

Conservative nonlinear oscillators in Abel's mechanical problem

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Recibido el 13 de diciembre de 2010; aceptado el 24 de marzo de 2011

We study a family of conservative, *truly nonlinear*, oscillators, arising from particular solutions of Abel's mechanical problem. An exact period-to-amplitude relation is produced for each instance. The lagrangian and hamiltonian formulations of such systems are discussed, along with their relations with the harmonic oscillator.

Keywords: Nonlinear oscillators; analytical mechanics; Abel's mechanical problem.

Se estudia una familia de osciladores no lineales, intrínsecamente no analíticos, que surgen como soluciones particulares del problema mecánico de Abel. Se incluye en cada caso la expresión exacta del periodo en función de la amplitud de las oscilaciones. Las formulaciones lagrangiana y hamiltoniana son discutidas, así como la relación que los mencionados sistemas guardan con el oscilador armónico.

Descriptores: Osciladores no lineales; mecánica analítica; problema mecánico de Abel.

PACS: 01.55.+b; 02.30.Xx; 02.30.Hq; 02.30.Em

1. Introduction

Examples of nonlinear oscillator models appear not only in physics, but also in engineering [1], mathematical biology [2-3] and related fields. In physics, specific applications may be found in quantum optics [4], solar physics [5] and laboratory plasmas [6], to name just a few possibilities. Some examples of current theoretical research are to be found in Refs 7 to 8.

As a consequence, there is an increasing interest in introducing this subject, in a clear and simple way, to undergraduate students. This interest is reflected in physics education journals [9-17].

The present paper's aim is to introduce little known instances of conservative nonlinear oscillators arising from Abel's mechanical problem, as a way of illustrating some of the main concepts and techniques of analytical mechanics. The only prerequisites for reading the present communication are: familiarity with ordinary differential equations, special functions and integral transforms, and a rough acquaintance with analytical mechanics. Thus, we believe it will be of interest for senior undergraduate and graduate students, and faculty members in charge of courses in analytical mechanics or mathematical methods for physicists.

Abel's mechanical problem can be stated as follows: [18] consider a wire bent into a smooth plane curve, and let a bead of mass m start from rest and slide without friction down the wire towards the origin under the action of its own weight. In this way, the time τ it takes the bead to reach the origin is function only of the initial height, y_i , at which the bead starts its motion. For a prescribed relation $\tau(y_i)$ it is possible, in

principle, to deduce the equation

$$f(x, y) = 0 \quad (1)$$

that describes the curve in which the bead slides. The central idea we present is that if you can smoothly match at the origin a curve with known $\tau(y_i)$ relation with another curve with the same prescribed behaviour, you will obtain a (generally anharmonic) oscillator with a known period to amplitude relation.

Expression (1) represents an holonomic constraint (as it is an equation, and not, for example, an inequality) so that the system is amenable to a lagrangian treatment. Moreover, we will be able to produce not only lagrangian and hamiltonian functions for each and every curve we construct, but also the effective potential that acts on the bead in each case. Interestingly, the systems we here present are not only examples of conservative nonlinear oscillators, but are also intrinsically non-analytical, *i.e.* that when subject to small amplitude oscillations, our systems do not conform to the simple harmonic oscillator model. These systems have been called *truly nonlinear oscillators* in recent literature [19].

The rest of this communication is divided as follows: in Sec. 2 we present a family of curves, particular solutions of Abel's mechanical problem. In Sec. 3 we derive the Lagrangian of a bead sliding in one of such curves. In Sec. 4 we construct curves in which the movement of the bead is that of a conservative nonlinear oscillator. In this last section we also discuss some of the salient features of the behavior of the resulting oscillators. Finally, Sec. 5 is reserved for conclusions.

2. A family of curves

We now return to Abel's mechanical problem, sketched in Sec. 1: a massive bead slides down towards the origin, in a frictionless motion, along a smooth plane curve σ . The conservation of the mechanical energy of the bead dictates that, when starting its motion from rest at a height y_i , it will reach the origin in a lapse of time $\tau(y_i)$ given by

$$\tau(y_i) = \int_0^{y_i} \frac{\frac{ds}{dy}}{\sqrt{2g}(y_i - y)^{\frac{1}{2}}} dy \quad (2)$$

(where

$$ds(y) = \sqrt{1 + \left(\frac{dx_\sigma}{dy}\right)^2} dy$$

is the differential of the arch length along σ), provided that the bead is subject to a potential:

$$V(y) = mgy.$$

Indeed, in a frictionless motion along a wire, the conservation of energy dictates that

$$\frac{m}{2} \left(\frac{ds}{dt}\right)^2 + mgy = E \quad (3)$$

is constant in time, as

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad (4)$$

is just the rapidity of the bead.

By considering initial conditions $y(t = 0) = y_i$ and $v(t = 0) = 0$ we get, from (3):

$$\frac{ds}{dt} = \sqrt{g(y_i - y)}$$

which, when solved for t , gives us (2).

Equation (2) is an example of Abel's integral equation. It can be inverted, in order to obtain ds/dy , by using the Laplace transform method. The steps are laid out in Ref. 18, among other places. The result, which constitutes the solution of Abel's mechanical problem, may be written in the following form:

$$\frac{ds}{dy}(y) = \frac{\sqrt{2g}}{\pi} \frac{d}{dy} \int_0^y \frac{\tau(\nu)}{(y - \nu)^{1/2}} d\nu \quad (5)$$

We now examine the implications of imposing on curve σ a condition of the type

$$\tau(y_i) = Ky_i^\beta \quad (6)$$

for a fixed $\beta \in \mathbb{R}$ and a real constant $K > 0$. For $\beta = 0$, (6) corresponds to the isochronous condition (*i.e.* the independence of the period from the amplitude in a periodic motion)

and the resulting curve turns out to be in this case (half of) Huygens's tautochrone (also known as the isochrone curve), while the solutions for $\beta = 1/2$ are inclined planes. For the general case we plug (6) in (5) to obtain:

$$\frac{ds}{dy}(y) = \frac{\sqrt{2g}}{\pi} K \frac{d}{dy} \int_0^y \frac{\nu^\beta}{(y - \nu)^{1/2}} d\nu \quad (7)$$

The integral on the r.h.s. of (7) is divergent for $\beta \leq -1$, and for $\beta > -1$ one can apply the Laplace transform method and the convolution theorem to get

$$\frac{ds}{dy}(y) = \sqrt{\frac{2g}{\pi}} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 3/2)} K \frac{d}{dy} y^{\beta+1/2} \quad (8)$$

where Γ stands for the gamma function (as we will be tacitly using some of the properties of this function it may be convenient to have a textbook such as [20] at hand).

Equation (8) tells us that s will be finite at the origin only if $\beta \geq -1/2$, and the case $\beta = -1/2$ itself has no geometrical (let alone physical) interpretation, as it would imply that the arch-length traveled from origin to any given point on the curve would be same, irrespective of the end point. Thus, physically acceptable bounded curves exist only for $\beta > -1/2$. With this caveat in mind, from (8) we may conclude

$$\begin{aligned} x_\sigma(y) &= \pm \int_0^y \sqrt{\left(\frac{ds_\sigma}{dy'}\right)^2 - 1} dy' \\ &= \pm \int_0^y \sqrt{\left(\frac{y'}{H}\right)^{2\beta-1} - 1} dy' \end{aligned} \quad (9)$$

where, by definition:

$$H := \left(\sqrt{\frac{2g}{\pi}} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1/2)} K \right)^{\frac{2}{1-2\beta}} \quad (10)$$

In order to retain the physical interpretation of Eq. (9) the integrand on the r.h.s. must remain real-valued for values of y arbitrarily close to $y = 0$, and so the expression is only valid for $\beta \leq 1/2$. Moreover, for any acceptable value of β (*i.e.* $-1/2 < \beta \leq 1/2$) we get the restriction

$$y < H \quad (11)$$

so that the curve ends at height H (a more thorough analysis shows us that dy/dx diverges at height H).

A curve σ is thus identified by three different parameters: β , \pm and H . From now on we will write $\sigma_{\beta,H,\pm}$ to refer to a

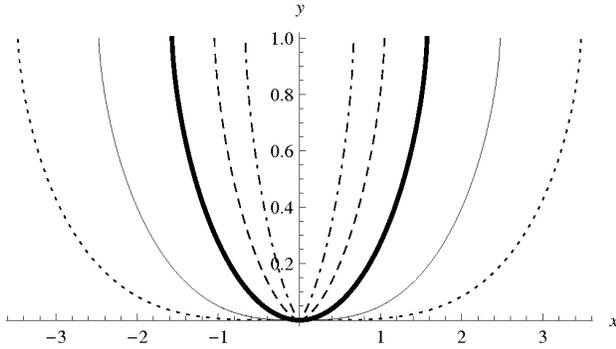


FIGURE 1. Some Σ curves. Chained curve: $\alpha = 0.24$, dashed: $\alpha = 0.54$, continuous thick: $\alpha = 1$ (the tautochrone), thin continuous: $\alpha = 1.86$, dotted: $\alpha = 2.85$. In all cases $H = 1\text{m}$, $g = 9.8\text{m/s}^2$.

specific curve. The solution (9) can be written, with a suitable change of the integration variable, in a very compact way:

$$x_{\beta,H,\pm}(y) = \pm H \int_0^{y/H} \sqrt{\eta^{2\beta-1} - 1} d\eta \quad y \in [0, H] \quad (12)$$

Notice that $\sigma_{\beta,H,-}$ is just the reflection of $\sigma_{\beta,H,+}$, with the vertical axis through the origin acting as the reflection axis.

By plugging (11) in (8) and applying the condition $s_{\pm,\beta,H}(y=0) = 0$ (meaning that we will measure the arch-length from the origin) we find the arch-length traveled from any given point $(x_{\pm,\beta,H}(y), y) \in \sigma_{\beta,H,\pm}$ to the origin:

$$s_{\beta,H,\pm}(x, y) = s_{\beta,H,\pm}(y) = \frac{H}{\beta + 1/2} \left(\frac{y}{H}\right)^{\beta+1/2} \quad (13)$$

so that $s_{\beta,H,-}(y) = s_{\beta,H,+}(y) = s_{\beta,H}(y)$, as would be expected.

Finally, by plugging (13) and (11) in (6) we find:

$$\begin{aligned} \tau_{\beta,H}(s_i) &= \sqrt{\frac{\pi H}{2g} \frac{\Gamma(\beta + 1/2)}{\Gamma(\beta + 1)}} \\ &\times \left\{ \frac{\beta + 1/2}{H} \right\}^{\beta/(\beta+1/2)} s_i^{\beta/(\beta+1/2)} \end{aligned} \quad (14)$$

The meaning of this equation is the following: if a bead moves along the curve $\sigma_{\beta,H,\pm}$ starting from rest at an arch-length s_i away from the origin, it will reach the origin in a time $\tau_{\beta,H}(s_i)$ given by (14).

3. A family of Lagrangians

In order to obtain the equation of motion of a particle moving on a $\sigma_{\beta,H,\pm}$ curve, we could start from the Lagrangian of a free-falling bead

$$L(x, y, \dot{x}, \dot{y}) = T - V = \frac{m}{2} \{(\dot{y})^2 + (\dot{x})^2\} - mgy \quad (15)$$

along with condition (12), which is that a of scleromic holonomic constraint, so that Lagrange's method of indeterminate multipliers could be applied directly. Actually, it is

much easier to reduce the number of variables, eliminating x through (12) in order to obtain from (15) the Lagrangian

$$L_{\beta,H}(y, \dot{y}) = \frac{m}{2} \dot{y}^2 \left(\frac{y}{H}\right)^{2\beta-1} - mgy \quad (16)$$

The Euler-Lagrange equation:

$$\left(\frac{d}{dt} \frac{\partial}{\partial \dot{y}} - \frac{\partial}{\partial y}\right) L_{\beta,H}(y, \dot{y}) = 0 \quad (17)$$

then gives us:

$$\frac{d^2y}{dt^2} + \frac{2\beta-1}{2y} \left(\frac{dy}{dt}\right)^2 + \frac{gH^{1-2\beta}}{y^{1-2\beta}} = 0 \quad (18)$$

valid for $0 \leq y \leq H$. This seems to be quite unmanageable an equation. But by applying the point-transformation $y \rightarrow s$, with the new coordinate s defined through (13) we obtain the transformed Lagrangian

$$\mathcal{L}_{\beta,H}(s, \dot{s}) := \frac{m}{2} \dot{s}^2 - mgH \left\{ \frac{\beta+1/2}{H} s \right\}^{\beta/(\beta+1/2)} \quad (19)$$

so that the Euler-Lagrange equation

$$\left(\frac{d}{dt} \frac{\partial}{\partial \dot{s}} - \frac{\partial}{\partial s}\right) \mathcal{L}_{\beta,H}(s, \dot{s}) = 0 \quad (20)$$

now renders

$$\frac{d^2s}{dt^2} + g(\beta + 1/2)^{2\beta/(2\beta+1)} H^{\frac{2\beta-1}{2\beta+1}} s^{\frac{1-2\beta}{2\beta+1}} = 0. \quad (21)$$

Equation (21) can be written in the form:

$$\frac{d^2s}{dt^2} = -Q_{\alpha,H} s^\alpha \quad (s > 0) \quad (22)$$

with the exponent

$$\alpha := \frac{1-2\beta}{1+2\beta}$$

taking values in $[0, \infty)$, and constant $Q_{\alpha,H} > 0$ defined through:

$$Q_{\alpha,H} := g \frac{(1+\alpha)^{(1-\alpha)/2}}{H^\alpha}.$$

For $\alpha > 0$ the nonlinear Eq. (22) is that of a point-like particle of mass m under the influence of the restoring force

$$f_{\alpha,H}(s) = -mQ_{\alpha,H} s^\alpha$$

(we will not mention anymore the case $\alpha = 0$, which is simply free fall). Thus, Eq. (22) would be that of a anharmonic oscillator, if it would not be for the annoying fact that s is only defined in the half-line. If we chose to stick with Eq. (22) we would have to impose suitable boundary conditions at $s = 0$ in order to have a well posed problem. Instead, we will learn in Sec. 4 how to circumvent this, by joining $\sigma_{\beta,H,+}$ with $\sigma_{\beta,H,-}$ at the origin, as laid out in Sec. 1.

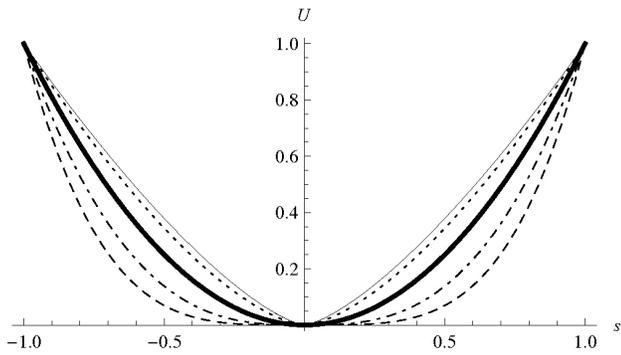


FIGURE 2. Examples of U_{Σ} potentials. Thin continuous curve: $\alpha = 0.24$ ($Q = 12.4 \text{ Jm}^{-1.24}/\text{kg}$.) dotted curve: $\alpha = 0.54$ ($Q = 15.4 \text{ Jm}^{-1.54}/\text{kg}$.) continuous thick curve: a harmonic oscillator, $\alpha = 1$ ($Q = 20.0 \text{ Jm}^{-2}/\text{kg}$.) chained: $\alpha = 1.86$, ($Q=28.6 \text{ Jm}^{-2.86}/\text{kg}$.) dashed: $\alpha = 2.85$ ($Q=38.5 \text{ Jm}^{-3.85}/\text{kg}$.) In all cases $m = 0.1 \text{ kg}$.

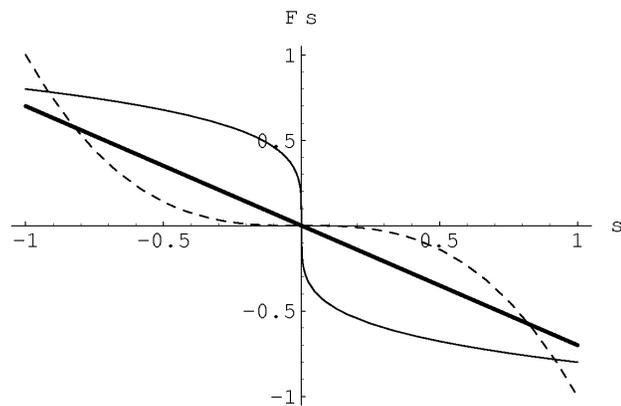


FIGURE 3. Some examples of F_{Σ} . Gray curve: $\alpha = 0.24$ ($mQ=1\text{Nm}^{-0.24}$), black line, corresponding to a harmonic oscillator: $\alpha = 1$ ($mQ=0.7\text{N/m}$), dashed: $\alpha = 2.85$ ($mQ=1\text{Nm}^{-2.85}$).

4. A family of nonlinear oscillators

From (12) it is easy to obtain the limits:

$$\begin{aligned} \lim_{y \rightarrow 0^-} \frac{dx_{\beta,H,-}}{dy} &= +\infty \\ \lim_{y \rightarrow 0^-} \frac{dx_{\beta,H,+}}{dy} &= -\infty \end{aligned} \quad (23)$$

that warrant that the $\sigma_{\beta,H,-}$ path can be smoothly joined at the origin with $\sigma_{\beta,H,+}$. We will call the resulting path:

$$\Sigma_{\beta,H} = \text{Im}(\sigma_{+,\beta,H}) \cup \text{Im}(\sigma_{-,\beta,H}) \quad (24)$$

Here, $\text{Im}(\sigma)$ stands for the path which is the image of curve σ .

Each one of the $\Sigma_{\beta,H}$ paths has one, and only one, minimum around which a bead will describe a periodic motion. The relation between the period of motion and the amplitude in each path can be deduced from (14) so that it is clear that

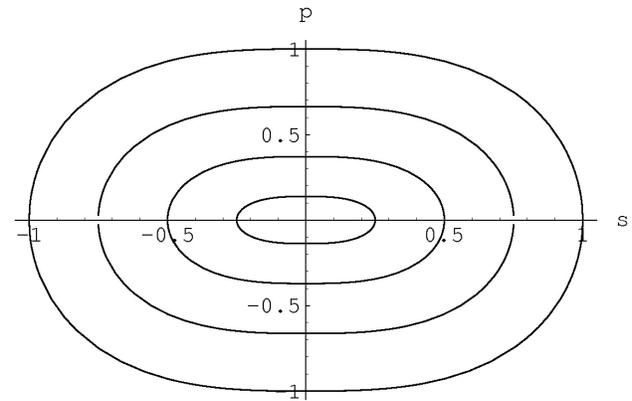


FIGURE 4. The phase-space portrait of a *truly nonlinear* oscillator of Eq. (31), with $\alpha = 2.86$, $m = 0.1\text{kg}$, and $Q=38.6 \text{ Jm}^{-3.86}/\text{kg}$. Amplitudes $s_i = 1, 0.75, 0.5$ and 0.25 m . p_S is graphed in kg m/s and s in meters.

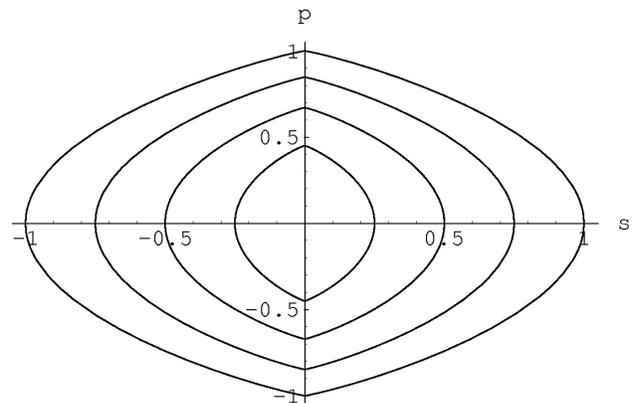


FIGURE 5. The phase-space portrait of a *truly nonlinear* oscillator of Eq. (31), with $\alpha = 0.24$, $m = 0.1\text{kg}$, and $Q=12.4 \text{ Jm}^{-1.24}/\text{kg}$. Amplitudes $s_i = 1, 0.75, 0.5$ and 0.25 m . p_S is graphed in kg m/s and s in meters.

(with the exception of the tautochrone, *i.e.* $\beta = 0$) the motion in a $\Sigma_{\beta,H}$ is anharmonic.

To be more precise: $T_{\beta,H}$, the period of motion in a $\Sigma_{\beta,H}$ curve, is related with the amplitude of motion, s_i , through $T_{\beta,H}(s_i) = 4\tau_{\beta,H}(s_i)$ with $\tau_{\beta,H}$ as given in (14). In terms of exponent α this turns out to be:

$$\begin{aligned} T_{\alpha,H}(s_i) &= \sqrt{\frac{8\pi H}{g}} \frac{\Gamma\left(\frac{1}{1+\alpha}\right)}{\Gamma\left(\frac{3+\alpha}{2(1+\alpha)}\right)} \\ &\times \left\{ (\alpha + 1)H \right\}^{\alpha/2-1/2} s_i^{1/2-\alpha/2}. \end{aligned} \quad (25)$$

In any given $\Sigma_{\beta,H}$ path, the generalized coordinate $S_{\beta,H}$ can be defined through

$$S_{\beta,H}(\mathbf{x}) := \begin{cases} -s_{-,\beta,H}(x, y) & \text{if } (x, y) \in \sigma_{\beta,H,-} \\ s_{+,\beta,H}(x, y) & \text{if } (x, y) \in \sigma_{\beta,H,+} \end{cases} \quad (26)$$

From now on we will drop, for expediency, the β, H subscripts.

Just as $s_{\pm}(x, y)$ assigns to each point in σ_{\pm} one and only one non-negative real value, so that s_{\pm} is a coordinate for σ_{\pm} , $S(x, y)$ assigns to each point in Σ one and only one real value, so that S is a coordinate for Σ .

In order to write the equation of motion of a bead in a Σ path we just need to notice that Eq. (22) can be written at once, for σ_+ and σ_- simultaneously, as:

$$\frac{d^2 S}{dt^2} = F_{\Sigma}(S) \tag{27}$$

where $F_{\Sigma}(S)$ is given by

$$F_{\Sigma}(S) = \begin{cases} mQ|S|^{\alpha} & \text{if } S < 0 \\ 0 & \text{if } S = 0 \\ -mQ|S|^{\alpha} & \text{if } S > 0 \end{cases} \tag{28}$$

The term F_{Σ} , which is the component in the S direction of the total force (constraint plus gravitational) acting on the bead, is derived from potential

$$U_{\Sigma}(S) := \frac{mQ}{\alpha + 1} |S|^{\alpha+1},$$

that is

$$F_{\Sigma}(S) = -\frac{d}{dS} U_{\Sigma}(S), \quad \forall S \in \mathbb{R}. \tag{29}$$

Notice: there is no problem with the derivative of the potential at $S = 0$, the potential is smooth there. The derivative of the force, though, is discontinuous at $S = 0$ for oscillators with $\alpha < 1$ (see Figs. 2 and 3.)

The Lagrangian of the bead moving on a Σ path can then be written down:

$$L_{\Sigma}(S, \dot{S}) = \frac{m}{2} (\dot{S})^2 - \frac{mQ}{\alpha + 1} |S|^{\alpha+1}, \tag{30}$$

along with its Hamiltonian:

$$H_{\Sigma}(S, p_S) = \frac{p_S^2}{2m} + \frac{mQ}{\alpha + 1} |S|^{\alpha+1}. \tag{31}$$

Let us pause for a moment and reflect in the following: by definition, the generalized momentum associated with S is

$$p_S := \frac{\partial}{\partial \dot{S}} L_{\Sigma}, \tag{32}$$

which along with (30) gives

$$p_S = m \dot{S}.$$

Also, we have that the magnitude of the velocity of a bead moving in any curve whatsoever is related with traveled the arch-length through (4), so that the term $p_S^2/2m$ is the kinetic energy of the bead. This did not need to be so: if we had derived a Hamiltonian $H_{\Sigma}(y, p_y)$ the kinetical energy term would have been quite difficult to recognize. In this sense, the arch-length is the ‘‘natural’’ coordinate for a curve.

From (31) and Hamilton’s equations we get the equation of motion of the bead:

$$\frac{d^2 S}{dt^2} + g \frac{\text{sgn}(S)|S|^{\alpha}}{Q^{\alpha}} = 0 \tag{33}$$

which is a quasi-linear second order differential equation, with no exact solutions.

Yet, a qualitative analysis can be made of the solutions of (32). Indeed, from (28) we recognize $U_{\Sigma}(S)$ as the potential energy of the bead when moving on path Σ . We thus have:

$$\frac{p_S^2}{2m} + \frac{mQ}{\alpha + 1} |S|^{\alpha+1} = E \tag{34}$$

where E is the total energy of the bead. From the very beginning we chose to study only conservative systems (no friction.) That Hamiltonian (31) has no explicit time dependency is just a confirmation of this fact.

Equation (33) tells us that our systems are in a sense just distorted versions of the harmonic oscillator, with the $1 - \alpha$ as an indicator of the degree of distortion. The phase-space portraits for some selected values of α are shown in Figs. 3, 4 and 5. All orbits are simply closed, no matter the values of α and E . With the exception of the case for $\alpha = 1$ (the harmonic oscillator,) non of the orbits are ellipses. For $\alpha > 1$ the orbits resemble ellipses compressed in the p_S direction, for $\alpha < 1$ the orbits resemble ellipses compressed in the S direction. The value of $|\alpha - 1|$ gives us an idea of the degree of compression. In the limit when $\alpha \rightarrow 0$ (corresponding to a free-falling bead elastically bouncing at the origin) the orbits collapse into rectangles.

At the same time, the oscillators just presented (with the exception of $\alpha = 1$) are truly nonlinear in the following sense: for $\alpha = 2n, n \in \mathbb{N}$, the Maclaurin series of U_{Σ} contains just one term, which is proportional to S^{α} , and for any other $\alpha \in \mathbb{R}^+$ the potential is not analytical in any neighbourhood around the minimum $S = 0$. This precludes the possibility of a standard small oscillations treatment.

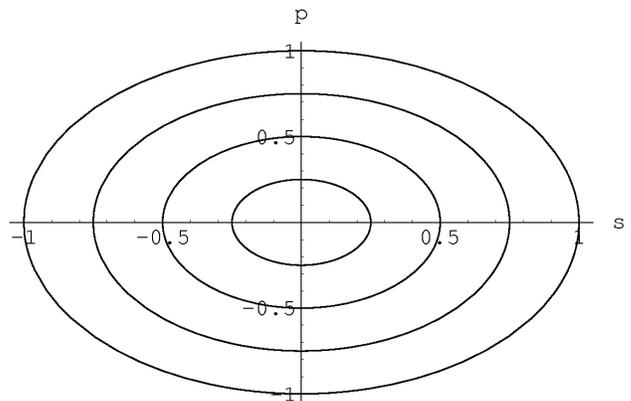


FIGURE 6. The phase-space portrait of an harmonic oscillator, showed for comparison.

5. Conclusions

In the preceding pages we have presented a family of conservative oscillators arising from Abel's mechanical problem. These oscillators are amenable to a lagrangian treatment, as the solutions of Abel's mechanical problem always result in holonomic scleronomic constraints. We have produced the exact period-to-amplitude relation (25), valid for all discussed cases. The oscillators turn out to be *truly nonlinear*, impervious to the usual perturbative treatment.

The material included in this paper may complement and enrich any exposition of classical and/or analytical mechanics at university intermediate level as it:

- Presents little known instances of conservative nonlinear oscillators.

- Provides physically relevant, and fairly simple, examples of the use of lagrangian mechanics.
- Shows us that there are still interesting things to learn in classical mechanics at a fairly elementary level.
- Illustrates the use of mathematical tools such as the Laplace transform and the Gamma function in physical problems.

Acknowledgments

The support of SNI-CONACYT (Mexico) is duly acknowledged.

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