s^P \cap \Pi_s^D$, $\widehat{\Pi}_1^2 = \Pi_s^P \cap \Pi_n^D$, $\widehat{\Pi}_1^3 = \Pi_n^P \cap \Pi_s^D$, $\widehat{\Pi}_1^4 = \Pi_n^P \cap \Pi_n^D$, $\widehat{\Pi}_5^1 = \Pi_s^P \cap \Pi_{IC}^D$, $\widehat{\Pi}_5^2 = \Pi_n^P \cap \Pi_{IC}^D$, $\widehat{\Pi}_6^1 = \Pi_{IC}^P \cap \Pi_s^D$ and $\widehat{\Pi}_6^2 = \Pi_{IC}^P \cap \Pi_n^D$.

Lemma 2.7. $\widehat{\Pi}_1^1 \supsetneq \Pi_1^1$ *and* $\widehat{\Pi}_1^4 \subsetneq \Pi_1^4$.

Theorem 2.8. *Let $\pi \in \Pi$ with primal and dual problems be consistent and bounded. The following assertions are true:*
(i) If $c \in \operatorname{int} M$ and π satisfies Slater Condition, then $\pi \in \widehat{\Pi}_1^1$;
(ii) If $\pi \in \widehat{\Pi}_1^2$, then π does not satisfy the Slater Condition;
(iii) If $\pi \in \widehat{\Pi}_1^3$, then $c \in M \setminus \operatorname{int} M$;
(iv) If $\pi \in \widehat{\Pi}_1^4$, then $c \in M \setminus \operatorname{int} M$ and π does not satisfy the Slater Condition.

Proof. i) Suppose that $c \in \operatorname{int} M$ and π satisfies the Slater Condition. First, if $c \in \operatorname{int} M$, then $F^* \neq \varnothing$ and F^* is bounded [Corollary 2.2]. On the other hand, if π satisfies the Slater Condition, then $\Lambda^* \neq \varnothing$ and Λ^* is bounded [Theorem 2.3]. Therefore, if $c \in \operatorname{int} M$ and π satisfies the Slater Condition, then $\pi \in \widehat{\Pi}_1^1$.

ii) If $\pi \in \widehat{\Pi}_1^2$, then $\Lambda^* = \varnothing$, by Theorem 2.3 π does not satisfy the Slater condition.

iii) If $\pi \in \widehat{\Pi}_1^3$, then $F^* = \varnothing$, by Corollary 2.2 $c \notin \operatorname{int} M$, in addition from hypothesis $c \in M$. So, we conclude that $c \in M \setminus \operatorname{int} M$.

iv) If $\pi \in \widehat{\Pi}_1^4$, then $F^* = \varnothing$ and $\Lambda^* = \varnothing$ this implies $c \in M \setminus \operatorname{int} M$ and π does not satisfy the Slater Condition. $\square$

With the following examples, we show that the above conditions, are only sufficient or necessary, respectively. The examples also show that $\widehat{\Pi}_1^i \neq \varnothing$ for $i = 1, 2, 3, 4$, which justifies the previous partition and Theorem. In ordinary linear programming we have $\widehat{\Pi}_1^2 = \widehat{\Pi}_1^3 = \widehat{\Pi}_1^4 = \varnothing$, according to the Duality Theorem [[8], Theorem 4.4].

In the Example 2.9, $\pi^1 \in \widehat{\Pi}_1^2$ and π^1 does not satisfy the Slater Condition, while in the Example 2.10, π^2 does not satisfy the Slater Condition and $\pi^2 \notin \widehat{\Pi}_1^2$. This shows that the condition (ii), stated in Theorem 2.8, is not a sufficient one.

Example 2.9. *Consider the optimization problem in $\mathbb{R}^2$*

$$
\begin{aligned}
P_1: \quad & \min_{x \in \mathbb{R}^2} x_2 \\
& s.t.\ \ x_1 + r x_2 \geq -r^2,\ r \in [0,1], \\
& \quad -x_1 + s x_2 \geq -s^2,\ s \in [0,1].
\end{aligned}
$$

If $r = 0 = s$, then $x_1 \geq 0$ and $-x_1 \geq 0$, we conclude that $x_1 = 0$. Now, as $x_1 = 0$, if $r,\ s \in (0,\ 1]$ then $r x_2 \geq -r^2$ y $s x_2 \geq -s^2$, i.e., $x_2 \geq -r$ y $x_2 \geq -s$, it follows that $x_2 \geq 0$. Therefore:

$$
F_1 = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 \geq 0 \right\},\ v^P(\pi^1) = 0,
$$

and $F_1^* = \{\binom{0}{0}\}$. As $\dim F_1 = 1$, $F_1 \subset \mathbb{R}^2$ and π^1 is continuous, we have that π^1 does not satisfy the Slater Condition. In Figure 1, we show F_1:

The dual problem of P_1 is:

$$
D_1: \max_{\lambda,\ \gamma \in \mathbb{R}_+^{([0,\ 1])}} \left(\sum_{r \in [0,\ 1]} -\lambda_r r^2 + \sum_{s \in [0,\ 1]} -\gamma_s s^2 \right),
$$

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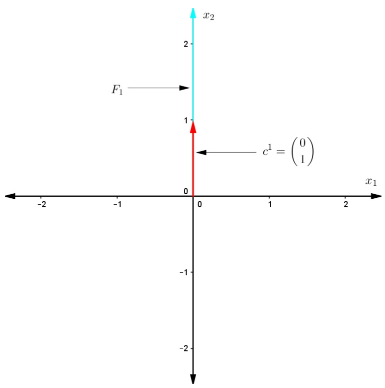

Fig. 1. Feasible set of P_1

$$s.t.\quad \sum_{r\in[0,\ 1]} \lambda_r \begin{pmatrix}1\\ r\end{pmatrix} + \sum_{s\in[0,\ 1]} \gamma_s \begin{pmatrix}-1\\ s\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix},$$

which is equivalent to:

$$-\left(\min_{\lambda,\ \gamma\in\mathbb{R}_+^{([0,\ 1])}} \left(\sum_{r\in[0,\ 1]} \lambda_r r^2 + \sum_{s\in[0,\ 1]} \gamma_s s^2\right)\right),$$

$$s.t.\quad \sum_{r\in[0,\ 1]} \lambda_r \begin{pmatrix}1\\ r\end{pmatrix} + \sum_{s\in[0,\ 1]} \gamma_s \begin{pmatrix}-1\\ s\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

If:

$$v_1 := \min_{\lambda,\ \gamma\in\mathbb{R}_+^{([0,\ 1])}} \left\{ \sum_{r\in[0,\ 1]} \lambda_r r^2 + \sum_{s\in[0,\ 1]} \gamma_s s^2 \right|,$$

$$\left. \sum_{r\in[0,\ 1]} \lambda_r \begin{pmatrix}1\\ r\end{pmatrix} + \sum_{s\in[0,\ 1]} \gamma_s \begin{pmatrix}-1\\ s\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix} \right\},$$

we have $0 \leq v_1$. *Now, if* $t_0 \in (0,\ 1]$, *then* $\overline{\theta}^0 = (\lambda^0;\gamma^0)$ *is in* Λ_1, *if and only if:*

$$\lambda^0_{t_0}\begin{pmatrix}1\\ t_0\end{pmatrix} + \lambda^0_{t_0}\begin{pmatrix}-1\\ t_0\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix},$$

where $\lambda^0_{t_0} = \gamma^0_{t_0}$ *and* $\lambda^0_r = 0 = \gamma^0_s$ *for all* $r,\ s \in [0,1]\setminus\{t_0\}$. *I.e.,* $\lambda^0_{t_0}$ *has to satisfy the equality:*

$$1 = \lambda^0_{t_0}\ t_0 + \lambda^0_{t_0}\ t_0.$$

So, we conclude that $\overline{\theta}^0 \in \Lambda_1$, *if and only if* $\lambda^0_{t_0} = \frac{1}{2t_0}$. *Then:*

$$v_1 \leq \sum_{r\in[0,\ 1]} \lambda^0_r r^2 + \sum_{s\in[0,\ 1]} \gamma^0_s s^2 = \frac{t_0^2}{2t_0} + \frac{t_0^2}{2t_0} = t_0. \tag{1}$$

If t_0 *approaches to zero in (1), we have* $v_1 \leq 0$. *Therefore* $v^D(\pi^1) = 0$. *On the other hand,* $v^D(\pi^1) = 0$, *if and only if* $\lambda_r r^2 = 0 = \lambda_s s^2$, *for all* $(r,s) \in [0,1]\times[0,1]$. *Then, the only possible optimal solutions have the form* $\overline{\theta}^1 = (\lambda^1;\gamma^1)$, *where* $\lambda^1_0, \gamma^1_0 \in \mathbb{R}_+$ *and* $\lambda^1_r = 0 = \gamma^1_s$ *for all* $r,\ s \in (0,1]$, *but* $\overline{\theta}^1 \notin \Lambda_1$, *because:*

$$\lambda^1_0\begin{pmatrix}1\\ 0\end{pmatrix} + \gamma^1_0\begin{pmatrix}-1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix},$$

is impossible. Therefore $\Lambda_1^* = \varnothing$. *All these show that* $\pi^1 \in \widehat{\Pi}^2_1$ *and* $\widehat{\Pi}^2_1 \neq \varnothing$.

Example 2.10. *We consider, the following problem:*

$$\begin{aligned} P_2: \quad & \min_{x\in\mathbb{R}^2} x_2 \\ & s.t.\ \ x_1 \geq 0, \\ & \quad -x_1 \geq 0, \\ & \quad\quad x_2 \geq 0. \end{aligned}$$

In Figure 2, we show the feasible set of P_2.

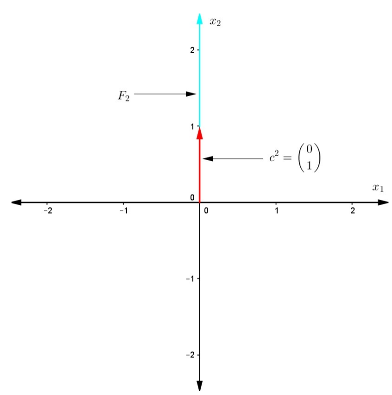

Fig. 2. Feasible set of P_2

We observe that P_2 *is consistent,* $v^P(\pi^2) = 0$ *and* $F_2^* = \{\binom{0}{0}\}$. *In addition, as* $\dim F_2 = 1$ *and* $F_2 \subset \mathbb{R}^2$, *we have that* π^2 *does not satisfy the Slater Condition.*

The dual problem of P_2 *is:*

$$\begin{aligned} D_2: \quad & \max_{\lambda_1,\lambda_2,\lambda_3\geq 0} (\lambda_1 0 + \lambda_2 0 + \lambda_3 0), \\ & s.t.\ \ \lambda_1\binom{1}{0} + \lambda_2\binom{-1}{0} + \lambda_3\binom{0}{1} = \binom{0}{1}. \end{aligned}$$

From the problem it follows that $v^D(\pi^2) = 0$. Now, $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^{'}$ with $\lambda_1, \lambda_2, \lambda_3 \geq 0$ is in $\Lambda_2 = \Lambda_2^*$, if and only if:

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

i.e., $0 = \lambda_1 - \lambda_2$ and $1 = \lambda_3$, or equivalently $\lambda_1 = \lambda_2$ and $1 = \lambda_3$. Then:

$$\Lambda_2 = \Lambda_2^* = \{(\lambda_1, \lambda_1, 1)^{'} \in \mathbb{R}^3 \ : \ \lambda_1 \geq 0\}.$$

Therefore, $\pi^2 \notin \widehat{\Pi}_1^2$.

In the Example 2.11, $\pi^3 \in \widehat{\Pi}_1^3$ and $c^3 \in M_3 \setminus \operatorname{int} M_3$. While, in the Example 2.12, $\pi^4 \notin \widehat{\Pi}_1^3$ and $c^4 \in M_4 \setminus \operatorname{int} M_4$. This shows that the condition (iii), stated in Theorem 2.8, is not a sufficient condition.

Example 2.11. *Consider in $\mathbb{R}^2$ the problem:*

$$P_3: \quad \min_{x \in \mathbb{R}^2} x_1, \quad s.t. \ \ x_1 + t^2 x_2 \geq 2t, \ t \in [0,1].$$

In Figure 3, we show the feasible set of P_3:

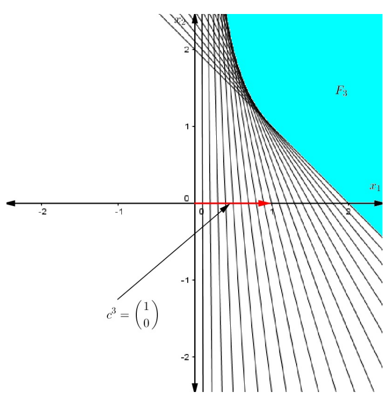

Fig. 3. Feasible set of P_3

π^3 satisfies the Slater Condition and $\binom{2}{2}$ is a Slater point. In fact $1 + t^2 > t$ for all $t \in [0,1]$ if and only if $2 + 2t^2 > 2t$ for all $t \in [0,1]$, so, P_3 is consistent. In Figure 3, we observe that $v^P(\pi^3) = 0$ but $F_3^* = \varnothing$, i.e., P_3 is not solvable. Moreover, in Figure 4, we show that $c^3 \in M_3 \setminus \operatorname{int} M_3$.

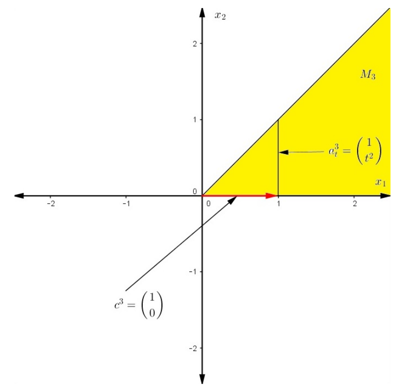

Fig. 4. First moment cone of P_3

The dual problem of P_3 is:

$$D_3: \quad \max_{\lambda \in \mathbb{R}_+^{([0,1])}} \sum_{t \in [0,1]} \lambda_t 2t, \quad s.t. \ \sum_{t \in [0,1]} \lambda_t \binom{1}{t^2} = \binom{1}{0}.$$

The function $\lambda \in \mathbb{R}_+^{([0,1])}$ defined as:

$$\lambda_t := \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t \in (0,1], \end{cases}$$

is a feasible solution for the problem D_3 with $v^D(\pi^3) = 0$. It follows that D_3 is solvable. Therefore, $\pi^3 \in \widehat{\Pi}_1^3$ and $\widehat{\Pi}_1^3 \neq \varnothing$.

Example 2.12. *The primal problem P_4 is formulated as follows:*

$$P_4: \quad \min_{x \in \mathbb{R}^2} x_1, \quad s.t. \ \ x_1 + t^2 x_2 \geq 0, \ t \in [0,1].$$

In Figure 5, we show the feasible set of P_4.

Since π^4 satisfies the Slater Condition, then P_4 is consistent. We observe that $v^P(\pi^4) = 0$ and $F_4^* = \{0\} \times \mathbb{R}^+$, therefore P_4 is solvable. On the other hand, as in the above example, $c^4 \in M_4 \setminus \operatorname{int} M_4$.

The dual problem of P_4 is now:

$$D_4: \quad \max_{\lambda \in \mathbb{R}_+^{([0,1])}} \sum_{t \in [0,1]} \lambda_t 0, \quad s.t. \ \sum_{t \in [0,1]} \lambda_t \binom{1}{t^2} = \binom{1}{0}.$$

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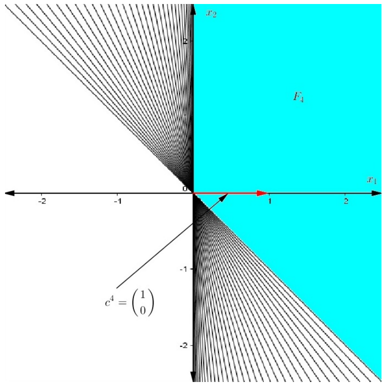

Fig. 5. Feasible set of P_4

Again, we have that the function $\lambda \in \mathbb{R}_+^{([0,1])}$ defined as:

$$\lambda_t := \begin{cases} 1, & \text{if } t = 0, \\ \\ 0, & \text{if } t \in (0,1], \end{cases}$$

is a feasible solution of the problem D_4 with $v^D(\pi^4) = 0$. Thus, we conclude that D_4 is solvable.

Observation 2.13. *With the previous example we also show that $\widehat{\Pi}_1^1 \neq \varnothing$. In the same example $\pi^4 \in \widehat{\Pi}_1^1$ and $c^4 \notin \operatorname{int} M$. On the other hand, in the Example 2.10 $\pi^2 \in \widehat{\Pi}_1^1$ and π^2 does not satisfy the Slater Condition. This means that both $c \in \operatorname{int} M$ and the Slater Condition are not necessary conditions for $\pi \in \widehat{\Pi}_1^1$.*

In the next Example 2.14, $\pi^5 \in \widehat{\Pi}_1^4$, $c^5 \in M_5 \setminus \operatorname{int} M_5$ and π^5 does not satisfy the Slater Condition. Later on, in the Example 2.15, $\pi^6 \in \widehat{\Pi}_1^1$, $\binom{0_n}{1} \notin \operatorname{cl} N_6$, $c^6 \in M_6 \setminus \operatorname{int} M_6$ and π^6 does not satisfy Slater Condition. This shows two things. First the condition (i) in Theorem 2.8 is not a necessary condition, and second the condition (iv), stated in Theorem 2.8, is not a sufficient one.

Example 2.14. *Consider in $\mathbb{R}^3$ the primal problem, with $\alpha > 0$:*

$$\begin{aligned} P_5: \quad & \min_{x \in \mathbb{R}^3} (x_1 + \alpha x_3), \\ s.t. \quad & x_1 + t^2 x_2 \geq 2t, \ t \in [0,1], \\ & s x_3 \geq -s^2, \ s \in [0,1], \\ & -r x_3 \geq -r^2, \ r \in [0,1]. \end{aligned}$$

If $s,\ r \in (0,1]$ then $x_3 \geq -s$ and $x_3 \leq r$, which means that $x_3 = 0$. So, $\dim F_5 \leq 2$. We can look at F_5 in $\mathbb{R}^2$, as the feasible set of P_3. Therefore, $v^P(\pi^5) = 0$ and $F_5^* = \varnothing$. In addition, as $\dim F_5 = 2$ and $F_5 \subset \mathbb{R}^3$, we have that π^5 does not satisfy the Slater condition. On the other hand, $c^5 \in M_5 \setminus \operatorname{int} M_5$. In fact:

$$\begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and, for all $\epsilon > 0$:

$$\begin{pmatrix} 1 \\ -\frac{\epsilon}{2} \\ \alpha \end{pmatrix} \notin M_5,$$

and:

$$\left\| \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} - \begin{pmatrix} 1 \\ -\frac{\epsilon}{2} \\ \alpha \end{pmatrix} \right\|_2 = \frac{\epsilon}{2} < \epsilon.$$

Therefore, $c^5 \in M_5 \setminus \operatorname{int} M_5$.

The dual problem of P_5 is:

$$D_5: \max_{\lambda,\beta,\gamma \in \mathbb{R}_+^{([0,1])}} \left(\sum_{t\in[0,1]} \lambda_t 2t + \sum_{s\in[0,1]} \beta_s(-s^2) + \sum_{r\in[0,1]} \gamma_r(-r^2) \right),$$

$$s.t. \sum_{t\in[0,1]} \lambda_t \begin{pmatrix} 1 \\ t^2 \\ 0 \end{pmatrix} + \sum_{s\in[0,1]} \beta_s \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} + \sum_{r\in[0,1]} \gamma_r \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix}.$$

This is equivalent to:

$$-\left(\min_{\lambda,\beta,\gamma \in \mathbb{R}_+^{([0,1])}} \left(-\left(\sum_{t\in[0,1]} \lambda_t 2t \right) + \sum_{s\in[0,1]} \beta_s s^2 + \sum_{r\in[0,1]} \gamma_r r^2 \right) \right),$$

$$s.t. \sum_{t\in[0,1]} \lambda_t \begin{pmatrix} 1 \\ t^2 \\ 0 \end{pmatrix} + \sum_{s\in[0,1]} \beta_s \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} + \sum_{r\in[0,1]} \gamma_r \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix}.$$

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From the above equality system, we have that:

$$\begin{pmatrix}1\\0\end{pmatrix} = \sum_{t\in[0,1]} \lambda_t \begin{pmatrix}1\\t^2\end{pmatrix} \text{ where } \lambda \in \mathbb{R}_+^{([0,1])}.$$

The only solution of the equation above is $\lambda^0 \in \mathbb{R}_+^{([0,1])}$ *with* $\lambda_0^0 = 1$ *and* $\lambda_t^0 = 0$, *for all* $t \in (0,1]$. *Then, feasible points of* D_5 *have the form:*

$$\overline{\theta} = (\lambda^0; \beta; \gamma),$$

where $\beta, \gamma \in \mathbb{R}_+^{([0,1])}$. *If we evaluate the objective function of the last problem at the points that have the above form, the problem will be reduced to:*

$$-\left(\min_{\beta,\gamma\in\mathbb{R}_+^{([0,1])}} \left(\sum_{s\in[0,1]} \beta_s s^2 + \sum_{r\in[0,1]} \gamma_r r^2\right)\right),$$

$$s.t. \qquad \sum_{s\in[0,1]} \beta_s s - \sum_{r\in[0,1]} \gamma_r r = \alpha. \tag{2}$$

From (2) it follows that:

$$v_5 := \min_{\beta,\gamma\in\mathbb{R}_+^{([0,1])}} \left\{ \sum_{s\in[0,1]} \beta_s s^2 + \sum_{r\in[0,1]} \gamma_r r^2 \;\middle|\; \sum_{s\in[0,1]} \beta_s s - \sum_{r\in[0,1]} \gamma_r r = \alpha \right\} \geq 0.$$

Note that for each $i \in (0,1]$*:*

$$\theta^0 := (\beta^0; \gamma^0),$$

where $\beta^0, \gamma^0 \in \mathbb{R}_+^{([0,1])}$, $\beta_i^0 = \frac{\alpha}{i}$, $\beta_s^0 = 0$ *for all* $s \in [0,1] \setminus \{i\}$ *and* $\gamma_r^0 = 0$ *for all* $r \in [0,1]$, *is a feasible point of (2). Moreover, if we evaluate* $\sum_{s\in[0,1]} \beta_s s^2 + \sum_{r\in[0,1]} \gamma_r r^2$ *in* θ^0, *we have that:*

$$v_5 \leq \sum_{s\in[0,1]} \beta_s^0 s^2 + \sum_{r\in[0,1]} \gamma_r^0 r^2 = \alpha i. \tag{3}$$

If $i \to 0$ *in (3), we have that* $v_5 \leq 0$. *Therefore* $v^D(\pi^5) = 0$. *On the other hand, if a feasible point of the problem* D_5 *is an optimal solution, the objective function evaluated at this point must satisfy:*

$$\sum_{s\in[0,1]} \beta_s s^2 + \sum_{r\in[0,1]} \gamma_r r^2 = 0.$$

Then, the possible optimal solutions of problem D_5 *have the following form:* $\overline{\theta}^0 = (\lambda^0; \beta^1; \gamma^1)$, *where* $\lambda^0, \beta^1, \gamma^1 \in \mathbb{R}_+^{([0,1])}$, $\lambda_0^0 = 1$, $\lambda_t^0 = 0$ *for all* $t \in (0,1]$, $\beta_0^1 \in \mathbb{R}_+$, $\beta_s^1 = 0$ *for all* $s \in (0,1]$, $\gamma_0^1 \in \mathbb{R}_+$ *and* $\gamma_r^1 = 0$ *for all* $r \in (0,1]$. *But* $\overline{\theta}^0 \notin \Lambda_5$, *because it implies* $\alpha = 0$, *which is a contradiction with the hypothesis* $\alpha > 0$. *Therefore,* $\Lambda_5^* = \varnothing$.

Example 2.15. *Consider the following problem in:* $\mathbb{R}^2$

$$\begin{aligned} P_6: \qquad & \min_{x\in\mathbb{R}^2} x_1, \\ & s.t.\ \ x_1 \geq 0, \\ & \quad -x_1 \geq 0, \\ & \qquad x_2 \geq 0. \end{aligned}$$

The feasible set of P_6 *is the same as the feasible set of the Example 2.10. Then* $v^P(\pi^6) = 0$ *and* $F_6^* = \{0\} \times \mathbb{R}_+$ *in addition, since* $\dim F_6 = 1$ *and* $F_6 \subset \mathbb{R}^2$, *we have that* π^6 *does not satisfy the Slater Condition. On the other hand, in Figure 6, we show that* $c^6 \in M_6 \setminus \operatorname{int} M_6$.

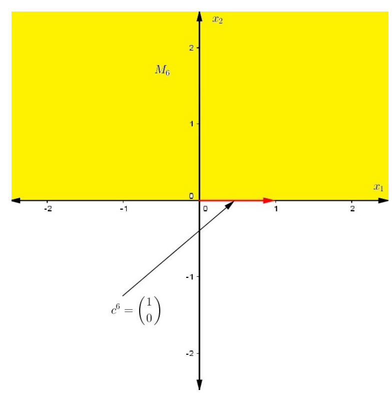

Fig. 6. First moment cone of P_6

The dual problem of P_6 *is:*

$$\begin{aligned} D_6: \qquad & \max_{\lambda_1,\lambda_2,\lambda_3\geq 0} (\lambda_1 0 + \lambda_2 0 + \lambda_3 0), \\ & s.t.\ \binom{1}{0} = \lambda_1\binom{1}{0} + \lambda_2\binom{-1}{0} + \lambda_3\binom{0}{1}. \end{aligned}$$

From the problem it follows that, $v^D(\pi^6) = 0$. *Now:*

$$\overline{\lambda} = \begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} \in \Lambda_6 = \Lambda_6^*,$$

if and only if:

$$\begin{pmatrix}1\\0\end{pmatrix} = \lambda_1\begin{pmatrix}1\\0\end{pmatrix} + \lambda_2\begin{pmatrix}-1\\0\end{pmatrix} + \lambda_3\begin{pmatrix}0\\1\end{pmatrix},$$

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i.e., $1 = \lambda_1 - \lambda_2$ *and* $0 = \lambda_3$*, or equivalently,* $\lambda_1 = 1 + \lambda_2$ *and* $\lambda_3 = 0$*. Then:*

$$\Lambda_6 = \Lambda_6^* = \left\{ \begin{pmatrix} 1+\lambda_2 \\ \lambda_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \ : \ \lambda_2 \geq 0 \right\},$$

which implies that $\Lambda_6^* \neq \varnothing$*. Therefore,* $\pi^6 \notin \widehat{\Pi}_1^4$*.*

Remember that in the first refined primal-dual partition, the sets $\Pi_S^P \cap \Pi_{IC}^D$ and $\Pi_{IC}^P \cap \Pi_S^P$ are empty. However, we will show that the sets $\Pi_s^P \cap \Pi_{IC}^D = \widehat{\Pi}_5^1$ and $\Pi_{IC}^P \cap \Pi_s^D = \widehat{\Pi}_6^1$, in the second general primal-dual partition, are nonempty. We shall present some necessary conditions for the fact that a given parameter π belongs to the state $\widehat{\Pi}_5^2 = \Pi_n^P \cap \Pi_{IC}^D$ and $\widehat{\Pi}_6^2 = \Pi_{IC}^P \cap \Pi_n^D$, respectively. Using also the definitions of the states, we shall characterize the cells $\widehat{\Pi}_5^1$ and $\widehat{\Pi}_6^1$.

Proposition 2.16. $\pi \in \widehat{\Pi}_5^1$*, if and only if* $c \notin M$ *and* F^* *is not bounded.*

Proof. Suppose that $\pi \in \widehat{\Pi}_5^1$ and $c \in M$ or F^* is bounded. First, if $c \in M$, we have the contradiction $\Lambda \neq \varnothing$. Second, if F^* is bounded, then $\pi \in \Pi_S^P \cap \Pi_{IC}^D$ and again we get to a contradiction. On the other hand, if $c \notin M$ and F^* is not bounded, then $\Lambda = \varnothing$ and $F^* \neq \varnothing$. Therefore, $\pi \in \widehat{\Pi}_5^1$. □

Now, $\widehat{\Pi}_5^i \subset \Pi_5$ for $i = 1, 2$. Then, for the Lemma 2.1:

$$c \notin M, \ (0_n, 1)' \ \notin \ \mathrm{cl}\, N \text{ and } (\{c\} \times \mathbb{R}) \cap \mathrm{cl}\, N \neq \varnothing$$

, is a necessary condition for π belongs to $\widehat{\Pi}_5^i$ for $i = 1, 2$, respectively.

Proposition 2.17. $\pi \in \widehat{\Pi}_6^1$*, if and only if* $\binom{0_n}{1} \in \mathrm{cl}\, N$ *and* Λ^* *is not bounded.*

Proof. Suppose that $\pi \in \widehat{\Pi}_6^1$ and $\binom{0_n}{1} \notin \mathrm{cl}\, N$ or Λ^* is bounded. First, if $\binom{0_n}{1} \notin \mathrm{cl}\, N$, we have the contradiction $F \neq \varnothing$. Second, if Λ^* is bounded, then $\pi \in \Pi_{IC}^P \cap \Pi_S^D$ and also we get again to a contradiction. On the other hand, if $\binom{0_n}{1} \in \mathrm{cl}\, N$ and Λ^* is not bounded, then $F = \varnothing$ and $\Lambda^* \neq \varnothing$. Therefore, $\pi \in \widehat{\Pi}_6^1$. □

Again, $\widehat{\Pi}_6^i \subset \Pi_6$ for $i = 1, 2$. Then, by the Lemma 2.1:

$$(0_n, 1)' \ \in \ \mathrm{cl}\, N, \ c \in M \text{ and } \{c\} \times \mathbb{R} \nsubseteq K$$

, is a necessary condition for π belongs to $\widehat{\Pi}_6^i$ for $i = 1, 2$, respectively.

With the following examples, we will show that the conditions (v) and (vi), presented in Lemma 2.1, are only necessary. We will also show that in continuous LSIP $\widehat{\Pi}_j^i \neq \varnothing$ for $i = 1, 2$ and $j = 5, 6$. However, in ordinary linear programming all these sets are empty [[8], Proposition 4.2].

Example 2.18. *Consider, in* $\mathbb{R}^2$*, the optimization problem:*

$$P_7: \quad \min_{x \in \mathbb{R}^2} x_2, \\ s.t. \ t^2 x_1 + t x_2 \geq t, \ t \in [0, 1].$$

The feasible set of P_7 *is presented in the Figure 7.*

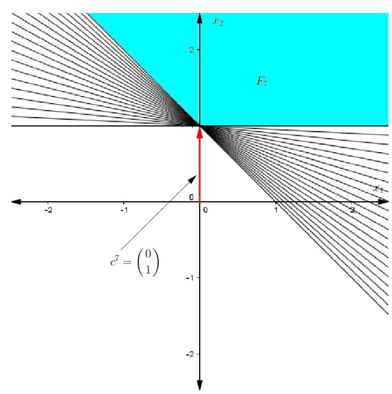

Fig. 7. Feasible set of P_7

Figure 7, shows that P_7 *is consistent and bounded, with an optimal value* $v^P(\pi^7) = 1$*, and an optimal set:*

$$F_7^* = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 = 1 \right\}.$$

From the previous equality it follows that F_7^* *is unbounded. The cone* M_7 *of* P_7*, is shown in the Figure 8.*

As $c^7 = \binom{0}{1}$*, we have that* $c^7 \in \mathrm{cl}\, M_7 \setminus M_7$*, then the*

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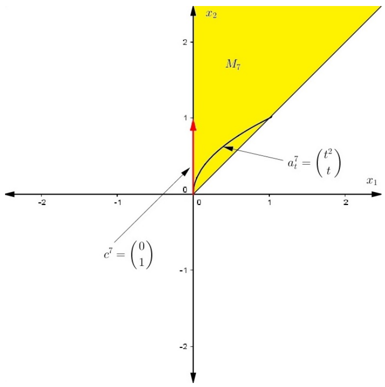

Fig. 8. First moment cone of P_7

dual problem D_7 is inconsistent. We conclude that $\pi^7 \in \widehat{\Pi}_5^1$, therefore $\widehat{\Pi}_5^1 \neq \varnothing$.

Example 2.19. *Consider now, the next problem in: $\mathbb{R}^2$:*

$$\begin{aligned} P_8 : \quad & \min_{x \in \mathbb{R}^2} x_1, \\ & s.t. \;\; tx_1 + t^3 x_2 \geq t^2, \; t \in [0,1]. \end{aligned}$$

The feasible set of P_8 is shown in the Figure 9.

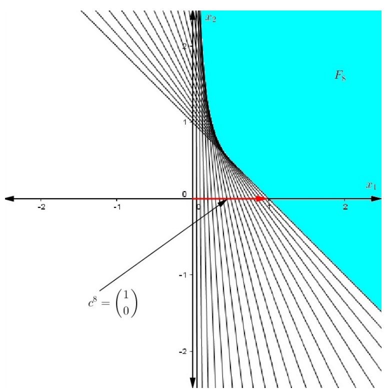

Fig. 9. Feasible set of P_8

Figure 9, shows that P_8 is consistent and bounded, with optimal value $v^P(\pi^8) = 0$, however $F^* = \varnothing$, i.e., P_8 is unsolvable. The cone M_8 of P_8, is presented in the Figure 10:

As $c^8 = \binom{1}{0}$, we have that $c^8 \in \operatorname{cl} M_8 \setminus M_8$, whereby the dual problem D_8 is inconsistent. So, we observe that $\pi_8 \in \widehat{\Pi}_5^2$, therefore $\widehat{\Pi}_5^2 \neq \varnothing$.

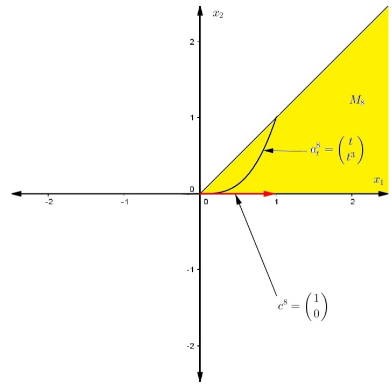

Fig. 10. First moment cone of P_8

Observation 2.20. *In the Example 2.19, $c^8 \notin M_8$, $(0_n, 1)' \notin \operatorname{cl} N_8$ (P_8 is consistent), also $(\{c^8\} \times \mathbb{R}) \cap \operatorname{cl} N_8 \neq \varnothing$ (P_8 is bounded), but $\pi^8 \notin \widehat{\Pi}_5^1$. It shows that $c \notin M$, $(0_n, 1)' \notin \operatorname{cl} N$ and $(\{c\} \times \mathbb{R}) \cap \operatorname{cl} N \neq \varnothing$ is not a sufficient condition for $\pi \in \widehat{\Pi}_5^1$. On the other hand, in the Example 2.18, $c^7 \notin M_7$, $(0_n, 1)' \notin \operatorname{cl} N_7$, and $(\{c^7\} \times \mathbb{R}) \cap \operatorname{cl} N_7 \neq \varnothing$, but $\pi^7 \notin \widehat{\Pi}_5^2$. It shows that $c \notin M$, $(0_n, 1)' \notin \operatorname{cl} N$ and $(\{c\} \times \mathbb{R}) \cap \operatorname{cl} N \neq \varnothing$ is not a sufficient condition for $\pi \in \widehat{\Pi}_5^2$, as well.*

Example 2.21. *We now study the following problem in $\mathbb{R}^2$:*

$$\begin{aligned} P_9 : \quad & \min_{x \in \mathbb{R}^2} x_1, \\ & s.t. \;\; x_1 + t^2 x_2 \geq 2t, \; t \in [0,1], \\ & \qquad -x_1 \geq 0. \end{aligned}$$

We observe that for each $t \in (0,1]$:

$$\left[\frac{1}{2t} \begin{pmatrix} 1 \\ t^2 \\ 2t \end{pmatrix} + \frac{1}{2t} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \frac{t}{2} \\ 1 \end{pmatrix}, \qquad (4)$$

is an element of N_9. If $t \to 0$ in (4), we have that $(0_2, 1)' \in \operatorname{cl} N_9$. Therefore P_9 is inconsistent.

The dual problem of P_9 is:

$$\begin{aligned} D_9 : \quad & \max_{\lambda \in \mathbb{R}_+^{([0,1])}, \gamma \in \mathbb{R}_+} \sum_{t \in [0,1]} \lambda_t 2t, \\ & s.t. \sum_{t \in [0,1]} \lambda_t \binom{1}{t^2} + \gamma \binom{-1}{0} = \binom{1}{0}. \end{aligned}$$

we have that $\overline{\theta} = (\lambda^0; \gamma)$*, where* $\gamma \in \mathbb{R}_+$ *and* $\lambda^0 \in \mathbb{R}_+^{([0,1])}$ *defined as:*

$$\lambda_t^0 := \begin{cases} 1+\gamma, & \text{if } t = 0, \\ 0, & \text{if } t \in (0,1], \end{cases}$$

is a feasible point of D_9*. Then* $v^D(\pi^9) = 0$ *and* Λ^* *is unbounded. This means that* $\pi^9 \in \widehat{\Pi}_6^1$*. Therefore* $\widehat{\Pi}_6^1 \neq \varnothing$*.*

Example 2.22. *Consider, in* $\mathbb{R}^2$*, the following problem:*

$$\begin{aligned} P_{10}: \quad & \min_{x \in \mathbb{R}^2} x_2, \\ & s.t.\ t^2 x_1 \geq t,\ t \in [0,1] \\ & \quad sx_2 \geq -s^2,\ s \in [0,1], \end{aligned}$$

We observed that for each $t \in (0,1]$*:*

$$\left[\frac{1}{t}\begin{pmatrix} t^2 \\ 0 \\ t \end{pmatrix}\right] = \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix}, \tag{5}$$

is an element of N_{10}*. If* $t \to 0$ *in (5), we observe that* $(0_2, 1)' \in \operatorname{cl} N_{10}$*. Therefore,* P_{10} *is inconsistent.*

The dual problem of P_{10} *is:*

$$\begin{aligned} D_{10}: \quad & \max_{\lambda,\ \gamma \in \mathbb{R}_+^{([0,\ 1])}} \left(\sum_{t \in [0,\ 1]} \lambda_t t + \sum_{s \in [0,\ 1]} \gamma_s(-s^2) \right), \\ & s.t. \sum_{t \in [0,\ 1]} \lambda_t \binom{t^2}{0} + \sum_{s \in [0,\ 1]} \gamma_s \binom{0}{s} = \binom{0}{1}. \end{aligned}$$

From the system above, we have that:

$$0 = \sum_{t \in [0,1]} \lambda_t t^2 \text{ con } \lambda \in \mathbb{R}_+^{([0,1])},$$

whose solutions have the form $\lambda^0 \in \mathbb{R}_+^{([0,1])}$ *with* $\lambda_0^0 \in \mathbb{R}_+$ *and* $\lambda_t^0 = 0$ *for all* $t \in (0,1]$*. Then, the feasible points of* (D_{10}) *look like:*

$$\overline{\theta}_{10} = (\lambda^0; \gamma),$$

where $\gamma \in \mathbb{R}_+^{([0,1])}$*. If we evaluate the objective function of the dual problem in points that have the above form, the problem is reduced to:*

$$\begin{aligned} & \max_{\gamma \in \mathbb{R}_+^{([0,\ 1])}} \sum_{s \in [0,\ 1]} \gamma_s(-s^2), \\ & s.t. \sum_{s \in [0,\ 1]} \gamma_s s = 1, \end{aligned}$$

which is equivalent to:

$$\begin{aligned} & -\min_{\gamma \in \mathbb{R}_+^{([0,\ 1])}} \sum_{s \in [0,\ 1]} \gamma_s s^2, \\ & s.t. \sum_{s \in [0,\ 1]} \gamma_s s = 1. \end{aligned}$$

we have that:

$$v_{10} := \min_{\gamma \in \mathbb{R}_+^{([0,\ 1])}} \left\{ \sum_{s \in [0,\ 1]} \gamma_s s^2 \;\middle|\; \sum_{s \in [0,\ 1]} \gamma_s s = 1 \right\} \geq 0.$$

Now, if $s_0 \in (0,\ 1]$*, then* $\gamma^0 \in \mathbb{R}_+^{([0,\ 1])}$ *satisfies:*

$$\sum_{s \in [0,\ 1]} \gamma_s^0 s = 1,$$

where $\gamma_{s_0}^0 = \frac{1}{s_0}$ *and* $\gamma_s^0 = 0$ *for all* $s \in [0,1] \setminus \{s_0\}$*. Hence:*

$$v_{10} \leq \sum_{s \in [0,\ 1]} \gamma_s^0 s^2 = \frac{1}{s_0} s_0^2 = s_0. \tag{6}$$

If $s_0 \to 0$ *in (6), we have* $v_{10} \leq 0$*. Therefore,* $v^D(\pi^{10}) = 0$*. On the other hand,* $v^D(\pi^{10}) = 0$*, if and only if:*

$$\sum_{t \in [0,1]} \lambda_t^0 t = \sum_{s \in [0,1]} \gamma_s s^2, \text{ for all } \overline{\theta}_{10} = (\lambda^0;\ \gamma) \in \Lambda_{10},$$

but, if $\overline{\theta}_{10} \in \Lambda_{10}$*,*

$$0 = \sum_{t \in [0,1]} \lambda_t^0 t,$$

it follows that:

$$0 = \sum_{s \in [0,1]} \gamma_s s^2 \text{ for all } \overline{\theta}_{10} = (\lambda^0;\ \gamma) \in \Lambda_{10}.$$

So, the possible optimal solutions of D_{10}*, have the form* $\overline{\theta}_{10}^0 = (\lambda^0; \gamma^1)$*, where* $\lambda^0, \gamma^1 \in \mathbb{R}_+^{([0,\ 1])}$*,* $\lambda_0^0, \gamma_0^1 \in \mathbb{R}_+$ *and* $\lambda_t^0 = 0 = \gamma_s^1$ *for all* $t,\ s \in (0,1]$*. If* $\overline{\theta}_{10}^0 \in \Lambda_{10}$*, then:*

$$\binom{0}{1} = \lambda_0^0 \binom{0}{0} + \gamma_0^1 \binom{0}{0},$$

which is impossible. Therefore, $\overline{\theta}_{10}^0 \notin \Lambda_{10}$*, whereby* $\Lambda_{10}^* = \varnothing$*. So, we conclude that* $\pi^{10} \in \widehat{\Pi}_6^2$*, and thus* $\widehat{\Pi}_6^2 \neq \varnothing$*.*

Observation 2.23. *We see in Example 2.22 that* $(0_n, 1)' \in \operatorname{cl} N_{10}$ *(P_{10} is inconsistent),* $c^{10} \in M_{10}$ *($\overline{\theta}_{10} = (\lambda^0;\ \gamma^0) \in \Lambda_{10}$, i.e, D_{10} is consistent) and* $\{c^{10}\} \times \mathbb{R} \nsubseteq K_{10}$ *($v^D(\pi_{10}) = 0$, i.e, D_{10} is bounded), but* $\pi^{10} \notin \widehat{\Pi}_6^1$*. This shows that* $(0_n, 1)' \in \operatorname{cl} N$*,* $c \in M$ *and* $\{c\} \times \mathbb{R} \nsubseteq K$ *is not a sufficient condition for* $\pi \in \widehat{\Pi}_6^1$*. On the other hand, in the Example 2.21,* $(0_n, 1)' \in \operatorname{cl} N_9$*,* $c^9 \in M_9$ *and* $\{c^9\} \times \mathbb{R} \nsubseteq K_9$*, but* $\pi^9 \notin \widehat{\Pi}_6^2$*. It demonstrates that* $(0_n, 1)' \in \operatorname{cl} N$*,* $c \in M$ *and* $\{c\} \times \mathbb{R} \nsubseteq K$ *is not a sufficient condition for the following statement* $\pi \in \widehat{\Pi}_6^2$*.*

2.1.1 Some Topological Properties of the Sets Generated by the Second General Primal-dual Partition

In [11], Goberna and Todorov presented the characterization of the interior of the sets Π_2, Π_3 and Π_4. They also studied the density properties and the interior of these and other sets of the first general primal-dual partition. In this section, we shall study some topological properties of the sets $\widehat{\Pi}_1^1$, $\widehat{\Pi}_1^2$, $\widehat{\Pi}_1^3$, $\widehat{\Pi}_1^4$, $\widehat{\Pi}_5^1$, $\widehat{\Pi}_5^2$, $\widehat{\Pi}_6^1$ and $\widehat{\Pi}_6^2$. In particular, we investigate the interior and some density properties of the mentioned sets of the new second general primal-dual partition.

Theorem 2.24. $\widehat{\Pi}_1^1$ *is dense in* Π_1*.*

Proof. Since $\Pi_1^1 \subseteq \widehat{\Pi}_1^1$ and Π_1^1 is dense in Π_1 [[10], Theorem 3.3], hence $\widehat{\Pi}_1^1$ is dense in Π_1. □

Theorem 2.25. *The sets* $\widehat{\Pi}_1^i$*,* $i = 1, 2, 3, 4$ *are neither closed nor open.*

Proof. Since these sets are cones with the null triplet belonging to $\widehat{\Pi}_1^1$, only $\widehat{\Pi}_1^1$ could be closed:

$\widehat{\Pi}_1^1$ is not closed. In fact, consider the sequence $\{\pi^r\}$ in $\widehat{\Pi}_1^1$, where $\pi^r := (\frac{1}{r} e_1, 1, e_1)$, obviously:

$$\lim_{r \to \infty} \pi^r = (0_n, 1, e_1),$$

but $(0_n, 1, e_1) \in \Pi_4$. Therefore, $\widehat{\Pi}_1^1$ is not closed.

$\widehat{\Pi}_1^1$ is not open. Indeed, consider $\pi := (0_n, 0, 0_n)$ in $\widehat{\Pi}_1^1$. Let $r > 0$, we define $\pi^r := (0_n, 0, \frac{r}{2} e_1)$. It is easy to see that, for all $r > 0$, $\pi^r \in \Pi_2$ and $d(\pi, \pi^r) = \frac{r}{2} < r$. This implies that $\widehat{\Pi}_1^1$ is not open.

We shall prove that the sets $\widehat{\Pi}_1^i$, $i = 2, 3, 4$ are not open. Since $\widehat{\Pi}_1^i \subset \Pi_1$, then $\operatorname{int} \widehat{\Pi}_1^i \subset \operatorname{int} \Pi_1$, but $\operatorname{int} \Pi_1 = \Pi_1^1$ [[10], Theorem 2], as $\Pi_1^1 \subset \widehat{\Pi}_1^1$, so it follows that $\operatorname{int} \widehat{\Pi}_1^i \subset \widehat{\Pi}_1^1$. However $\widehat{\Pi}_1^i \cap \widehat{\Pi}_1^1 = \varnothing$, and, the above inclusion is only possible if $\operatorname{int} \widehat{\Pi}_1^i = \varnothing$. Since $\widehat{\Pi}_1^i \neq \varnothing$ it follows that $\widehat{\Pi}_1^i$ is not open for every $i = 2, 3, 4$. □

Corollary 2.26. $\operatorname{int} \widehat{\Pi}_1^i$*,* $i = 2, 3, 4$ *are empty.*

Theorem 2.27. $\operatorname{int} \widehat{\Pi}_1^1 \neq \varnothing$*.*

Proof. Since $\Pi_1^1 \subset \widehat{\Pi}_1^1$ then $\operatorname{int} \Pi_1^1 \subset \operatorname{int} \widehat{\Pi}_1^1$.
but:

$$\operatorname{int} \Pi_1^1 = \Pi_1^1,$$

and:

$$\Pi_1^1 \neq \varnothing \text{ [Theorem 2.5]},$$

we conclude that $\operatorname{int} \widehat{\Pi}_1^1 \neq \varnothing$. □

Theorem 2.28. $\operatorname{int} \widehat{\Pi}_1^1$ *is dense in* Π_1*.*

Proof. Since $\Pi_1^1 \subset \widehat{\Pi}_1^1 \subset \Pi_1$, then:

$$\operatorname{int} \Pi_1^1 \subset \operatorname{int} \widehat{\Pi}_1^1 \subset \operatorname{int} \Pi_1,$$

it follows:

$$\overline{\operatorname{int} \Pi_1^1} \subset \overline{\operatorname{int} \widehat{\Pi}_1^1} \subset \overline{\operatorname{int} \Pi_1}.$$

As $\operatorname{int} \Pi_1^1$ is dense on Π_1 [[10], Theorem 3.3] and $\operatorname{int} \Pi_1$ is dense on Π_1 [[11], Theorem 2], we have that $\overline{\operatorname{int} \widehat{\Pi}_1^1} = \Pi_1$. Therefore, $\operatorname{int} \widehat{\Pi}_1^1$ is dense in Π_1. □

Theorem 2.29. $\operatorname{int} \widehat{\Pi}_1^1 = \Pi_1^1$*.*

Proof. Since Π_1^1 is open [[10], Theorem 3.3], it follows $\Pi_1^1 \subset \operatorname{int} \widehat{\Pi}_1^1$. We will only show that $\operatorname{int} \widehat{\Pi}_1^1 \setminus \Pi_1^1 = \varnothing$. Suppose the contrary, i.e., $\operatorname{int} \widehat{\Pi}_1^1 \cap (\Pi_1^1)^c \neq \varnothing$. Then, there exists $\pi \in \operatorname{int} \widehat{\Pi}_1^1$, such that $\pi \notin \Pi_1^1$, so it follows that $c \notin \operatorname{int} M$ or π does not satisfy the Slater Condition [Theorem 2.6].

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First, $c \notin \operatorname{int} M$ implies that $c \in M \setminus \operatorname{int} M$ because, by hypothesis $\pi \in \operatorname{int} \widehat{\Pi}_1^1$, but:

$$\operatorname{int} \widehat{\Pi}_1^1 \subset \widehat{\Pi}_1^1 \subset \Pi_1.$$

In addition, $c \notin \operatorname{int} M$ implies $M \neq \mathbb{R}^n$. From the above two implications, we conclude that, there exists a sequence $\{c^r\}$ from $\mathbb{R}^n \setminus M$ such that:

$$\lim_{r\to\infty} c^r = c.$$

We define $\pi^r := (a, b, c^r)$. The sequence $\{\pi^r\}$ is in Π_{IC}^D and satisfies:

$$\lim_{r\to\infty} \pi^r = \pi.$$

This is a contradiction, because, by hypothesis, $\pi \in \operatorname{int} \widehat{\Pi}_1^1$.

Second, if π does not satisfy the Slater Condition, then $\pi \notin \operatorname{int} \Pi_C^P$. Now, by hypothesis $\pi \in \operatorname{int} \widehat{\Pi}_1^1$, we have $\pi \in \Pi_C^P$. So, if π does not satisfy the Slater Condition and the hypothesis is true, $\pi \in \operatorname{bd} \Pi_C^P$. Therefore there exists a sequence $\{\pi^r\}$ on Π_{IC}^P such that:

$$\lim_{r\to\infty} \pi^r = \pi.$$

This is a contradiction, because $\pi \in \operatorname{int} \widehat{\Pi}_1^1$.

It follows that $\operatorname{int} \widehat{\Pi}_1^1 \setminus \Pi_1^1 = \varnothing$, or equivalently $\operatorname{int} \widehat{\Pi}_1^1 = \Pi_1^1$. $\square$

Observation 2.30. *Since* $\operatorname{int} \Pi_5 = \varnothing$ *and* $\operatorname{int} \Pi_6 = \varnothing$ *[[11], Theorem 2], we conclude that* $\operatorname{int} \widehat{\Pi}_j^i = \varnothing$ *for* $i = 1, 2$ *and* $j = 5, 6$.

We have proved that all parameters in Π_1 can be approached by parameters in $\widehat{\Pi}_1^1$. In addition, we have shown that the sets $\widehat{\Pi}_1^1$, $\widehat{\Pi}_1^2$, $\widehat{\Pi}_1^3$ and $\widehat{\Pi}_1^4$ are neither closed nor open, and that the interior of the sets $\widehat{\Pi}_1^2$, $\widehat{\Pi}_1^3$ and $\widehat{\Pi}_1^4$ are empty. The characterization of int $\widehat{\Pi}_1^1$ follows from the equality int $\widehat{\Pi}_1^1 = \Pi_1^1$ which was also proved in this section.

3 Conclusion and Future Work

To conclude, we would like to mention that the lack of necessary and sufficient conditions, characterizing the majority of the states of the second general primal-dual partition has, in some sense, its justification. Namely, there are no necessary and sufficient conditions for the solvability, neither for the primal nor for the dual linear semi-infinite optimization problems. On the contrary, for the solvability, considered in the first general primal-dual partition, the conditions could be found in lemma 2.4. Anyway, finding such necessary and sufficient conditions is still a challenging problem. Another open question is how will apply the developed theory in this article removing the continuity in the LSIP.

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