Radial Basis Functions for Phase Unwrapping

Funciones Radiales de Base para Desenvolvimiento de Fase

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Abstract. An important step in fringe pattern analysis is the so called phase unwrapping. Although this task can be performed easily using path dependent algorithms, most times, however, these algorithms are not robust enough specially in the presence of noise. On the other hand, path independent methods such as least-squares based or regularization based may be little convenient due to programming complexity or time consuming. In this paper we describe an alternative algorithm for phase unwrapping based in the determination of weights to linearly combine a set of radial basis functions (RBFs). As described, our algorithm is fast and can be easily implemented following a simple matrix formulation. Numerical and real experiments with good results show that our method can be applied in many kinds of optical tests.
Keywords: Phase unwrapping, Radial basis functions.

Resumen. Un importante paso en el análisis de patrones de franjas es el llamado desenvolvimiento de fase. Aunque esta tarea puede ser realizada fácilmente usando algoritmos dependientes del camino, muchas veces, sin embargo, estos algoritmos no son suficientemente robustos especialmente con la presencia de ruido. Por otro lado, los métodos independientes del camino tales como los basados en mínimos cuadrados o regularización pueden ser poco convenientes debido a la complejidad de programación o al tiempo de procesado. En este artículo describimos un algoritmo alternativo para desenvolvimiento de fase basado en la determinación de pesos para combinar linealmente un conjunto de funciones radiales de base (FRBs). Como se describe, nuestro algoritmo es rápido y puede ser fácilmente implementado siguiendo una formulación matricial simple. Experimentos numéricos y reales con buenos resultados muestran que nuestro método puede ser aplicado a muchos de los tipos de pruebas ópticas.
Palabras clave: Desenvolvimiento de fase, Funciones radiales de base.

1 Introduction

Optical methods are widely used for nondestructive testing and metrology. Most of them such as interferometric or holographic tests produce fringe patterns that contain the physical information to be determined. Because of this, fringe pattern analysis has been an important topic in the study of optical interferometry and optical metrology. In fringe pattern analysis the aim is to recover the phase that modulates a two-dimensional cosine function. Most proposed phase recovery techniques such as Fourier based [14], phase stepping [10] or regularization [15], provide a non-continuous phase, wrapped in the interval $[-\pi, \pi)$. For this reason, an important step in fringe pattern analysis is the phase unwrapping.

The unwrapping process can be, in many cases, a difficult task owing to phase inconsistencies or noise, and the application of path dependent algorithms [13] does not always provides proper results. The least-squares solution for the phase unwrapping problem, which was clearly described in matrix form by Hunt [6], is a robust alternative for many cases. Ghiglia and Romero [3] proposed a robust algorithm to find a solution in the presence of path-integral phase inconsistencies using the cosine transform. Some other robust methods based on regularization have been proposed [5,12]. The processing time or computational complexity of the above mentioned methods, however, may be a strong inconvenient. A different point of view to solve the phase unwrapping problem was proposed by Arines [1]. This author proposed to model the unwrapped phase by linearly combining an orthogonal base of polynomials. In this paper we follow a similar philosophy of reference [9] to propose an alternative method for phase unwrapping, which consists in modeling the unwrapped phase with a linear combination of RBFs. As we shall see, the determination of weights is easily described in a typical matrix formulation, and the required matrix inversion can be performed applying direct methods. This paper is organized in the following way: In section 2 we explain the fundamentals of the phase unwrapping problem. In section 3 we describe our phase unwrapping technique. Some practical considerations to implement our algorithm are discussed in section 4. Numerical and real experiments are presented in section 5. Finally, in section 6 we summarize some conclusions.

2 The phase unwrapping problem

In order to understand the phase unwrapping fundamentals we define the wrapped and the unwrapped phase as θ_r and ϕ_r respectively, where $r = (x, y)$ is the vector coordinate in a regular lattice. As we know that $\theta_r \in [-\pi, \pi)$, we can establish the following relation:

$$\theta_r = W[\phi_r] = \phi_r + 2\pi k_r, \tag{1}$$

where W represents the wrapping operator and k_r a field of integers such that $W[\phi_r] \in [-\pi, \pi)$. We should note that θ_r represents the observed phase (wrapped) and ϕ_r the real unknown phase (unwrapped) to be determined. The wrapped phase-difference vector field $\Delta\theta_r$, which can be computed from the wrapped phase map, is defined as

$$\Delta\theta_r = (\theta_r - \theta_s, \theta_r - \theta_t). \tag{2}$$

Where $s = (x-1, y)$ and $t = (x, y-1)$ are contiguous horizontal and vertical sites respectively. In a similar manner we can also define the unwrapped phase-difference field: $\Delta\phi_r = (\theta_r - \phi_s, \phi_r - \phi_t)$. As we are dealing with discrete phase fields, closely related to digital fringe images, the problem of the recovery of ϕ_r from θ_r can be properly solved if the sampling theorem is reached, that is, if the distance between two fringes is more than two pixels (the phase difference between two fringes is 2π). In phase terms the sampling theorem is reached if the phase difference between two pixels is less than π or, in general

$$\|\Delta\phi_r\| < \pi. \tag{3}$$

If this condition is satisfied, the following relation can be established:

$$\Delta\phi_r = W[\Delta\theta_r] = (W[\theta_r - \theta_s], W[\theta_r - \theta_t\,]). \tag{4}$$

Note that $W[\Delta\theta_r]$ can be obtained from the observed field θ_r.

By analyzing this equation, we see that ϕ_r can be achieved by two-dimensional integration of the vector field $W[\Delta\theta_r]$. One way to realize this integration is by means of a least-squares approach [2,7,8,9].

3 Radial basis functions for phase unwrapping

In the phase unwrapping technique proposed here, the determination of the unwrapped phase is visualized as a modeling of the two dimensional function ϕ_r through a linear combination of RBFs, which can be represented in the following form

$$\phi_r \simeq \sum_{k=1}^{n} a_k\,\psi_r^k, \qquad \psi_r^k = \exp\left[-\frac{(x-x_k)^2+(y-y_k)^2}{2\sigma^2}\right], \tag{5}$$

where a_k and ψ_r^k represent the weights to be determined and a set of uniformly distributed RBFs respectively. In the work reported by Montoya et al. [11] a detailed analysis of this kind of representation was done for wave front fitting. In general, the computation of the weights a_k for modeling any surface is a simple task using matrix inversion, however, in this case we have no access to ϕ_r but only to θ_r or $\nabla\theta_r$. The problem seems not to have solution in the actual form, however, analyzing equation (4) we can determine a_k by visualizing the problem according to

$$W[\Delta\theta_r] \simeq \sum_{k=1}^{n} a_k\,\psi_r^k, \tag{6}$$

where $\Delta\psi_r^k = (\psi_r^k - \psi_s^k, \psi_r^k - \psi_t^k)$. We may understand clearly that equation (6) is valid as a consequence of equation (4). In order to describe the computation of **a** $= a_k$ in matrix form, we define the following column vectors:

$$\Delta\Theta_x = \begin{bmatrix} \vdots \\ W[\theta_r - \theta_s] \\ \vdots \end{bmatrix},\quad \Delta\Theta_y = \begin{bmatrix} \vdots \\ W[\theta_r - \theta_t\,] \\ \vdots \end{bmatrix},\quad \Delta\Psi_x^k = \begin{bmatrix} \vdots \\ \psi_r^k - \psi_s^k \\ \vdots \end{bmatrix},\quad \Delta\Psi_y^k = \begin{bmatrix} \vdots \\ \psi_r^k - \psi_t^k \\ \vdots \end{bmatrix}. \tag{7}$$

The components of these vectors are the wrapped phase differences and the phase differences of the fields θ_r and ψ_r^k respectively.

The linear system derived from equation (6) can be written as

$$\begin{bmatrix} \Delta\Psi_x^1 & : & \Delta\Psi_x^2 & : & \cdots & : \Delta\Psi_x^n \\ \Delta\Psi_y^1 & : & \Delta\Psi_y^2 & : & \cdots & : \Delta\Psi_y^n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \simeq \begin{bmatrix} \Delta\Theta_x \\ \Delta\Theta_y \end{bmatrix}. \quad (8)$$

or, in compact form

$$\Psi a \simeq \Theta. \quad (9)$$

Where the size of matrix $\mathbf{\Psi}$ is $(2 \cdot M \cdot N) \times n$. $(M \times N)$ represents the size of the observed phase field θ_r, and n the number of RBFs used for modeling ϕ_r.

The recovery of **a** from equation (9) represents a least-squares problem, which can be solved considering that the mean-squared-error is given by

$$\varepsilon = \frac{1}{n}(\Theta\text{-}\Psi a)^{\mathrm{T}}(\Theta\text{-}\Psi a). \quad (10)$$

Differentiating this equation and equating it to zero, we determine that [4]

$$a = (\Psi^{\mathrm{T}}\Psi)^{-1}\Psi^{\mathrm{T}}\Theta \quad (11)$$

We can observe in this equation that vector **a** is obtained inverting a matrix of size $n \times n$.

4 Parameter selection process of the RBFs

In this section we briefly discuss some important details of our phase unwrapping method. As described in the previous section, the continuous phase ϕ_r can be approximated by means of equation (5) once the weights a_k were computed. However, it is necessary to define proper values of *n* and σ (equation 5), and the spatial distribution of the RBFs over the image. In order to determine optimal values of these parameters, a detailed analysis using Fourier theory was done by Montoya *et al.* [14]. In their study they conclude that these parameters can be determined analyzing the wide-band of the spectrum of the wave-front function (the phase function ϕ_r in our case). In the case of phase unwrapping, however, we have not a direct access to ϕ_r but only to $W[\phi_r]$. For practical purposes, in our technique the above mentioned parameters are determined heuristically, we distributed the RBFs over the image at regular intervals δ_x and δ_y along horizontal and vertical directions respectively. Figure 1 shows a typical spatial distribution of RBFs over the image. We observed that for many experiments, best results were obtained with a distribution of 7-10 RBFs along both directions, and the value of σ such that $\delta_x \leq \sigma \leq 2\delta_x$.

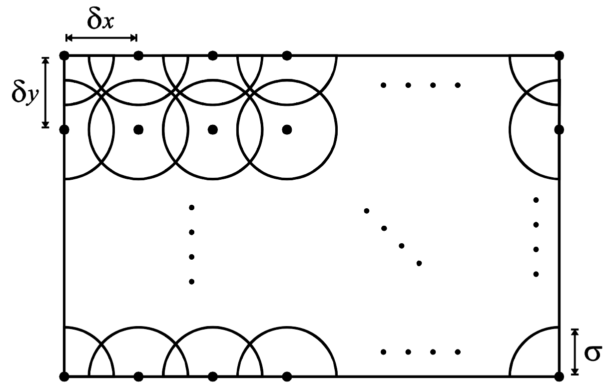
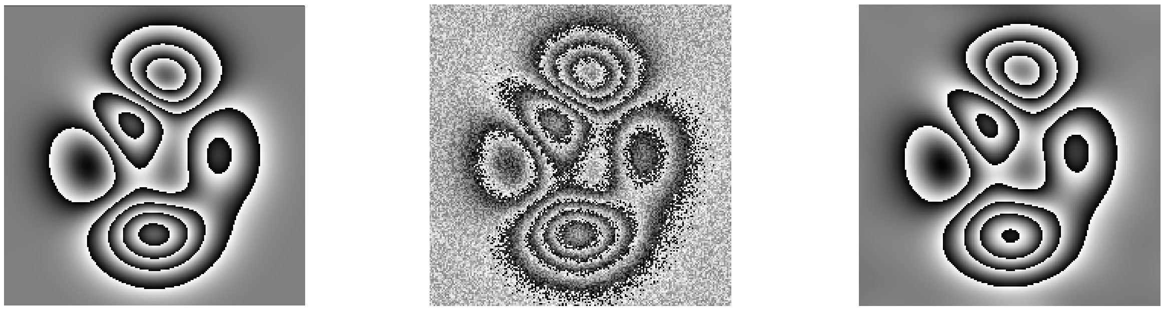


Fig. 1. Schematic diagram of the typical RBF's distribution over a rectangular image used in our experiments

5 Experimental results

In this section we present some experimental results using our phase unwrapping method. The algorithm was tested with synthetic and real wrapped phase fields. In the first experiment we simulated a 200×200 noiseless wrapped phase, which is shown in Figure 2(a) using a gray level codification (black represents $-\pi$ and white π). We proved our method with the same phase field but adding Gaussian noise with zero mean and variance=7.8 (Figure 2(b)). The result is shown in Figure 3(c) (it is shown wrapped for comparison purposes). The parameters used in this case were $n = 10 \cdot 10$ (100 RBFs) and $\sigma = \delta_x = \delta_y$. We can see that if the number of rows and columns of the image are 200, $\delta_x = \delta_y = \frac{200}{10}$.

The processing time in this experiment was about 9 seconds in a 2.66 GHz Pentium D based computer, using Matlab 6.5. According to the result, it is interesting to note that our technique strongly reduces the noise, however, with our experience we observed that our method fails with higher noise levels.



(a) (b) (c)

Fig. 2. (a) Gray level codification of a synthetic noiseless wrapped phase map. (b) The same phase corrupted with noise. (c) Result applying our unwrapping method (shown wrapped for comparison purposes)

In order to test our method with real data, we realized two experiments with two different wrapped phase fields obtained from interferometric fringe patterns. Figures 3(a) and 4(a) show them in a gray level codification. In these cases the size of the images were 400×400 and 450×450 respectively.

Owing to the smoothness of these data, we could easily represent the unwrapped phase with few RBFs, in both experiments we used the parameters $n = 7 \cdot 7$ (49 RBFs) and $\sigma = 1.8\delta_x$. The results applying our technique are shown in Figures 3(b) and 4(b).

(a) (b)

Fig. 3. (a) Gray level codification of a wrapped phase map recovered from a holographic interferogram. (b) Unwrapped phase map (shown wrapped for comparison purposes) obtained with the proposed method

(a) (b)

Fig. 4. (a) Gray level codification of a wrapped phase map recovered from an interferogram. (b) Unwrapped phase map (shown wrapped for comparison purposes) obtained with the proposed method

6 Conclusions

We have presented an alternative procedure to solve the problem of phase unwrapping. Although the application of our algorithm is restricted to smooth phase fields, it can be used in many metrological and interferometric tests without discontinuities or abrupt changes in the phase, obtaining good results. As our method does not requires sophisticated iterative matrix inversion but any direct method such as LU factorization, our algorithm can be easily implemented in a computer program, being fast and efficient.

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